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Construction of Properly Posed Set of Nodes for Graded Lagrange Interpolated in N - Dimensional Space

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Abstract: The constitution of the properly posed set of nodes for the multivariate polynomial graded interpolation is studied deeply in this paper. On the basis of Lagrange interpolation which along the algebraic curve without multiple factors, we proposed the approaches of graded Lagrange interpolation which along the algebraic curve without multiple factors. Furthermore, we advised a basic method of constructing the graded Lagrange interpolation in R^2.
Keywords: properly posed set of nodes; graded interpolation; Lagrange interpolation; multivariate interpolation.

INTRODUCTION

Multivariate interpolation is a classic and complex problem in recent years, and the wide application of multivariate interpolation makes the study of multivariate interpolation theory receive more and more attention. Multivariate polynomial interpolation is often used in the multivariate interpolation, while the multivariate polynomial interpolation has a connection with the single-element polynomial interpolation, but it is not a simple promotion of the monad situati-on because of its complexity in the multivariate polynomial, a primary problem is the existence and uniqueness of the multivariate interpolation polynomial.

The research on the fitness of Lagrange interpolation in bivariate polynomial space was first initiated by professor Liang Xuezhang who transformed multivariate interpolation problem into a geometric problem in reference [1], so that we can use some of the theories and methods of algebraic geometry.

In 1998, the Lagrange interpolation along the algebraic curve without multiple factors was further discussed by Liang Xuezhang and Lv chunmei in reference [2]. In 2003, using the concept of the weak *Gröbner* base, Professor Cui Li-hong gave a new method of constructing the properly posed set of nodes for Lagrange interpolation along the plane algebraic curve.

This paper is the further promotion and development of the above work. Based on the Lagrange interpolation along the algebraic curves without multiple factors, we proposed a, method for graded Lagrange interpolation along the graded algebraic curve without multiple factors, furthermore, using this result we give a basic method for constructing the properly posed set of nodes for graded Lagrange interpolation in $\boldsymbol{R}^2$. The preliminary knowledge that used in this paper will be introduced as follows:

Bezout's theorem [1] Set $p_1(x,y)$ *and* $p_2(x,y)$ respectively is $m-th$ *and* $n-th$ algebraic polynomials. If their public zero number more than mn, there will be a public factor in $p_1(x,y)$ *and* $p_2(x,y)$.

Define 1.1 [1] Let $A=\{q_i\}_{i=1}^{d_n}$ are distinct points of $\boldsymbol{R}^2$, $d_n=\binom{n+2}{2}$, for any given real array $\{f_i\}_{i=1}^{d_n}$, we seeks a polynomial $P(x,y)\in \boldsymbol{P}_n^{(2)}$, which meeting the following condition:

$$P(q_i) = f_i, \quad i = 1, \cdots, d_n \qquad (1.1)$$

If the equations (1.1) always exists a unique set of solution for any given real array $\{f_i\}_{i=1}^{d_n}$, then we say the problem is proper, and call the point set $A = \{q_i\}_{i=1}^{d_n}$ a properly posed set of nodes for $\boldsymbol{P}_n^{(2)}$. In contrast to single-element interpolation, condition $d_n = \binom{n+2}{2} = \dim \boldsymbol{P}_n^{(2)}$ is only a necessary condition that the nodes $A = \{q_i\}_{i=1}^{d_n}$ can be a properly posed set of nods for $\boldsymbol{P}_n^{(2)}$ rather than a sufficient condition. So the main problem of the bivariate Lagrange is research on the appropriate problem. So then, how can point set $A = \{q_i\}_{i=1}^{d_n}$ become the properly posed set of nodes for $\boldsymbol{P}_n^{(2)}$? The answer is in the literature [1] as follows:

Theorem 1.2 [1] $A = \{q_i\}_{i=1}^{d_n}$ can be the properly posed set of nodes for $\boldsymbol{P}_n^{(2)}$ if and only if relations that none of these points is on the algebraic curve $P(x, y) = 0$ which $P(x, y) \in \boldsymbol{P}_n^{(2)}$ and $P(x, y) \neq 0$.

Theorem 1.3 [1] If $A = \{q_i\}_{i=1}^{d_n}$ is a properly posed set of nodes for $\boldsymbol{P}_n^{(2)}$, and if none of these points is on the k-th (k=1,2) irreducible curve $Q(x, y) = 0$, then $A = \{q_i\}_{i=1}^{d_n}$ with $(n+3)k - 1$ points being distinct and selected freely in the irreducible curve $Q(x, y) = 0$ must constitute a properly posed set of nodes for $\boldsymbol{P}_{n+k}^{(2)}$.

The theorem has been used to construct many schemes of the bivariate interpolation.

Definition 1.4 [2] Suppose $Q(x, y) = 0$ is a $l-th$ algebraic curve without multiple factors in $\mathbf{C}^2$, $A = \{q_i\}_{i=1}^{d_n}$ are distinct points on $Q, k = nl - \frac{1}{2}(l^2 - 3l)(n \geq l)$. If relations $\forall P(x, y) \in \boldsymbol{P}_n^{(2)}, p(q_i) = 0, i = 1, 2, \cdots, k$ imply $P(x, y) = 0$ on the algebraic curve Q, then we call $A = \{q_i\}_{i=1}^{d_n}$ a properly posed set of nodes for polynomial interpolation of degree *n* along the $l-th$ algebraic curve without multiple factors Q.

Graded Lagrange Interpolation along graded algebraic curves without multiple factors

First, we give the following definition.

Definition 2.1 Let *n*,*l* are natural numbers, $Q(x, y) = 0$ is $l-th$ graded algebraic curves without multiple factors, $d_n(l)$ is defined as follows:

$$d_n(l) = (n+1)(n+1) - (n+1-l)(n+1-l) \qquad (2.1)$$

Among them $n \geq l$. Let $A = \{q_i\}_{i=1}^{d_n(l)}$ are distinct points on $Q(x, y) = 0$, for any given real array $\{f_i\}_{i=1}^{d_n(l)}$, we seeks a graded polynomial $P(x, y) \in \boldsymbol{P}_{n,n}^{(2)}$, which meeting the following condition:

$$P(q_i) = f_i, \quad i = 1, \cdots, d_n(l) \qquad (2.2)$$

If the equations (2.2) always exists a unique set of solution for any given real array $\{f_i\}_{i=1}^{d_n(l)}$, then we say the problem is proper, and call the point set $A = \{q_i\}_{i=1}^{d_n(l)}$ a properly posed set of nodes for graded polynomial interpolation of degree n which along the $l-th$ graded algebraic curves without multiple factors $Q(x,y)$ and write $A=\{q_i\}_{i=1}^{d_n(l)}\in \boldsymbol{I}_{n,n}^{(2)}(Q)$.

Lemma 2.2 $A=\{q_i\}_{i=1}^{d_n(l)}\in \boldsymbol{I}_{n,n}^{(2)}(Q)$ if and only if relations

$$P(x,y)\in \boldsymbol{P}_{n,n}^{(2)},\ P(q_i)=f_i,\ i=1,\cdots,d_n(l)$$

imply that there exists a $R(x,y)\in \boldsymbol{P}_{n-l,n-l}^{(2)}$,such that $P(x,y)=Q(x,y)R(x,y)$.

Proof: Here we Just prove the necessity.

If $A=\{q_i\}_{i=1}^{d_n(l)}\in \boldsymbol{I}_{n,n}^{(2)}(Q)$ and $P(q_i)=f_i$,$i=1,\cdots,d_n(l)$, by definition 2.1 that $P(x,y)\equiv 0$ on the graded algebraic curves without multiple factors $Q(x,y)=0$, therefore each factor of $P(x,y)\, and\, Q(x,y)$ has an infinite number of intersection, as each factor is irreducible, so by the theorem of Bezout, each factor of $Q(x,y)$ is the factor of $P(x,y)$, then

$$P(x,y)=Q(x,y)\cdot R(x,y) \qquad R(x,y)\in \boldsymbol{P}_{n-l,n-l}^{(2)} \tag{2.3}$$

Among them $n\geq l$.

We obtain the following theorem for a properly posed set of nodes for graded polynomial interpolation of degree n which along the graded algebraic curves without multiple factors.

Theorem 2.3 Suppose $Q(x,y)=0$ is $l-th$ graded algebraic curves without multiple factors in $\boldsymbol{R}^2$ and $Q(x,y)=0$ is factorized as.

$$Q(x,y)=Q_1(x,y)\cdots Q_m(x,y) \tag{2.4}$$

Where $Q_i(x,y)\ (i=1,\cdots,m)$ is distinct irreducible polynomial of degree l_i, respectively $l_1+l_2+\cdots+l_m=l$. If we choose $2(n+1)l_i+1$ distinct points freely on each factor of curve $Q_i(x,y)\ (i=1,\cdots,m)$ and delete properly altogether $r=l^2+m$ points from their aggregation,
then the remaining points constitute a properly posed set of nodes for graded polynomial interpolat
-ion of degree n which along the $l-th$ graded algebraic curves without multiple factors $Q(x,y)$.

Proof: Let $B=\{q_i\}_{i=1}^{k}$ be a set of points on the curve $Q(x,y)=0$,then $k=2(n+1)l+m$, delete properly altogether $r=l^2+m$ points from their aggregation, then the remaining $2(n+1)l-l^2$ points is exactly same as the properly posed set of nodes for graded polynomial interpolation of degree n on $Q(x,y)=0$.

Let $P(x,y)\in \boldsymbol{P}_{n,n}^{(2)}$ is solution of equations

$$P(q_i)=0,\quad i=1,\cdots,k \tag{2.5}$$

Then the $n-th$ graded algebra curve $P(x,y)=0$ and $m-th$ irreducible algebraic curve at least have $2(n+1)l_i+1$, $(i=1,\cdots,m)$ distinct intersection, so by the theorem of Bezout,

$$P(x,y)=Q_1(x,y)\cdots Q_m(x,y)\cdot R(x,y)=Q(x,y)\cdot R(x,y) \qquad (2.6)$$

where $R(x,y)\in \mathbf{P}^{(2)}_{n-l,n-l}$.

On the contrary, $P(x,y)$ that can represent (2.6) are all the solutions of equation (2.5) above. Suppose $T_1(x,y),\cdots,T_{s_1}(x,y)$ is a maximum linearly independent group $\boldsymbol{P}^{(2)}_{n,n}$, $(s_1=(n+1)(n+1))$, $R_1(x,y),\cdots,R_{s_2}(x,y)$ is a maximum linearly independent group of $\boldsymbol{P}^{(2)}_{n-l,n-l}$, ($s_2=(n+1-l)(n+1-l)$).Due to (2.2) is equivalent to (2.5), there are only s_2 linearly unrelated and $n-th$ graded polynomial $P_i(x,y)=Q(x,y)\cdot R_i(x,y)$, $(i=1,\cdots,s_2)$ which through the group B. This shows that the rank of coefficient matrix $[T_j(q_i)]$ $(i=1,\cdots,k;\ j=1,\cdots,s_1)$ for equations 2.2 is $s_1-s_2=2(n+1)l-l^2$. So there must be a s_1-s_2-order non-singular submatrix, the s_1-s_2 points involved in sub-matrix must be a properly posed set of nodes for $n-th$ graded polynomial interpolation along graded algebraic curves $Q(x,y)=0$.

Theorem 2.4 (Recursive Construction Theorem) Let μ be a properly posed set of nodes of $\boldsymbol{P}^{(2)}_{n,n}$, $|\mu|=(n+1)(n+1)$, if none of these points are on a $l-th(l\ge 1)$ graded algebra curve without multiple factors $Q(x,y)=0$, then for any $B\in \boldsymbol{I}^{(2)}_{n+l,n+l}(Q)$, $B\bigcup\mu$ is the properly posed set of nodes of $\boldsymbol{P}^{(2)}_{n+l,n+l}$.

Proof: The number of points in $B\bigcup\mu$ is

$$(n+1)^2+(n+l+1)^2-(n+l+1-l)^2=(n+1+l)^2$$

which exactly is the dimension of $\boldsymbol{P}^{(2)}_{n+l,n+l}$.Let us prove posedness.

Disproof: Suppose $B\bigcup\mu$ is not the properly posed set of nodes of $\boldsymbol{P}^{(2)}_{n+l,n+l}$, there must be $P(x,y)\in \boldsymbol{P}^{(2)}_{n+l,n+l}$,s.t. $P(x,y)=0$ through the point group $B\bigcup\mu$. In particular, the points of B are all at $P(x,y)=0$, none of these points are on the graded algebraic curve $Q(x,y)=0$.This shows that $P(x,y)\ and Q(x,y)$ have $2(n+1)l+l^2$ intersection, the intersection is more than $2(n+l)l$. By the Bezout Theorem, There must be $R(x,y)\in \boldsymbol{P}^{(2)}_{n,n}$, s.t. $P(x,y)=Q(x,y)R(x,y)$. And because $P(x,y)$ through the other points group μ on $Q(x,y)=0$, So $R(x,y)=0$ through the points of μ, and this conflict with μ are proper set of nodes of $\boldsymbol{P}^{(2)}_{n,n}$, so the assumption does not hold, the original proposition holds.

Example : Choose freely a point q_0 in the real plane, It must be the graded interpolation properly posed set of nodes of $\boldsymbol{P}^{(2)}_{0,0}$,do not through q_0 make an ellipse l_1, then q_0 and the $k=(0+2+1)^2-(0+2+1-2)^2=8$ points being distinct and selected freely in l_1 must be the properly posed set of nodes of $\boldsymbol{P}^{(2)}_{2,2}$. (As Picture 1)

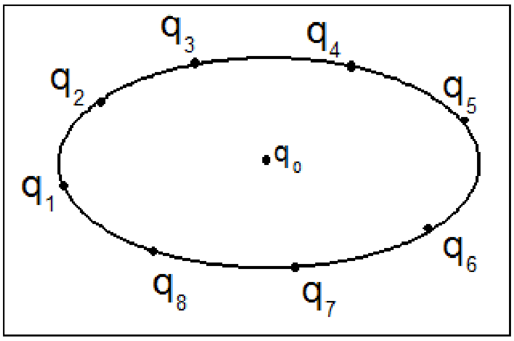

Fig-1:

CONCLUSION

In the present paper, we obtain the basic method for graded Lagrange interpolation along the graded algebraic curve without multiple factors, furthermore, using this result we get the recursive construction method for the properly posed set of nodes for graded Lagrange interpolation in $\boldsymbol{R}^2$. Thus, we can basically understand the geometric structure of the properly posed set of nodes for graded Lagrange interpolation in n-dimensional space.

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