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A Study of Elliptic, Parabolic and Hyperbolic Partial Differential Equation with Finite Difference Approximation

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Abstract **Review Article**

As we know that every partial differential equation can be solve by algebraic method or numerical method, so in this paper we will use finite difference method with Elliptic, parabolic and hyperbolic partial differential equation to solve numerically any partial differential equation.

Keywords: Elliptic, Parabolic, Hyperbolic, Partial Differential.

INTRODUCTION

A partial differential equation illustrate relevant between unfamiliar function and its partial derivative, partial differential equation appear repeatedly in most areas of physic and engineering. Further, in recent year we have seen an exhibitive augmentation in the use of partial differential equation in sectors such as chemistry, biology, and economy and computer science. Work out ordinary differential equation initiate finding a function or a set of a function of one implicit variable but partial differential equation for functions of two or more than two variables. Partial differential equations step up in the study of many branch of applied mathematics for example in fluid dynamics, heat transfer, boundary layer flow, electro-magnetic, elasticity and quantum mechanics. The analytical manner of these equations is more preoccupied process and requires application of advance mathematical methods. On the other hand it is commonly easier to generate enough approximation solution by uncomplicated and impressive numerical method. Various numerical methods have been suggested for the work out of partial differential equation, but just the finite difference methods have become famous and effective occupied than others.

In this paper I have discussed about finite difference approximation to solve Elliptic, Parabolic and Hyperbolic partial differential equation. As we understand that every partial differential equation can be solve analytically or numerically, so I have search my work on the numerical parts to solve PDEs by finite difference scheme with all three equations. The work that I have done are driving Jacobi's method and Gauss-Sidle method with Elliptic part and deriving of explicit and implicit of both Parabolic and Hyperbolic partial differential equation by finite difference scheme and at the end of this paper I have used truncation error with fundamental thermo of numerical analysis as well. The general form of partial differential equation as follows

$$a\frac{\partial^2\varphi}{\partial x^2}+b\frac{\partial^2\varphi}{\partial x\partial y}+c\frac{\partial^2\varphi}{\partial y^2}+d\frac{\partial\varphi}{\partial x}+e\frac{\partial\varphi}{\partial y}+f\varphi=g \qquad (1)$$

This can be written also as follows

$$a\varphi_{xx}+b\varphi_{xy}+c\varphi_{yy}+d\varphi_{x}+e\varphi_{y}+f\varphi=g$$

Where a,b,c... are all function of x and y, also equation (1) can be classified as follows.

1) if $b^2-4ac>0$ Then the PDE belongs to the elliptic equations Laplace equation.

2) if $b^2-4ac=0$ Then the PDE belongs to the parabolic equations or Heat condition equation.

3) if $b^2 - 4ac < 0$ Then the PDE belongs to the hyperbolic equations or Wave equation.

It is simple to see that Laplace equation is the form of Elliptic type, the wave equation is of the form hyperbolic type and the Heat equation is the form of parabolic type [5].

Finite difference for Laplace equation

As we see form interdiction that the Laplace equation is an equation which is given as the form of

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \; or \; \varphi_{xx}+\varphi_{yy}=0 \; or \; \nabla^2\varphi=0 \tag{2}$$

Now we have from finite difference approximation

$$\varphi_{xx}=\frac{1}{h^2}[\varphi(x+h,y)-2\varphi(x,y)+\varphi(x-h,y)]+o(h^2) \tag{3}$$

$$\varphi_{yy}=\frac{1}{k^2}[\varphi(x,y+k)-2\varphi(x,y)+\varphi(x,y-k)]+o(k^2) \tag{4}$$

Now substituting equation (3) and (4) in equation (2) then we get

$$\frac{1}{h^2}[\varphi(x+h,y)-2\varphi(x,y)+\varphi(x-h,y)+\frac{1}{k^2}[\varphi(x,y+k)-2\varphi(x,y)+\varphi(x,y-k)=0 \tag{5}$$

If $h=k$ the from above equation we get

$$\frac{1}{h^2}[\varphi(x+h,y)-2\varphi(x,y)+\varphi(x-h,y)+\varphi(x,y+k)-2\varphi(x,y)+\varphi(x,y-k)]=0 \; or$$

$$\varphi(x+h,y)+\varphi(x-h,y)+\varphi(x,y+k)+\varphi(x,y-k)-4\varphi(x,y)=0$$

$$\varphi(x,y)=\frac{1}{4}[\varphi(x+h,y)+\varphi(x-h,y)+\varphi(x,y+k)+\varphi(x,y-k)] \tag{6}$$

Were h is the grid size.

We can also show the above relevant in the following figure which will shows that the value of φ at any point is the mean of its values at the four vicinity points.

As a suitable notation for grid points and analogous values of the solution is by choosing $x=ih$ and $y=jh$ where $i=0,1,2,3...$ *and* $j=,1,2,3...$

$\varphi(x,y)=\varphi_{i,j}$

$\varphi(x+h,y)=\varphi_{i+1,j}$

$\varphi(x-h,y)=\varphi_{i-1,j}$

$\varphi(x,y+h)=\varphi_{i,j+1}$

$\varphi(x,y-h)=\varphi_{i,j-1}$

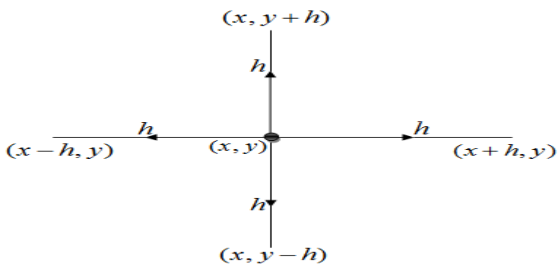

Figure.1

Now equation (b) can be written as

$$\varphi_{i,j}=\frac{1}{4}[\varphi_{i+1,j}+\varphi_{i-1,j}+\varphi_{i,j+1}+\varphi_{i,j-1}] \tag{7}$$

Which shows that the quantity of φ at any interior grid points (nod point) are the average of its values at four neighboring points to the right, left, below and above [17].

Relation (18) is known as standard five points formula which will shown in the below figure [17]
Some time we can use from finite difference approximation to briefly write as follows.

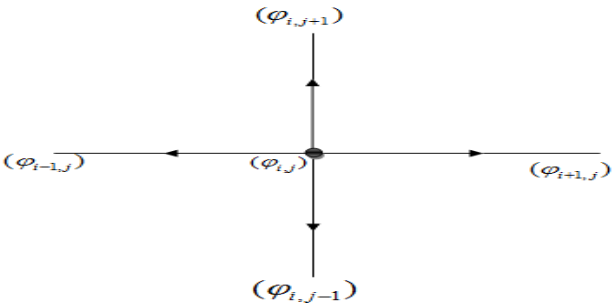

Figure.2

$$\varphi(x+h,y+k)+\varphi(x-h,y+k)+\varphi(x+h,y-k)+\varphi(x-h,y-k)=4\varphi(x,y)+o(h^4)$$

$$=>\varphi(x,y)=\frac{1}{4}[\varphi(x+h,y+k)+\varphi(x-h,y+k)+\varphi(x+h,y-k)+\varphi(x-h,y-k)]\ or$$

$$\varphi_{i,j}=\frac{1}{4}[\varphi_{i+1,j+1}+\varphi_{i-1,j+1}+\varphi_{i+1,j-1}+\varphi_{i-1,j-1}] \tag{8}$$

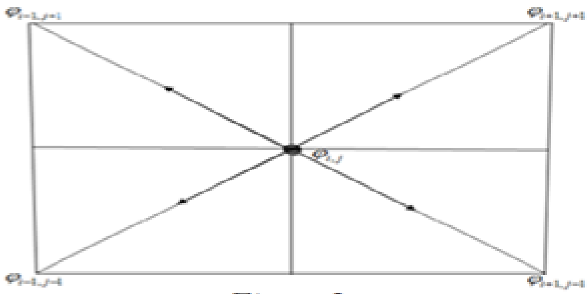

Figure 3

Which shows that the quantity of $\varphi_{i,j}$ is the average of four neighboring diagonal grid points and relation (8) is known as diagonal five points formula which is shown in the below figure.

Note: We should observed that relation (8) is less accurate than relation (7) yet it answer as a wisely good approximation for acquire the starting quantities at the grid points [10].

Computation of Laplace equation

Laplace equation is the form of $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0\ or\ \varphi_{xx}+\varphi_{yy}=0\ or\ \nabla^2\varphi=0$, now consider a rectangular region $\lambda\ in\ xy\ plane$.

Divide this region in square grid points of sides $\Delta x=h\ and\ \Delta y=k$ which is shown in the below figure [10].

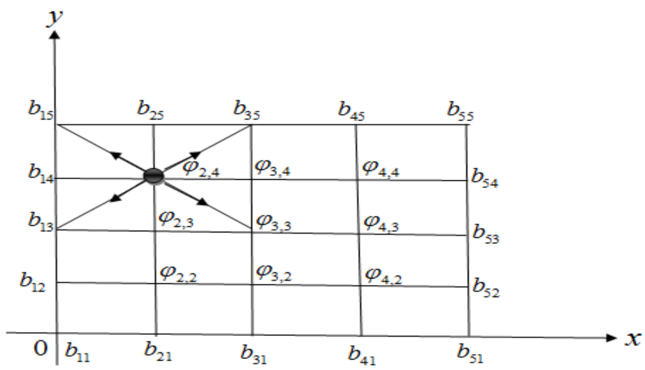

Figure.4

Now to distinguish the initial values of φ at interior grid points, we first use the diagonal five point formula. So we have

$$\varphi_{3,3} = \frac{1}{4}[b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1}]$$

$$\varphi_{2,4} = \frac{1}{4}[b_{1,5} + b_{3,3} + b_{3,5} + b_{1,3}]$$

$$\varphi_{4,4} = \frac{1}{4}[b_{3,5} + b_{5,3} + b_{5,5} + b_{3,3}]$$

$$\varphi_{4,2} = \frac{1}{4}[b_{3,3} + b_{5,1} + b_{5,3} + b_{3,1}]$$

$$\varphi_{2,2} = \frac{1}{4} = [b_{1,3} + b_{3,1} + b_{3,3} + b_{1,1}]$$

The quantities at the remaining interior points are $\varphi_{3,2}, \varphi_{3,4}, \varphi_{4,3}$ *and* $\varphi_{3,2}$ compute by standard five point formula, so we have

$$\varphi_{2,3} = \frac{1}{4}[b_{1,3} + b_{3,3} + b_{2,4} + b_{2,2}]$$

$$\varphi_{3,4} = \frac{1}{4}[b_{2,4} + b_{4,4} + b_{3,5} + b_{3,3}]$$

$$\varphi_{4,3} = \frac{1}{4}[b_{3,3} + b_{5,3} + b_{4,4} + b_{4,2}]$$

$$\varphi_{3,2} = \frac{1}{4}[b_{2,2} + b_{4,2} + b_{3,3} + b_{3,1}]$$

When we distinguished all the quantities of $\varphi_{i,j}$ once then we can find the approximation by the following iteration and we repeat the method since we get trivial difference among tow sequential iteration [15].

Jacobi's Methods: From Jacobi's method we can represent the iterative quantity of $\varphi_{i,j}$ by $\varphi_{i,j}^{(n+1)}$ then the iterative formula to compute (7) is $\varphi_{i,j}^{(n+1)} = \frac{1}{4}[\varphi_{i-1,j}^{(n)} + \varphi_{i+1,j}^{(n)} + \varphi_{i,j-1}^{(n)} + \varphi_{i,j+1}^{(n)}]$ which output better quantities of $\varphi_{i,j}$ at interior point and is called Jacobi's formula [10].

Gauss'- Seidel Method: From the Gauss' Seidel method we can represent the iterative quantity of $\varphi_{i,j}^{(n+1)}$ then the iterative formula to compute (d) is $\varphi_{i,j}^{(n+1)} = \frac{1}{4}[\varphi_{i-1,j}^{(n+1)} + \varphi_{i+1,j}^{(n)} + \varphi_{i,j+1}^{(n+1)} + \varphi_{i,j-1}^{(n)}]$ which output better quantities of $\varphi_{i,j}$ at interior point and is called Gauss'-Seidel method [5]

Remark: Gauss'-Seidel method is smooth and can be agreeable to computer calculation and also its convergence is slow but the solution is lengthy, So Gauss'-Sidle method convergence twice as fast as Jacobi's method and the accuracy of computing depend on the mesh size, that means if we smaller the h we can find better accuracy [12].

Parabolic partial differential equation

We know that parabolic partial differential equation describe Heat condition equation and in this part we are going to use finite difference approximation to solve parabolic partial equation and also we are going to use finite difference approximation to derive explicit and implicit methods with truncation error of parabolic partial differential equation. Also we have used stability, consistency and convergence as well [4].

Finite difference for parabolic equation

In this part we would consider the computing of Heat equation and Heat equation is one of the most famous partial differential equation.

One dimension Heat equation is the form of

$$\frac{\partial \varphi}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial x^2} \tag{9}$$

Heat equation is usually discussed for x in some specific distance $0 \le x \le l$ and time $t \ge 0$ initial temperature $\varphi(0,t) = f(x)$ is given.

The bounder condition at $x = 0$ and $x = l$ for all $t \ge 0$ are gives $\varphi(0,t) = 0$ *and* $\varphi(l,t) = 0$.

Explicit method of parabolic partial differential equation

Consider the Heat condition equation $\varphi_t = \frac{1}{c^2}\varphi_{xx}$ approximate by forward time t>0 and central difference space (x) then as follows we have

$$\varphi_t \approx \frac{\varphi(x,t+k) - \varphi(x,t)}{k}$$

$$\varphi_{xx} \approx \frac{\varphi(x+h,t) - 2\varphi(x,t) + \varphi(x-h,t)}{h^2}$$

Choosing $x = ih$ *and* $y = ih$ *where* $i = 0,1,2,3...$ *and* $j = 0,1,2,3...$ we get

$$\varphi_t \approx \frac{\varphi(i,j+1) - \varphi(i,j)}{k}$$

$$\varphi_{xx} \approx \frac{\varphi(i+1,j) - 2\varphi(i,j) + \varphi(i-1,j)}{h^2}$$

Now substituting the value of φ_t and φ_{xx} in Heat conduction equation then we get

$$\frac{\varphi(i,j+1) - \varphi(i,j)}{k} = c^2\,\frac{\varphi(i+1,j) - 2\varphi(i,j) + \varphi(i-1,j)}{h^2} \tag{10}$$

Interpretation of (10) initiate four points, which are sign in the following figure such that on left hand side of ((10), we have used forward difference since we don't have information for negative t at the beginning [11].

From (10) we can calculate $\varphi_{i,j+1}$ which is

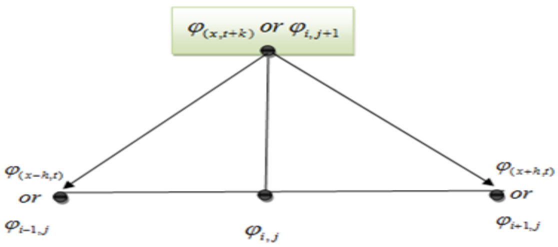

Figure .5

$$\varphi_{i,j+1} = \varphi_{i,j} + \frac{c^2 k}{h^2}[\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}]$$

Putting $\frac{c^2 k}{h^2} = r$ we get

$$\varphi_{i,j+1} = \varphi_{i,j} + r[\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}] => \varphi_{i,j+1} = (1-2r)\varphi_{i,j} + r[\varphi_{i+1,j} + \varphi_{i-1,j}] \quad (11)$$

Computing of such equation is determined for values of x from $0\ to\ l$ and time t from $0\,to\ \infty$.

So the computation is not determined in a close domain but progress in an open terminate region from initial values, satisfying the boundary condition. The above formula (11) is known as Schmidt's formula enables us to define the value of φ at the $(i, j+1)^{Th}$ nodal point in duration the value of the known value at the points $x_{i-1}, x_i\ and\ x_{i+1}$ at the immediate t_j .

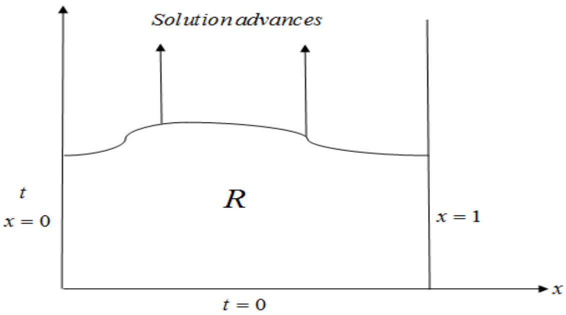

Figure .6

This relevant discus two time levels j and $j+1$ so therefore it is called two level formulas and is reliable for $0<r \le \frac{1}{2}$. In specific if $r = \frac{1}{2}$ then the formula (11) can be change in the form of below

$$\varphi_{i,j+1} = \frac{1}{2}[\varphi_{i+1,j} + \varphi_{i-1,j}] \quad (12)$$

This relevant is called **Bender Schmidt relation** [13].

Remark: In explicit method we have one quantity at level above to the right hand side of equation (11), so for each value of each time step we get explicitly the values [12].

Implicit method of parabolic partial differential equation

Consider the Heat condition $\varphi_t = \frac{1}{c^2}\varphi_{xx}$ we use the backward difference with φ_t and central difference with φ_{xx} in Heat equation then we have

$$\varphi_t \approx \frac{\varphi(i,j) - \varphi(i,j+1)}{k} \quad and \quad \varphi_{xx} \approx \frac{\varphi(i+1,j) - 2\varphi(i,j) + \varphi(i-1,j)}{h^2}$$

$$\frac{\varphi(i,j) - \varphi(i,j-1)}{k} = c^2\,\frac{\varphi(i+1,j) - 2\varphi(i,j) + \varphi(i-1,j)}{h^2} \quad (13)$$

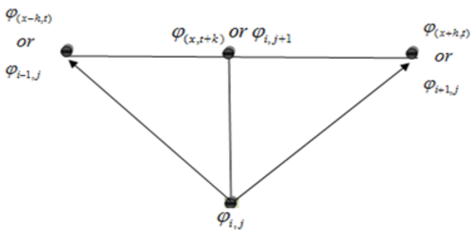

Figure .7

Now written the above approximation at level (j+1)th then we get

$$\frac{\varphi(i,j+1) - \varphi(i,j)}{k} = c^2\,\frac{\varphi(i+1,j+1) - 2\varphi(i,j+1) + \varphi(i-1,j+1)}{h^2}$$

$$\varphi(i,j+1)-\varphi(i,j)=\frac{c^2k}{h^2}[\varphi(i+1,j+1)-2\varphi(i,j+1)+\varphi(i-1,j+1)]$$

If $\frac{c^2k}{h^2}=r$ the we have

$$\varphi(i,j+1)-\varphi(i,j)=r[\varphi(i+1,j+1)-2\varphi(i,j+1)+\varphi(i-1,j+1)]$$

$$\varphi(i,j)=(1+2r)\varphi(i,j+1)-r[\varphi(i-1,j+1)+\varphi(i+1,j+1)] \tag{14}$$

This result is called implicit method formula

Remark: In implicit method we get system of equation, because we have more quantity at level explicit then implicit [2].

Truncation error on parabolic partial differential equation

Consider the Heat condition $\varphi_t=\varphi_{xx}$ Approximation by $\mathrm{F}_{i,j}\varphi=0$ or $\mathrm{L}\varphi\approx\mathrm{F}_{i,j}\varphi=0$. Let $\bar{\varphi}$ be the exact solution $\mathrm{L}\varphi-\mathrm{F}_{i,j}\varphi=T_{i,j}=0$ is locale truncation error then

$$L_{i,j}=\frac{\varphi(i,j)-\varphi(i,j+1)}{k}+o(k)-\frac{\varphi(i-1,j)-2\varphi(i,j)+\varphi(i+1,j)}{h^2}+o(h^2)\approx 0\ or$$

$$T_{i,j}=\frac{\bar{\varphi}(i,j+1)-\bar{\varphi}(i,j)}{k}-\frac{\bar{\varphi}(i-1,j)-2\bar{\varphi}(i,j)+\bar{\varphi}(i+1,j)}{h^2}=0$$

Now by Taylor series we have

$$\bar{\varphi}_{i+1,j}=\bar{\varphi}(x_{i+1,j},t_j)=\bar{\varphi}(x_i,t_j)+h\frac{\partial\bar{\varphi}}{\partial x_i}+\frac{h^2}{2}\frac{\partial^2\bar{\varphi}}{\partial x_i^2}+\frac{h^3}{6}\frac{\partial^3\bar{\varphi}}{\partial x_i^3}+\frac{h^4}{12}\frac{\partial^4\bar{\varphi}}{\partial x_i^4}+ \tag{15}$$

$$\bar{\varphi}_{i-1,j}=\bar{\varphi}(x_{i-1,j},t_j)=\bar{\varphi}(x_i,t_j)-h\frac{\partial\bar{\varphi}}{\partial x_i}+\frac{h^2}{2}\frac{\partial^2\bar{\varphi}}{\partial x_i^2}-\frac{h^3}{6}\frac{\partial^3\bar{\varphi}}{\partial x_i^3}+\frac{h^4}{12}\frac{\partial^4\bar{\varphi}}{\partial x_i^4}+ \tag{16}$$

$$\bar{\varphi}_{i,j+1}=\bar{\varphi}(x_i,t_j)+k\frac{\partial\bar{\varphi}}{\partial x_i}+\frac{k^2}{2}\frac{\partial^2\bar{\varphi}}{\partial x_i^2}+\frac{k^3}{6}\frac{\partial^3\bar{\varphi}}{\partial x_i^3}+\frac{k^4}{12}\frac{\partial^4\bar{\varphi}}{\partial x_i^4}+ \tag{17}$$

Now we substitute equation 15, 16 and 17 in equation 14 then we get

$$T_{i,j}=\frac{1}{k}\{\bar{\varphi}(x_i,t_j)+k\frac{\partial\bar{\varphi}}{\partial x_i}+\frac{k^2}{2}\frac{\partial^2\bar{\varphi}}{\partial x_i^2}+\frac{k^3}{6}\frac{\partial^3\bar{\varphi}}{\partial x_i^3}+\frac{k^4}{12}\frac{\partial^4\bar{\varphi}}{\partial x_i^4}+...-\bar{\varphi}(i.j)\}-\frac{1}{h^2}\{\bar{\varphi}(x_i,t_j)-h\frac{\partial\bar{\varphi}}{\partial x_i}+\frac{h^2}{2}\frac{\partial^2\bar{\varphi}}{\partial x_i^2}-\frac{h^3}{6}\frac{\partial^3\bar{\varphi}}{\partial x_i^3}+\frac{h^4}{12}\frac{\partial^4\bar{\varphi}}{\partial x_i^4}$$

$$+...-2\bar{\varphi}(i,j)+\bar{\varphi}(x_i,t_j)+h\frac{\partial\bar{\varphi}}{\partial x_i}+\frac{h^2}{2}\frac{\partial^2\bar{\varphi}}{\partial x_i^2}+\frac{h^3}{6}\frac{\partial^3\bar{\varphi}}{\partial x_i^3}+\frac{h^4}{12}\frac{\partial^4\bar{\varphi}}{\partial x_i^4}+...\}=0$$

After some calculation it can be reduce as follows

$$T_{i,j}=(\frac{\partial\bar{\varphi}}{\partial t}-\frac{\partial^2\bar{\varphi}}{\partial x^2})+\frac{k}{2}\frac{\partial^2\bar{\varphi}}{\partial x^2}-\frac{h^2}{12}\frac{\partial^4\bar{\varphi}}{\partial x^4}+\frac{k^2}{6}\frac{\partial^3\bar{\varphi}}{\partial x^3}-\frac{h^4}{360}\frac{\partial^6\bar{\varphi}}{\partial x^6}$$

Since $\bar{\varphi}$ is the exact solution of $\frac{\partial\varphi}{\partial t}=\frac{\partial^{2\varphi}}{\partial t^2}$ the leading non zero form (principle part of the local truncation error is $\frac{k}{2}\frac{\partial^2\bar{\varphi}}{\partial x^2}-\frac{h^2}{12}\frac{\partial^4\bar{\varphi}}{\partial x^4}+...=>T_{i,j}=\frac{k}{2}\frac{\partial^2\bar{\varphi}}{\partial x^2}-\frac{h^2}{12}\frac{\partial^4\bar{\varphi}}{\partial x^4}+\frac{k^2}{6}\frac{\partial^3\bar{\varphi}}{\partial x^3}-\frac{h^4}{360}\frac{\partial^6\bar{\varphi}}{\partial x^6}$ Lead form $\mathrm{c}_1k+c_2h^2\varepsilon o(k+h^2)\therefore T_{i,j}\varepsilon o(k+h^2)$.

Hence it is the local truncation error [6].

Notes: By local truncation error we can minimize the error as much as we can [26].

Convergence: A one step finite difference scheme approximated partial differential equation is a convergence scheme if the computing of the finite difference scheme $\varphi_{i,j}$ convergence to $\bar{\varphi}_{(x,t)}$ such as any computation of the partial differential equation$(\Delta x,\Delta t)\to 0$ [7].

Stability: The error caused by a small disturbance in the numerical method remains bounded, that means it could happen unconditionally in the entire domain or it could happen conditionally within a range [16].

Consistency: Given a partial differential equation $L\varphi = f$ approximated by the finite difference scheme $L_{i,j}\varphi = f$ is consistent with the partial differential equation if $L\varphi - L_{i,j}\gamma \to 0 \; as \; \Delta x, \Delta t \to 0$ for γ smooth enough, that means scheme approximating a partial differential maybe stable but has a solution that converge to the solution of differential equation has the mesh length in consistency, for example we can compute the below equation [14].

$$L = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} \; and \; L\varphi = \frac{\partial \varphi}{\partial t} + a\frac{\partial \varphi}{\partial x}$$

Now $L_{i,j}$ is a forward space and forward time then we have

$$L_{i,j} = \frac{\varphi(i,j+1) - \varphi(i,j)}{\Delta t} + a\frac{\varphi(i+1,j) - \varphi(i,j)}{\Delta x} \tag{18}$$

$$\varphi_{i,j+1} = \varphi_{i,j} + \Delta t\frac{\partial \varphi}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} + O(\Delta t)^2 \tag{19}$$

$$\varphi_{i+1,j} = \varphi_{i,j} + \Delta x\frac{\partial \varphi}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 \varphi}{\partial x^2} + O(\Delta x)^2 \tag{20}$$

Now we substitute (19) and (20) in equation (18) then we get

$$L_{i,j} = \frac{\partial \varphi}{\partial t} + a\frac{\partial \varphi}{\partial x} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 \varphi}{\partial x^2} + O(\Delta t)^2 + O(\Delta x)^2 \; \therefore L_{i,j} - L_{i,j}\varphi = -\frac{(\Delta t)^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} - a\frac{(\Delta x)^2}{2!}\frac{\partial^2 \varphi}{\partial x^2} + O(\Delta t)^2 + O(\Delta x)^2$$

Hear $L_{i,j} - L_{i,j}\varphi \to 0 \; as \; \Delta x, \Delta t \to 0$ Therefore the scheme is consistency [18].

Fundamental theorem of numerical analysis ($\text{consistdncy} + \text{stability} <=> \text{convergence}$)

Suppose a liner finite difference scheme is consistent with a well defined linear initial value problem then stability guaranties convergence as mash length goes to zero, for example we have as follows

Suppose $\frac{\partial \varphi}{\partial \text{t}} - \frac{\partial^2 \varphi}{\partial x^2} = 0$ Be approximated by $\frac{\varphi_{i,j+1} - \varphi_{i,j}}{2k} - \frac{\varphi_{i+1,j} - (\frac{3}{2}\varphi_{i,j+1} + \frac{1}{2}\varphi_{i,j-1}) + \varphi_{i-1,j}}{h^2} = 0$ (21)

Now substituting the value of $\varphi_{i,j+1} \; and \; \varphi_{i,j-1}$ by using finite difference method in (14) then we get

$$\varphi_{i,j+1} - \varphi_{i,j-1} = \varphi + k\frac{\partial \varphi}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} + \frac{k^3}{3!}\frac{\partial^3 \varphi}{\partial t^3} + \frac{k^4}{4!}\frac{\partial^4 \varphi}{\partial t^4} + \ldots - \{\varphi - k\frac{\partial \varphi}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} - \frac{k^3}{3!}\frac{\partial^3 \varphi}{\partial t^3} + \frac{k^4}{4!}\frac{\partial^4 \varphi}{\partial t^4} + \ldots\}$$

$$=> \varphi_{i,j+1} - \varphi_{i,j-1} = 2k\frac{\partial \varphi}{\partial t} + \frac{k^3}{3}\frac{\partial^3 \varphi}{\partial t^3} + \ldots$$

Now we have

$$\frac{3}{2}\varphi_{i,j+1} + \frac{1}{2}\varphi_{i,j-1} = \frac{3}{2}[\varphi + k\frac{\partial \varphi}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} + \frac{k^3}{3!}\frac{\partial^3 \varphi}{\partial t^3} + \ldots] + \frac{1}{2}[\varphi - k\frac{\partial \varphi}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} - \frac{k^3}{3!}\frac{\partial^3 \varphi}{\partial t^3} + \ldots] = 2\varphi + k\frac{\partial \varphi}{\partial t} + 2k^2\frac{\partial^2 \varphi}{\partial t^2} + \ldots$$

Now we have $\varphi_{i,j+1} \; and \; \varphi_{i,j-1}$ as follows

$$\varphi_{i,j+1} + \varphi_{i,j-1} = 2\varphi + \frac{h^2}{12}\frac{\partial^2 \varphi}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 \varphi}{\partial x^4} + \ldots$$

By using all above relation we get

$$T_{i,j} = \frac{\partial \varphi}{\partial t} + \frac{k^2}{6}\frac{\partial^3 \varphi}{\partial t^3} + \frac{k^5}{120}\frac{\partial^5 \varphi}{\partial t^5} - \frac{1}{h^2}[2\varphi + h^2\frac{\partial^2 \varphi}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 \varphi}{\partial t^4} + ... - 2\varphi - k\frac{\partial \varphi}{\partial t} - 2k^2\frac{\partial^2 \varphi}{\partial t^2} + ...$$

$$T_{i,j} = (\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2}) + \frac{k}{h^2}\frac{\partial \varphi}{\partial t} + 2\frac{k^2}{h^2}\frac{\partial^2 \varphi}{\partial t^2} + \frac{k^2}{6}\frac{\partial^3 \varphi}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 \varphi}{\partial x^4} + ...$$

Case (1): If $k = rh$ Then it is reduces to $T_{i,j} = (\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2}) + \frac{r}{h}\frac{\partial \varphi}{\partial t} + 2r^2\frac{\partial^2 \varphi}{\partial t^2} + \frac{r^2h^2}{6}\frac{\partial^3 \varphi}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 \varphi}{\partial x^4} + ...$

As $h \to 0$ *then* $\frac{r}{h}\frac{\partial \varphi}{\partial t} \to \infty$ Therefore it is inconsistency.

Case (2): If $k = rh^2$ Then it is reduces to $T_{i,j} = (\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2}) + r\frac{\partial \varphi}{\partial t} + r^2\frac{\partial^2 \varphi}{\partial t^2} + ...$

As $r \to 0$ *then* $h \to 0$ hence it is consistent [15].

Hyperbolic partial differential equation

We know that hyperbolic parabolic partial differential equation describe Wave equation and in this part we are going to use finite difference approximation to solve hyperbolic parabolic partial equation and also we are going to use finite difference approximation to derive explicit and implicit methods with truncation error of hyperbolic parabolic partial differential equation. Also I have used stability, consistency and convergence as well [5].

Finite difference for hyperbolic equation

As we have defined in interdiction, the second order linear partial differential equation $a\varphi_{xx} + b\varphi_{xy} + c\varphi_{yy} + d\varphi_x + e\varphi_y + f\varphi = g$, it is called hyperbolic partial differential equation if $b^2 - 4ac < 0$ is holed.

The simplest instance of hyperbolic equation is the one dimensional wave equation. The study of waves is one of the famous areas in engineering and all vibration problems covered by wave equation. Consider the elastic string vibrating with length L, exist on x-axis the interval $[0,L]$, suppose that $\varphi(x,t)$ denote displacement of the string in the vertical plane, then the vibration of the elastic string is covered one dimensional wave equation

$$\varphi_{xx} = c^2\varphi_{tt} \qquad (22)$$

Where $0 \le x \le L, t \ge 0$ such that c^2 is a constant and depend on the property of string and for solving hyperbolic equation we need to prescribe the following conditions

- Initial condition or initial displacement $\varphi(x,0) = f(x)$ $0 \le x \le L$ and initial velocity $\varphi_t(x,0) = g(x,0)$ $0 \le x \le L$
- Boundary condition $\varphi(0,t) = 0$ *and* $\varphi(L,t) = 0$ $t \ge 0$ we consider the case when the ends string is fixed.

Since the initial and boundary condition are both prescribe then the problem is called initial and boundary value problem.

Likewise of Heat condition equation, we are going to derive Waveequation then explain explicit and implicit hyperbolic partial differential equation [13].

Derivation of one dimensional of wave equation

Consider the following cortisone coordinate (x, y) with the deflection of string in the vertical discretion at a given time t and given spatial location x.

Let take an infinitesimal points PQ in the length of spring vibration, then we will have the following assumption

1. The mass per unit length is constant and its shows by ρ .
2. The string perfectly should be elastic and often no resistance to bending.
3. The gravitational force is negligible.

4. The string perfectly small\negligible transverse deflection x direction.
5. The vertical deflections are small at all times and therefore slopes are small.

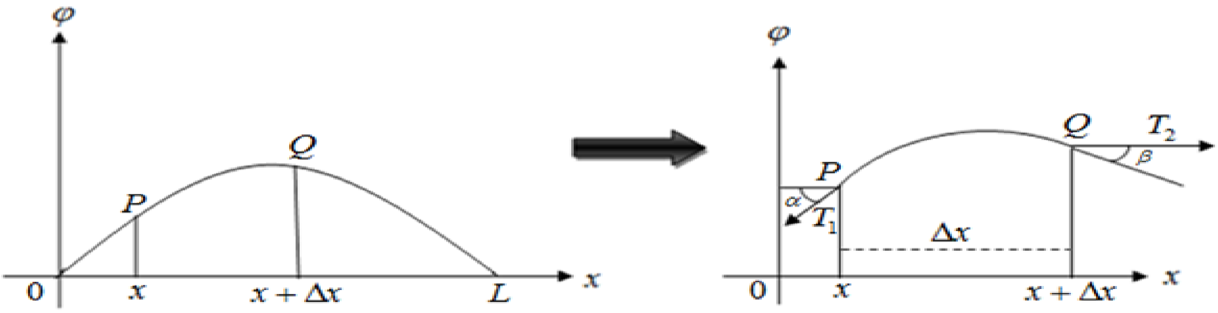

Figure .8

Now from assumption (3) to holds we need the horizontal component force to be equal, so we have

$$T_1 \cos\alpha = T_2 \cos\beta = T \tag{23}$$

If the angel α *or* β are small, then *cos* of but tens to one the $T_1 = T_2 = T$ It means that the horizontal component are equal. Now according to the vertical direction we have, sum of the force in vertical direction equal to the mass with the acceleration in vertical direction.

$\sum F = Ma$ Where F shows the force in vertical direction, M shows mass and a shows the acceleration in vertical direction

$T_2 \sin\beta - T_1 \sin\alpha = M \dfrac{\partial^2\varphi}{\partial t^2}$ Where $a = \dfrac{\partial^2\varphi}{\partial t^2}$ is acceleration and also $M = \rho \cdot \Delta x$

$$T_2 \sin\beta - T_1 \sin\alpha = M \frac{\partial^2\varphi}{\partial t^2} \Rightarrow T_2 \sin\beta - T_1 \sin\alpha = \rho \cdot \Delta x \frac{\partial^2\varphi}{\partial t^2}$$

Now we divide but said by T , then we get

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{\rho \cdot \Delta x}{T} \frac{\partial^2\varphi}{\partial t^2} \tag{24}$$

Now from equation (16) we put the value of T in L.H.S of (24) then we get

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \cdot \Delta x}{T} \frac{\partial^2\varphi}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho \cdot \Delta x}{T} \frac{\partial^2\varphi}{\partial t^2} \tag{25}$$

$Tag(\beta)$ Is slope of the string at $Q = (x + \Delta x) = \left.\dfrac{\partial\varphi}{\partial x}\right|_{x+\Delta x}$ and $Tag(\alpha)$ is slope of the string at $P = x = \left.\dfrac{\partial\varphi}{\partial x}\right|_{x}$ now from (25) we get

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial\varphi}{\partial x}\right|_{x} = \frac{\rho \cdot \Delta x}{T} \frac{\partial^2\varphi}{\partial t^2}$$

Now divide but said by Δx , then we get

$$\frac{1}{\Delta x}\left[\left.\frac{\partial\varphi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial\varphi}{\partial x}\right|_{x}\right] = \frac{\rho}{T} \frac{\partial^2\varphi}{\partial t^2} \tag{26}$$

An question arise hear that how much did the slope $\frac{\partial \varphi}{\partial x}$ change between the point P and Q it means that the second derivative of deflection with respect to x arise, therefore we get

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 \varphi}{\partial t^2} => \frac{\partial^2 \varphi}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 \varphi}{\partial x^2} \text{ or } \varphi_{tt} = c^2 \varphi_{xx} \text{ , where } c^2 = \frac{T}{\rho} \tag{27}$$

Hence equation (27) is called **wave** equation [2].

Explicit method for wave equation

The one dimensional wave equation is given by $\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2}$. Suppose the step sizes for x *and* t equal to h *and* k such that $x_i = x_0 + ih$ and $t_i = t_0 + jk$, now by using central difference, we write the approximation

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{h^2}[\varphi(x+h,t) - 2\varphi(x,t) + \varphi(x-h,t)] => \varphi_{xx} = \frac{1}{h^2}[\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}] \tag{28}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{k^2}[\varphi(x,y+k) - 2\varphi(x,y) + \varphi(x,y-k)] => \varphi_{tt} = \frac{1}{k^2}[\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}] \tag{29}$$

Now applying equation (28) and (29) in Wave equation then we get

$$\frac{1}{k^2}[\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}] = c^2 \frac{1}{h^2}[\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}]$$

$$\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1} = \frac{k^2c^2}{h^2}[\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}] => \varphi_{i,j+1} = r[\varphi_{i+1,j} + \varphi_{i-1,j}] + (2-2r)\varphi_{i,j} - \varphi_{i,j-1} \tag{30}$$

Where $r = \frac{k^2c^2}{h^2}$

Equation (30) is called finite difference explicit method of hyperbolic partial differential equation [5].

Observation

1) If $r = \frac{k^2c^2}{h^2} = 1$ Then the coefficient of $\varphi_{i,j}$ vanish and formula (30) take the form

$$\varphi_{i,j+1} = \varphi_{i+1,j} + \varphi_{i-1,j} - \varphi_{i,j-1} \tag{31}$$

2) If $r = \frac{1}{c}$, Then the solution of (30) is stable and coincides with the solution of Wave equation.

3) If $r < \frac{1}{c}$, Then the solution is stable but inaccurate.

4) If $r > \frac{1}{c}$, Then the solution is unstable.

5) If $r \le 1 \; i \cdot e \; k \le h$, Then the formula (30) is convergence [16].

Truncation error in hyperbolic partial differential equation

Consider the below explicit wave equation, express by Taylor series

$$\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1} = r^2[\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}] \tag{32}$$

$$\varphi_{i+1,j} = \varphi(x_{i+1},t) + \varphi + h\frac{\partial \varphi}{\partial t} + \frac{h^2}{2!}\frac{\partial^2 \varphi}{\partial t^2} + \frac{h^3}{3!}\frac{\partial^3 \varphi}{\partial t^3} + \frac{h^4}{4!}\frac{\partial^4 \varphi}{\partial t^4} \tag{33}$$

$$\varphi_{i-1,j} = \varphi(x_{i-1,j},t_j) = \varphi(x_i,t_j) - h\frac{\partial \varphi}{\partial x_i} + \frac{h^2}{2}\frac{\partial^2 \varphi}{\partial x_i^2} - \frac{h^3}{6}\frac{\partial^3 \varphi}{\partial x_i^3} + \frac{h^4}{12}\frac{\partial^4 \varphi}{\partial x_i^4} + \tag{34}$$

$$\varphi_{i,j+1} = \varphi(x_i, t_j) + k\frac{\partial\varphi}{\partial x_i} + \frac{k^2}{2}\frac{\partial^2\varphi}{\partial x_i^2} + \frac{k^3}{6}\frac{\partial^3\varphi}{\partial x_i^3} + \frac{k^4}{12}\frac{\partial^4\varphi}{\partial x_i^4} + \quad (35)$$

$$\varphi_{i,j-1} = \varphi - k\frac{\partial\varphi}{\partial x_i} + \frac{k^2}{2}\frac{\partial^2\varphi}{\partial x_i^2} - \frac{k^3}{6}\frac{\partial^3\varphi}{\partial x_i^3} + \frac{k^4}{12}\frac{\partial^4\varphi}{\partial x_i^4} + \quad (36)$$

Now substitute equations (33), (34), (35) and (36) in equation (32) then we get

$$\backslash\varphi + h\frac{\partial\varphi}{\partial t} + \frac{h^2}{2!}\frac{\partial^2\varphi}{\partial t^2} + \frac{h^3}{3!}\frac{\partial^3\varphi}{\partial t^3} + \frac{h^4}{4!}\frac{\partial^4\varphi}{\partial t^4} + \ldots - 2\varphi + \varphi - k\frac{\partial\varphi}{\partial x_i} + \frac{k^2}{2}\frac{\partial^2\varphi}{\partial x_i^2} - \frac{k^3}{6}\frac{\partial^3\varphi}{\partial x_i^3} + \frac{h^4}{4!}\frac{\partial^4\varphi}{\partial t^4} + \ldots$$

From L.H.S we get $[k^2\frac{\partial^2\varphi}{\partial t^2} + \frac{k^4}{12}\frac{\partial^4\varphi}{\partial t^4} + \ldots]$ and R.H.S will take the form

$$= r^2[\{\varphi + h\frac{\partial\varphi}{\partial x} + \frac{h^2}{2!}\frac{\partial^2\varphi}{\partial x^2} + \frac{h}{3!}\frac{\partial^3\varphi}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4\varphi}{\partial t^4} + \cdots\} - 2\varphi + \{\varphi - h\frac{\partial\varphi}{\partial x} + \frac{h^2}{2!}\frac{\partial^2\varphi}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3\varphi}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4\varphi}{\partial x^4} + \}..]$$

From R.H.S we get $\frac{k^2c^2}{h^2}\left[h^2\frac{\partial^2\varphi}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4\varphi}{\partial x^4} + \cdots\right] = k^2c^2\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4\varphi}{\partial x^4} + \cdots\right]$, where $r^2 = \frac{c^2k^2}{h^2}$

Where all the process on the right hand sides is evaluated at (x_i, t_j), so the truncation error is given by

$$T.E = \varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1} - r^2(\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j})$$
$$= \left[k^2\frac{\partial^2\varphi}{\partial t^2} + \frac{k^4}{12}\frac{\partial^4\varphi}{\partial t^4} + \cdots\right] - k^2c^2\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4\varphi}{\partial x^4} + \cdots\right]$$
$$k^2\left[\frac{\partial^2\varphi}{\partial t^2} - c^2\frac{\partial^2\varphi}{\partial x^2}\right] + \frac{k^4}{12}\frac{\partial^4\varphi}{\partial t^4} - \frac{k^2h^2c^2}{12}\frac{\partial^4\varphi}{\partial x^4} + \cdots$$

Now by using differential equation we have

$$\frac{\partial^2\varphi}{\partial t^2} = c^2\frac{\partial^2\varphi}{\partial x^2} \Rightarrow \frac{\partial^4\varphi}{\partial t^4} = c^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\varphi}{\partial x^2}\right) = c^4\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\varphi}{\partial x^2}\right) = c^4\frac{\partial^4\varphi}{\partial x^4}$$

We obtain form truncation error

$$T.E = \frac{k^4c^4}{12}\frac{\partial^4\varphi}{\partial x^4} - \frac{k^4h^4c^4}{12}\frac{\partial^4\varphi}{\partial x^4} + \cdots = \frac{k^4h^4c^4}{12}\left[(r^2-1)\frac{\partial^4\varphi}{\partial x^4} + \cdots\right]$$

Since $k = \frac{hr}{c}$, so the order of method is given by

$order = \frac{1}{k^2}(T.E) = O(h^2 + k^2)$ [3].

Implicit method for wave equation

As we see from explicit method that it has disadvantages, because explicit method has a stability condition on the mesh ratio parameter $r = \frac{ck}{h}$.

Explicit method is stable for $r \leq 1$ and this condition limit the measures that can be for the length h and k ,so explicit method is not useful because the time uses is too high, therefore in such case we use implicit method**,** hence we derive the following two implicit methods [6].

1. At the first step we write the following approximation at (x_i, t_j)

$$\frac{\partial^2\varphi}{\partial t^2}_{i,j} = \frac{1}{k^2}\omega_t^2\varphi_{i,j} \quad 37$$

$$\frac{\partial^2\varphi}{\partial x^2}_{i,j} = \frac{1}{2h^2}\omega_x^2[\varphi_{i,j+1} + \varphi_{i,j-1}] \quad 38$$

So the difference approximation to the wave equation at the mesh point (x_i, t_j) is given by

$$\frac{1}{k^2}\omega_t^2\varphi_{i,j} = \frac{c^2}{2h^2}\omega_x^2[\varphi_{i,j+1} + \varphi_{i,j-1}] \; OR \; \omega_t^2\varphi_{i,j} = \frac{r^2}{2}\omega_x^2[\varphi_{i,j+1} + \varphi_{i,j-1}] \; OR$$

$$\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1} = \frac{r^2}{2}\left[\omega_x^2\varphi_{i,j+1} + \omega_x^2\varphi_{i,j-1}\right] \; OR$$

$$\varphi_{i,j+1} - \frac{r^2}{2}\omega_x^2\varphi_{i,j+1} = 2\varphi_{i,j} - \varphi_{i,j-1} + \frac{r^2}{2}\omega_x^2\varphi_{i,j-1} \qquad 39$$

Now we can expand the central difference approximation and write

$$\omega_x^2\varphi_{i,j+1} = \varphi_{i+1,j+1} - 2\varphi_{i,j+1} + \varphi_{i-1,j+1} \qquad 40$$

$$\omega_x^2\varphi_{i,j-1} = \varphi_{i+1,j-1} - 2\varphi_{i,j-1} + \varphi_{i-1,j-1} \qquad 41$$

Now we substitute the (40) and (41) in (39) then we get

$$\varphi_{i,j+1} - \frac{r^2}{2}\left(\varphi_{i+1,j+1} - 2\varphi_{i,j+1} + \varphi_{i-1,j+1}\right) = 2\varphi_{i,j} - \varphi_{i,j-1} + \frac{r^2}{2}\left(\varphi_{i+1,j-1} - 2\varphi_{i,j-1} + \varphi_{i-1,j-1}\right)$$

$$\varphi_{i,j+1} - \frac{r^2}{2}\varphi_{i+1,j+1} - \frac{r^2}{2}2\varphi_{i,j+1} + \frac{r^2}{2}\varphi_{i-1,j+1}$$

$$= 2\varphi_{i,j} - \varphi_{i,j-1} + \frac{r^2}{2}\varphi_{i+1,j-1} - \frac{r^2}{2}2\varphi_{i,j-1} + \frac{r^2}{2}\varphi_{i-1,j-1}$$

$$-\frac{r^2}{2}\varphi_{i+1,j+1} + (1+r^2)\varphi_{i,j+1} - \frac{r^2}{2}\varphi_{i-1,j+1}$$

$$= 2\varphi_{i,j} + \frac{r^2}{2}\varphi_{i+1,j-1} - (1+r^2)\varphi_{i,j-1} + \frac{r^2}{2}\varphi_{i-1,j-1} \qquad 42$$

Hence equation (42) shows that the implicit method [12].

Notes (1): For finding truncation error we can use Taylor expansion in equation (42), then we get $O\left(k^4 + \frac{k^2}{h^2}\right)$, so the order of this method is $O(k^2 + h^2)$.

2. The second step we use approximation (1) for $\frac{\partial^2\varphi}{\partial t^2}$ and the following approximation for

$$\frac{\partial^2\varphi}{\partial x^2}.\frac{\partial^2\varphi}{\partial x^2}_{i,j} = \frac{1}{2h^2}\omega_x^2\left[\varphi_{i,j+1} - \varphi_{i,j} + \varphi_{i,j-1}\right] \qquad 43$$

So the difference approximation to the wave equation at the mesh point (x_i, t_j) is given by

$$\frac{1}{k^2}\omega_t^2\varphi_{i,j} = \frac{c^2}{2h^2}\omega_x^2\left[\varphi_{i,j+1} - \varphi_{i,j} + \varphi_{i,j-1}\right] => \omega_t^2\varphi_{i,j} = r^2\omega_x^2\left[\varphi_{i,j+1} - \varphi_{i,j} + \varphi_{i,j-1}\right]$$

$$\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j+1} = \omega_x^2\varphi_{i,j+1} - \omega_x^2\varphi_{i,j} + \omega_x^2\varphi_{i,j-1}$$

$$\varphi_{i,j+1} - r^2\omega_x^2\varphi_{i,j+1} = 2\varphi_{i,j} - \varphi_{i,j+1} - r^2\omega_x^2\varphi_{i,j} + r^2\omega_x^2\varphi_{i,j-1} \qquad 44$$

Now if we expand the central difference in equation(65), then we get the implicit method.

$$\omega_x^2\varphi_{i,j+1} = \varphi_{i+1,j+1} - 2\varphi_{i,j+1} + \varphi_{i-1,j+1} \quad and \quad \omega_x^2\varphi_{i,j-1} = \varphi_{i+1,j-1} - 2\varphi_{i,j-1} + \varphi_{i-1,j-1}[25]$$

Notes (2): Hear also for finding truncation error we can use by Taylor expansion in equation (42), then we get $O\left(k^4 + \frac{k^2}{h^2}\right)$, so the order of this method is $O(k^2 + h^2)$ [8].

Remark: We saw that, implicit method often has very strong stability properties, analysis of stability equation (42) and (44) shows that the method is stable for all values of the grid ratio parameter r, hence the method is unconditionally stable, this shows that there is no restriction on the value of the grid length h and k [12].

CONCLUSION

On the basis of the above discussion we get the result obtained by finite difference approximation to provide the solution of Elliptic, Parabolic and Hyperbolic partial differential equation, so we know that every partial differential equation can be solved analytically or numerically and finite difference approximation is one of the most important ways to solve numerically any partial differential equations.

On the above three equation we are used finite difference scheme to derive Gauss-Sidle and Jacobi's method by in Elliptic equation, and derive explicit and implicit formula of both Parabolic and Hyperbolic equations. In the other hand we find that Gauss'-Seidel method is smooth and can be agreeable to computer calculation and also its convergence is slow but the solution is lengthy, So Gauss'-Sidle method convergences twice as fast as Jacobi's method and the accuracy of computing depend on the mesh size that means if we smaller the h we can find better accuracy. Also with the explicit method for each value of each time step we get explicitly the values but with the implicit method we get system of equation, because we have more quantity at level explicit then implicit.

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