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Investigating the Robustness of Filters for Integrated Processes in Business Cycles

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Abstract **Original Research Article**

Time series are frequently filtered to remove unwanted characteristics, such as trends and seasonal components, or to estimate components driven by stochastic cycles from a specific range of periods in a business cycle. A polynomial function of time is the most common deterministic time trend while an integrated process is the most common stochastic trend. The different filters implemented in this paper allow for different orders of deterministic time trends or integrated processes. The robustness of the filters is evaluated by plotting their gain function against the gain function of a simulated ideal filter. Implementing the filters on Nigerian gross domestic products (GDP), the results show that the gain of the Baxter-King (BK) filter deviates markedly from the square-wave gain of the ideal filter. The gain in Christiano–Fitzgerald (CF) filter is closer to the gain of the ideal filter than the BK filter. The gain in Hodrick–Prescott (HP) filter goes to one for those cycles at frequencies above six periods, whereas the other gain functions go to zero. The Butterworth (BW) filter does a reasonable job of filtering out the high-periodicity stochastic cycles but the low- periodicity stochastic cycles is not been completely removed.

Keywords: Filters, Business-cycle components, Ideal filter, Gain function, Stochastic cycles.

1. INTRODUCTION

Time series may contain deterministic trends or stochastic trends. A polynomial function of time is the most common deterministic time trend. An integrated process is the most common stochastic trend. An integrated process is a random variable that must be differenced one or more times to be stationary (Hamilton, 1994).

One common approach to filtering is the frequency domain method used by Hassler, *et al.*, (1994) and Rush, *et al.*, (1997). This method works as follows. First, one takes a discrete Fourier transform of the economic data, computing the periodic components associated with a finite number of "harmonic" frequencies. Second, one "zeros out" the frequencies that lie outside of the band of interest. Third, one computes the inverse Fourier transform to get the time domain filtered series, $\{ \tilde{y}_1,..., \tilde{y}_T \}$. There are two major drawbacks with this explicitly domain procedure, relative to our time domain method. First, since there are likely "stochastic trends" in most economic time series, arising from unit root component, it is necessary to first de-trend the series prior to taking the Fourier transform: in order to accomplish band-pass filtering, one must therefore make a choice of de-trending method. Working with annual data, Hassler *et al.*, (1994) used the Hodrick-Prescott filter with $\lambda = 10$ for this initial de-trending step. Working with quarterly data, Rush, *et al.*, (1997) argued for a much larger value, $\lambda = 10{,}000$ in the initial detrending step so as to avoid distorting business cycle outcomes. Second, the results of the frequency domain method at all dates are dependent on the sample length *T*. Consider, for example, the business cycle outcome $\tilde{y}_t$ obtained from a study of quarterly economic data in a study of length T_1, e.g. the observation on cyclical output in 1970:2, obtained used data through 1985. When the sample length is extended to T_2, the discrete Fourier transform of $\{ y_1, y_2 ..., y_T \}$ must be recomputed and each of its elements will change. Consequently, so too will each of the elements of the inverse Fourier transform of the filtered series, i.e., the cyclical observations, {

$\tilde{y}_1,...,\tilde{y}_T$ }. Thus, the outcome for cyclical output in 1970:2 will necessarily be different when data is added from 1986 to 1994. We need some concepts from the frequency-domain approach to time-series analysis to motivate how Baxter and King (1999) defined "as close as possible". The intuitive explanation presented here glosses over many technical details discussed by Priestley (1981), Hamilton (1994), Fuller (1996), and Wei (2006). As with much time-series analysis, the basic results are for covariance-stationary processes with additional results handling some nonstationary cases. We present some useful results for covariance stationary processes and discuss how to handle nonstationary series below.

The autocovariances γ_j, $j \in \{0,1,...,\infty\}$, of a covariance-stationary process y_t specify its variance and dependence structure. In the frequency-domain approach to time-series analysis, y_t and the autocovariances are specified in terms of independent stochastic cycles that occur at frequencies $\omega \in [-\pi, \pi]$. The spectral density function $f_y(\omega)$ specifies the contribution of stochastic cycles at each frequency ω relative to the variance of y_t, which is denoted by $\sigma_y^{\ 2}$. The variance and the autocovariances can be expressed as an integral of the spectral density function. Formally,

$$\gamma_j = \int_{-\pi}^{\pi} e^{iwj} f_y(\omega) d\omega \quad , \qquad (1)$$

where *i* is the imaginary number $i = \sqrt{-1}$.

The equation can be manipulated to show what fraction of the variance of y_t is attributable to stochastic cycles in a specified range of frequencies (Hamilton (1994). The equation implies that if $f_y(\omega) = 0$ for $\omega \in [-\omega_1, \omega_2]$, stochastic cycles at these frequencies contribute zero to the variance and autocovariances of y_t. The goal of time-series filters in this paper is to transform the original series into a new series $y_t^{\ *}$ for which the spectral density function of the filtered series $f_{y^*}(\omega)$ is zero for unwanted frequencies and equal to $f_y(\omega)$ for desired frequencies.

1.1 Band-pass Filters: The Baxter-King and Christiano-Fitzgerald Filter

For an infinitely long series, there is an ideal band-pass filter for which the gain function is 1 for $\omega \in [-\omega_0, \omega_1]$, and 0 for all other frequencies. It just so happens that this ideal band-pass filter is asymmetric moving average (SMA) filter with coefficients that sum to zero. Baxter and King (1999) derived the coefficients of this ideal band-pass filter and then define the BK filter to be the SMA filter with $2q + 1$ terms that are as close as possible to those of the ideal filter. There is a trade-off in choosing q: larger values of *q* cause the gain of the BK filter to be closer to the gain of the ideal filter, but larger values also increase the number of missing observations in the filtered series. Although Baxter and King (1999) minimized the error between the coefficients in their filter and the ideal band-pass filter, Christiano and Fitzgerald (2003) minimized the mean squared error between the estimated component and the true component, assuming that the raw series is a random-walk process. Christiano and Fitzgerald (2003) gave three important reasons for using their filter:

1. The true dependence structure of the data affects which filter is optimal.
2. Many economic time series are well approximated by random-walk processes.
3. Their filter does a good job passing through stochastic cycles of desired frequencies and blocking stochastic cycles from unwanted frequencies on a range of processes that are close to being a random-walk process.

The CF filter obtains its optimality properties at the cost of an additional parameter that must be estimated and a loss of robustness. The CF filter is optimal for a random-walk process. If the true process is a random walk with drift, then the drift term must be estimated and removed. The CF filter is not symmetric, so it will not remove second-order deterministic or second-order integrated processes. This filter is designed to be as close as possible to the random-walk optimal filter under the constraint that it be an SMA filter with constraints that sum to zero.

1.2 High-pass Filters: The Hodrick-Prescott and Butterworth Filter

The Hodrick and Prescott (1997) is a one-parameter high pass filter. It is a trend-removal technique that could be applied to data that came from a wide class of data-generating processes. In their view, the technique specified a trend in the data, and the data were filtered by removing the trend. The smoothness of the trend depends on a parameter λ. The trend becomes smoother as $\lambda \to \infty$.

Hodrick and Prescott (1997) recommended setting λ to 1,600 for quarterly data. King and Rebelo (1993) showed that removing a trend estimated by the HP filter is equivalent to a high-pass filter. They derived the gain function of this high-pass filter and showed that the filter would make integrated processes of order 4 or less stationary, making the HP filter comparable with the band-pass filters.

A two-parameter high pass filter is Butterworth filter. The gain functions of these filters are as close as possible to being a flat line at 0 for the unwanted periods and a flat line at 1 for the desired periods (Butterworth, 1930; Bianchi and Sorrentino, 2007).

Pollock (2000) showed that Butterworth filters can be derived from some axioms that specify properties we would like a filter to have. Although the Butterworth and BK filters share the properties of symmetry and phase neutrality, the coefficients of Butterworth filters do not need to sum to zero. Phase-neutral filters do not shift the signal forward or backward in time; Pollock (1999). Although the BK filter relies on the detrending properties of SMA filters with coefficients that sum to zero, Pollock (2000) showed that Butterworth filters have detrending properties that depend on the filters' parameters.

Yogo (2008) discussed the use of multiresolution wavelet analysis to decompose an economic time series into trend, cycle, and noise. The method is illustrated with GDP data, and the business-cycle component of the wavelet-filtered series closely resembles the series filtered by the approximate bandpass filter.

Hessler (2023) applied unobserved components (UC) models with real and financial trends and business and credit cycles to assess different measures of the credit cycle used by policymakers. The results suggest that the slope of the financial trend better predicts the credit to GDP ratio in the United States than the estimated business and credit cycles and the Basel gap. This suggests policymakers should consider permanent shocks to the financial sector when gauging the state of financial stability.

Klarl (2020) investigated the response of CO_2 emissions to the business cycle for the U.S. on a monthly basis between 1973 and 2015. The study uses different filtering methods and a Markov-switching approach to find that emissions elasticity with respect to GDP is not constant over time and that emissions are significantly more elastic during recessions than in normal times. The results suggest that environmental policy instruments should account for this asymmetric response of emissions due to changes in GDP.

Dutra *et al.*, (2022) aimed at identifying the most reliable measure of financial cycles by applying filters similar to Gross Domestic Product (GDP) for Business Cycles.

The Christiano and Fitzgerald (2003)'s filter was used to estimate and extract cycles from the original time series of four financial variables: Credit, House Prices, Share Prices, and Interest Rates. The results of three methods, namely the Concordance Index, the Granger Causality Test, and the AUROC Test, showed that Share Prices is the most accurate proxy to measure and estimate financial cycles. The study concluded that Share Prices have a higher capacity to predict financial and economic crises than GDP.

Donayre (2022) examined the behavior of Okun's law across business cycles using U.S. data for 1949-2020. The study relaxes the two-regime assumption of existing models of their relationship and documents three phases of the business cycle, revealing a steepening of Okun's relationship across three endogenously-determined regimes that align closely with expansions, mild recessions and deep recessions. The variation in Okun's coefficient correlates with changes in the average deviation of nominal wages from the median, uncovering the need for differentiated policy responses across recessions.

2. RESEARCH METHODOLOGY

A linear filter of y_t can be written as

$$y_t^{\,*} = \sum_{j=-\infty}^{\infty} \alpha_j y_{t-j} = \alpha(L) y_t \qquad (2)$$

where we let y_t be an infinitely long series as, required by some of the results below. To see the impact of the filter on the components of y_t at each frequency ω, we need an expression for $f_{y^*}(\omega)$ in terms of $f_y(\omega)$ and the filter weights α_j. Wei (2006) shows that for each ω,

$f_{y^*}(\omega) = \left|\alpha(e^{iw})\right|^2 f_y(\omega)$ (2), where $\left|\alpha(e^{iw})\right|$ is known as the gain of the filter.

Equation (1) makes explicit that the squared gain function $\left|\alpha(e^{iw})\right|^2$ converts the spectral density of the original series, $f_y(\omega)$, into the spectral density of the filtered series, $f_{y^*}(\omega)$. In particular, (1) says that for each frequency ω, the spectral density of the filtered series is the product of the square of the gain of the filter and the spectral density of the original series. As expected, the gain function provides a crucial interpretation of what a filter is doing. We want a filter for which $f_{y^*}(\omega)$ = 0 for unwanted frequencies and for which $f_{y^*}(\omega) =$ $f_y(\omega)$ for desired frequencies. So we seek a filter for which the gain is 0 for unwanted frequencies and for which the gain is 1 for desired frequencies.

2.1 METHODOLOGY

Baxter and King (1999) showed that there is an infinite-order SMA filter with coefficients that sum to zero that can extract the specified components from a non-stationary time series. The components are specified in terms of the minimum and maximum periods of the stochastic cycles that drive these components in the frequency domain. This ideal filter is not feasible, because the constraints imposed on the filter can only be satisfied using an infinite number of coefficients. Therefore, Baxter and King (1999) derived a finite approximation to this ideal filter. The infinite-order, ideal band-pass filter obtains the cyclical component with the calculation.

$$c_t = \sum_{j=-\infty}^{\infty} b_j y_{t-j} \qquad (3)$$

Letting p_l and p_h be minimum and maximum periods of the stochastic cycles of interest, the weights in this calculation are given by

$$b_j = \begin{cases} \pi^{-1}(\omega_h - \omega_l) \quad \text{if} \quad j = 0 \\ \\ (j\pi)^{-1}\{\sin(j\omega_h) - \sin(j\omega_l)\} \text{ if } \quad j \neq 0 \end{cases} \qquad (4)$$

where

$\omega_l = 2\pi / p_l$ and $\omega_h = 2\pi / p_h$ are the lower and higher cutoff frequencies, respectively. For the default case of non-stationary time series with finite length, the ideal band-pass filter cannot be used without modification. Baxter and King (1999) derived modified weights for a finite order SMA filter with coefficients that sum to zero. As a result, Baxter and King (1999) estimated c_t by

$$c_t = \sum_{j=-q}^{+q} \hat{b}_j y_{t-j} \qquad (5)$$

The coefficients $\hat{b}_j$ in this calculation are equal $\hat{b}_j = b_j - \bar{b}_q$ where $\hat{b}_{-j} = \hat{b}_j$ and $\bar{b}_q$ is the mean of the ideal coefficients truncated at $\pm q$:

$$\bar{b}_q = (2q+1)^{-1} \sum_{j=-q}^{q} b_j \qquad (6)$$

Note that $\sum_{j=-q}^{+q} \hat{b}_j = 0$ and that the first and last *q* values of the cyclical component cannot be estimated using this filter.

Pollock (2000) showsed that the gain of the Butterworth high-pass filter is given by

$$\varphi(\omega) = \left[1 + \left\{ \frac{\tan(\omega_c/2)}{\tan(\omega/2)} \right\}^{2m} \right]^{-1} \qquad (7)$$

Where *m* is the order of the filter, $\omega_c = 2\pi / p_h$ is the cutoff frequency, and p_h is the maximum period. The model represents the series to be filtered, y_t , in terms of zero mean, covariance stationary, and independent and identically distributed shocks v_t and ε_t:

$$y_t(\omega) = \frac{(1+L)^m}{(1+L)^m} v_t + \varepsilon_t \qquad (8)$$

Since the time series has finite length, the ideal band-pass filter cannot be computed exactly. Christiano and c_t = 0. On the other extreme, as $\lambda \to \infty$, the solution approaches the least-squares fit to the $\tau_t = \beta_0 + \beta_1 t$

Fitzgerald (2003) derive the finite-length CF band-pass filter that minimizes the mean squared error between the filtered series and the series filtered by an ideal band-pass filter that perfectly separates out the components. This filter is not symmetric nor do the coefficients sum to zero. The formula for calculating the value of cyclical component c_t for t = 2, 3, . . . , *T* - 1 using the asymmetric version of the CF filter can be expressed as

$$c_t = b_0 y_t + \sum_{j=1}^{T-t-1} b_j y_{t-j} + \tilde{b}_{T-tyT} + \sum_{j=1}^{t-2} b_j y_{t-j} + \tilde{b}_{t-1y1} \qquad (9)$$

where $b_0, b_1, \ldots$ are the weights used by the ideal band-pass filter. $\tilde{b}_{T-1}$ and $\tilde{b}_{t-1}$ are linear functions of the ideal weights used in this calculation. The CF filter uses two different calculations for ebt depending upon whether the series is assumed to be stationary or nonstationary. For the default nonstationary case with 1 < t <*T*, Christiano and Fitzgerald (2003) set $\tilde{b}_{T-t}$ and $\tilde{b}_{t-1}$ to

$$\tilde{b}_{T-t} = -\frac{1}{2} b_0 - \sum_{j=1}^{T-t-1} b_j \quad \text{and} \quad \tilde{b}_{t-1} = -\frac{1}{2} b_0 - \sum_{j=1}^{t-2} b_j \qquad (10)$$

which forces the weights to sum to zero.

For HP filter, Hodrick and Prescott (1997), the following optimization problem for τ_t

$$\min_{\tau_t} \left[\sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \{(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\}^2 \right]$$

(11)

where the smoothing parameter λ is set fixed to a value. If λ = 0, the solution degenerates to $\tau_t = y_t$

3. Data Analysis

The data used is the Nigerian GDP which covers 1981Q1 – 2012Q4. All analyses were carried out using STATA 12 (S.E)

Table 1: Natural log of GDP

11.058	11.060	11.312	11.386	11.544	11.897	12.191
11.042	11.072	11.323	11.386	11.544	11.696	12.258
11.032	11.071	11.317	11.387	11.545	11.766	11.984
11.057	11.073	11.314	11.431	11.540	11.944	12.071
11.028	11.078	11.311	11.427	11.591	11.978	12.268
11.025	11.067	11.324	11.427	11.591	11.764	12.340
11.018	11.062	11.345	11.426	11.591	11.816	12.051
11.049	11.064	11.340	11.459	11.585	11.998	12.143
10.969	11.073	11.338	11.456	11.632	12.040	12.339
10.957	11.136	11.347	11.456	11.638	11.819	12.415
10.952	11.136	11.361	11.453	11.636	11.869	12.112
10.967	11.138	11.355	11.488	11.633	12.061	12.205
10.962	11.149	11.355	11.484	11.728	12.115	12.402
10.942	11.209	11.360	11.484	11.728	11.864	12.483
10.938	11.205	11.366	11.479	11.726	11.924	
10.948	11.206	11.364	11.494	11.722	12.121	
11.056	11.215	11.364	11.497	11.649	12.184	
11.051	11.317	11.368	11.498	11.726	11.913	
11.052	11.314	11.389	11.493	11.866	11.996	

Table 2: Results of some filters revealing the first 30 quarters

GDP	ln_gdp	gdp_bk	bkgain	gdp_cf	cfgain	gdp_hp	hpgain	gdp_bw	bwgain
63433.08	11.05774		0.011868	0.023778	0.001128	0.037473	0.000246	0.019705	0.000241
62446.97	11.04207		0.046866	0.020558	0.003526	0.024879	0.003914	0.009903	0.003845
61818.98	11.03197		0.103208	0.018531	0.005218	0.017822	0.019497	0.005624	0.019182
63353.25	11.05648		0.178050	0.019391	0.003395	0.045327	0.059091	0.035887	0.058291
61555.17	11.02769		0.267659	0.021072	0.004405	0.01941	0.132838	0.012692	0.131487
61383.79	11.0249		0.367653	0.019515	0.013313	0.019308	0.240870	0.015255	0.239327
60930.50	11.01749		0.473272	0.012126	0.033952	0.014299	0.369906	0.012823	0.368851
62857.10	11.04862		0.579668	-0.00015	0.330028	0.047448	0.500000	0.048410	0.500000
58056.49	10.96917		0.682200	-0.01382	0.641061	-0.03048	0.615248	-0.027290	0.616463
57335.02	10.95667		0.776704	-0.02583	0.602503	-0.04209	0.708675	-0.036940	0.710918
57041.70	10.95154		0.859729	-0.03541	0.759094	-0.04709	0.780415	-0.040320	0.783357
57947.59	10.96729		0.928723	-0.04319	0.785753	-0.03207	0.833941	-0.024040	0.837268
57654.91	10.96223	-0.06581	0.982147	-0.04846	0.901113	-0.03878	0.873413	-0.029910	0.876888
56515.58	10.94227	-0.06800	1.019529	-0.04772	1.049189	-0.06135	0.902489	-0.052060	0.905955
56260.04	10.93774	-0.05840	1.041434	-0.03659	1.124617	-0.06949	0.924019	-0.060130	0.927378
56824.20	10.94772	-0.03599	1.049374	-0.01402	1.287113	-0.0641	0.940098	-0.054970	0.943298
63303.30	11.05569	-0.00784	1.045649	0.015028	1.209317	0.038384	0.952232	0.047056	0.955248
63021.69	11.05123	0.015811	1.033144	0.041138	1.137297	0.027617	0.96149	0.035718	0.964314
63095.2	11.0524	0.028627	1.01509	0.055783	0.96787	0.021734	0.968632	0.029248	0.971266
63593.08	11.06026	0.030107	0.994805	0.056357	0.826104	0.021866	0.974203	0.028844	0.976655
64371.74	11.07243	0.023285	0.975444	0.046912	0.897845	0.025671	0.978593	0.032202	0.980874
64245.64	11.07047	0.013503	0.95976	0.034128	1.003057	0.014742	0.982086	0.020918	0.984209
64426.51	11.07328	0.003937	0.949909	0.022046	1.06611	0.007996	0.984893	0.013891	0.986869
64740.56	11.07814	-0.00633	0.947295	0.010106	1.102702	0.002717	0.987168	0.008362	0.989009
64039.07	11.06725	-0.01961	0.952482	-0.00408	1.022031	-0.0189	0.989026	-0.01354	0.990744
63716.38	11.0622	-0.03506	0.965164	-0.02016	1.015462	-0.03526	0.990557	-0.03027	0.992161
63815.59	11.06375	-0.04505	0.9842	-0.03335	1.004703	-0.04558	0.991826	-0.04111	0.993327
64425.93	11.07327	-0.04349	1.007713	-0.03768	0.935672	-0.04848	0.992885	-0.0447	0.994292
68564.07	11.13552	-0.0315	1.033235	-0.03147	1.000869	0.000876	0.993776	0.003762	0.995095
68577.13	11.13571	-0.01792	1.057898	-0.01994	1.135305	-0.01221	0.994528	-0.01041	0.995768

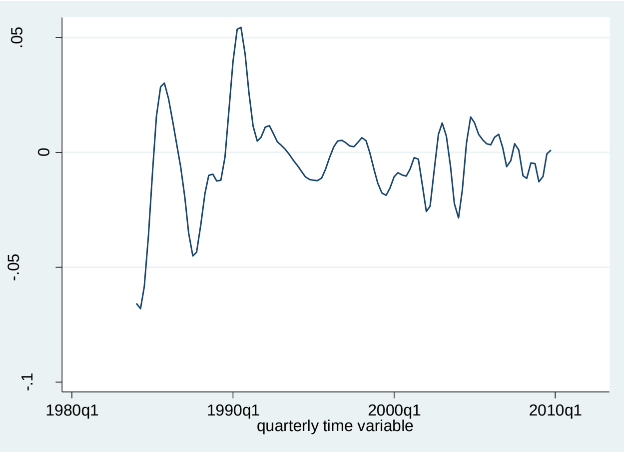

Fig. 1: The estimated business-cycle component

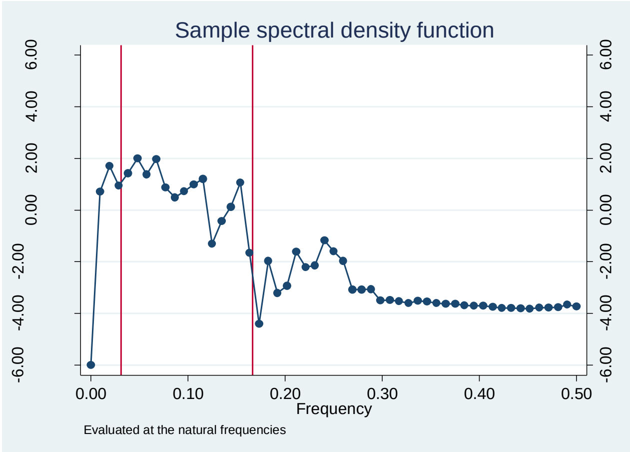

Fig. 2: Periodogram of the BK showing the information about the periodic component of the data

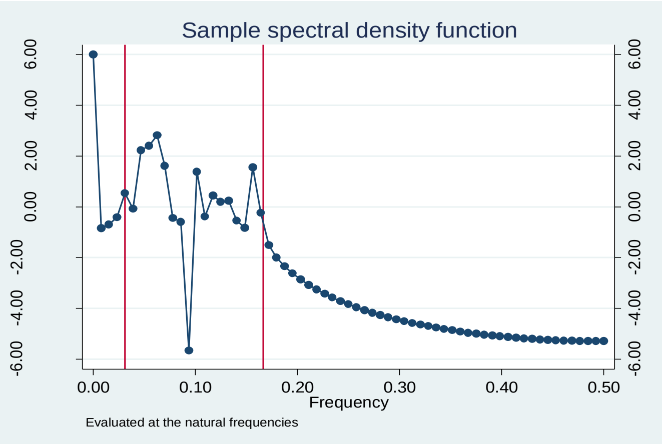

Fig. 3: The gain of the BK filter deviates from the business cycle component

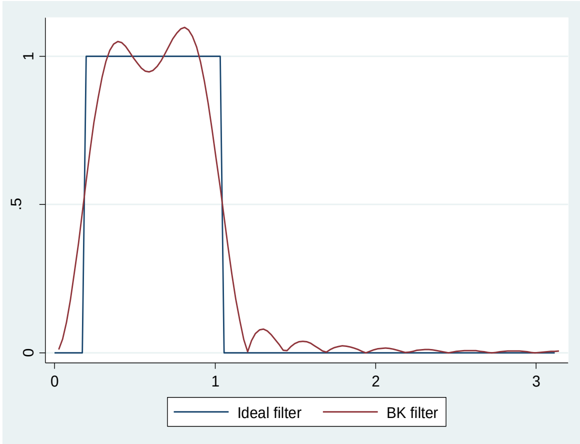

Fig. 4: The periodogram of the CF estimates of the business cycle component square-wave gain of the ideal filter

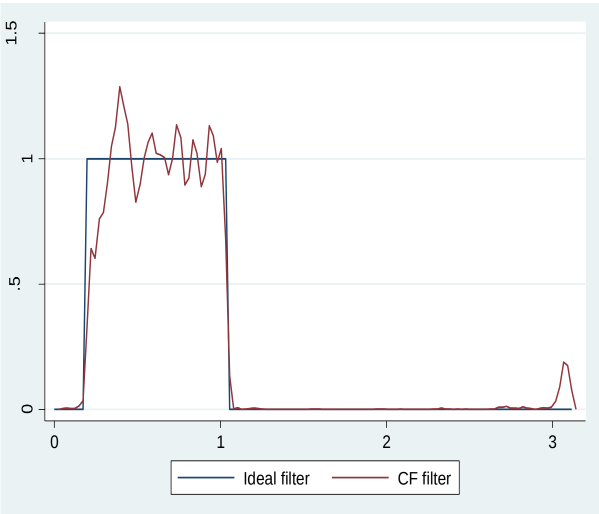

Fig. 5: Comparison of the BK gain and CF with ideal filter of the business cycle component

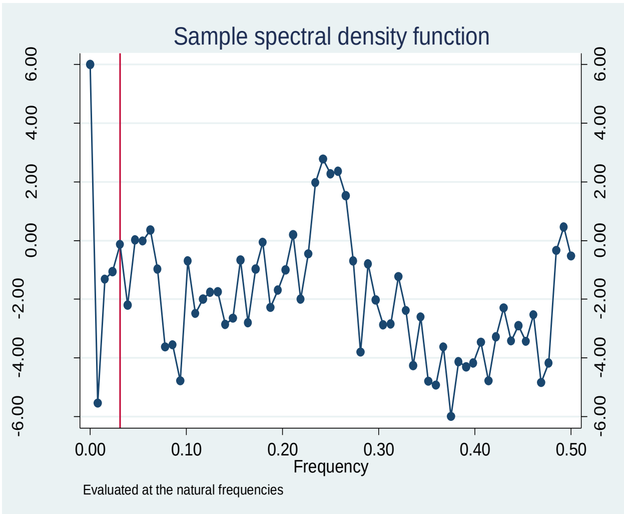

Fig. 6: The periodogram of the HP estimates

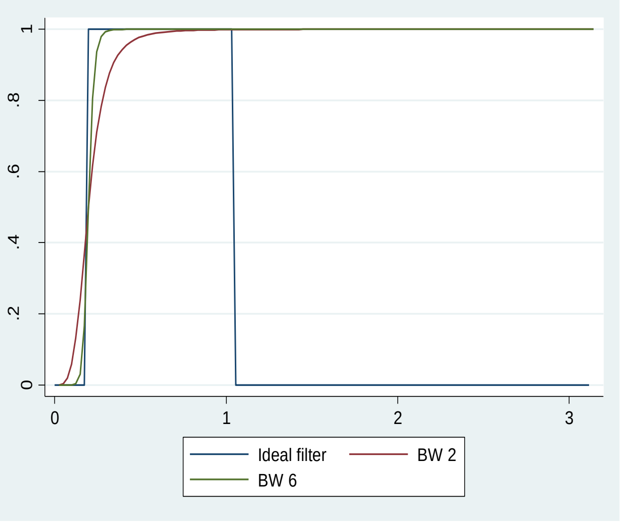

Fig. 7: Gain of CF filter closest to the gain of the ideal filter increases with the order of the filter

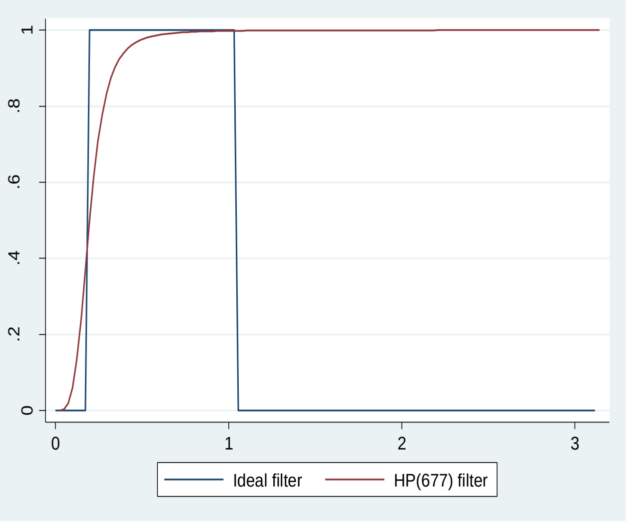

Fig. 8: Slope of gain function of BW filter

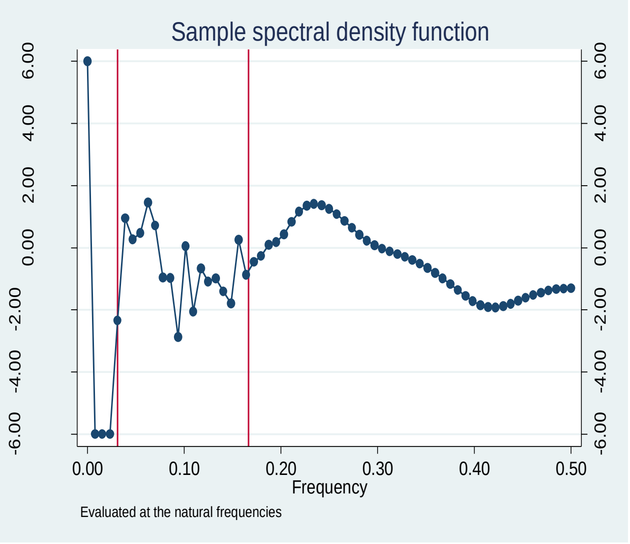

Fig. 9: Comparison of gain function of various filters

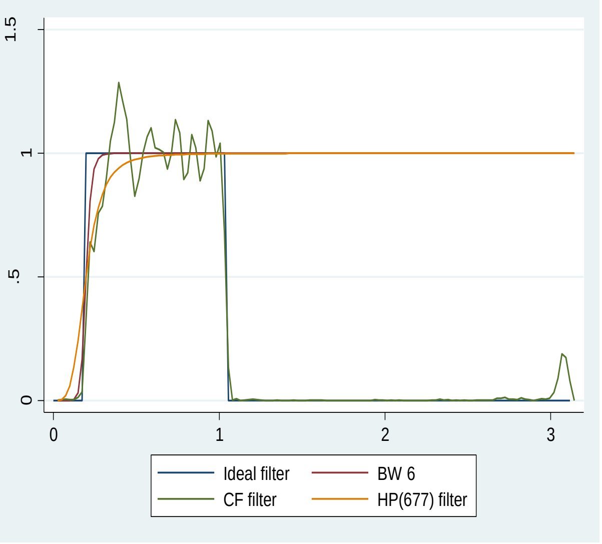

Fig. 10: The periodogram of the BW estimates of the business cycle component

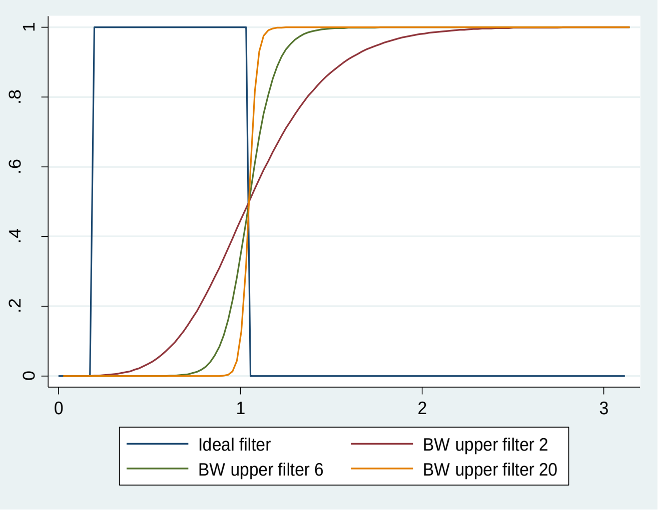

Fig. 11: Comparison of gain function of various BW filters

4. DISCUSSION OF RESULTS

The graph in Fig. 1 cannot really show the evidence as to well the component have been estimated but the periodogram in Fig. 2 shows this. It is an estimator of the spectral density function. The results are natural frequencies, which are the standard frequencies divided by 2π. If the filter completely removed the stochastic cycles corresponding to the unwanted frequencies, the periodogram would be a flat line at the minimum value of -6 outside the range identified by the vertical lines. That the periodogram takes on values greater than -6 outside the specified range indicates the inability of the BK filter to pass through only stochastic cycles at frequencies inside the specified band. The graph in Fig. 3 reveals that the gains of the BK filter deviates markedly from the square-wave gain of the ideal filter.

Increasing the symmetric moving average will cause the gain of the BK filter to more closely approximate the gain of the ideal filter at the cost of lost observations in the filtered series. In Fig. 4, the periodogram of the CF estimates of the business-cycle component indicates that the CF filter did a better job than the BK filter of passing through only the desired stochastic cycles. Comparing the graph in Fig.5 with the graph of the BK gain function reveals that the CF filter is closer to the gain of the ideal filter than is the BK filter. The graph also reveals that the gain of the CF filter oscillates above and below 1 for desired frequencies. Fig. 7 shows that by comparing the gain graphs, gain of the CF filter is closest to the gain of the ideal filter. Both the BK and the HP filters allow some low-frequency stochastic cycles to pass through. The plot also illustrates that the HP filter is a high-pass filter because its gain is 1 for those stochastic cycles at frequencies above 6 periods, whereas the other gain functions go to zero. The graph in fig. 8 reveals that the slope of the gain function increases with the order of the filter, and in Fig. 9, although the slope of the gain function from the CF filter is closer to being vertical at the cutoff frequency, the gain function of the Butterworth filter does not oscillate above and below 1 after it first reaches the value of 1. The flatness of the Butterworth filter below and above the cutoff frequency is not an accident, it is one of the filter's properties. The periodogram in Fig. 10 reveals that the two-pass process has passed the original through a band-pass filter. It also reveals that the two-pass process did a reasonable job of filtering out the stochastic cycles corresponding to the unwanted frequencies. Finally in Fig. 11, because the cutoff period is 6, the gain functions for $m = 2$ and $m = 6$ are much flatter than the gain functions for $m = 2$ and $m = 6$ in when the cutoff period was 32. The gain function for $m = 20$ is reasonably close to vertical, so we used it. For any given cutoff period, the computation eventually becomes unstable for larger values of m. For instance, when the cutoff period is *32*, $m = 20$ is not numerically feasible.

5. CONCLUSION AND RECOMMENDATION

In a business cycle, which estimate is better depends on whether the oscillations around 1 in the graph of the CF gain function cause more problems than the non-vertical slopes at the cutoff periods that occur in the BW6 gain function of that same graph and the BW upper filter 20 gain function graphed above. The choice between the BK or the CF filter is one between robustness or efficiency. The BK filter handles a broader class of stochastic processes, but the CF filter produces a better estimate of c_t if y_t is close to a random-walk process or a random-walk-plus-drift process.

REFERENCES

- Baxter, M., & King, R. G. (1999). Measuring business cycles: approximate band-pass filters for economic time series. *Review of economics and statistics*, *81*(4), 575-593.
- Bianchi, G., & Sorrentino, R. (2007). "Electronic filter simulation and sesign", New York: McGraw–Hill.
- Butterworth, S. (1930). On the theory of filter amplifiers. *Experimental Wireless and the Wireless Engineer*, **7**(536).
- Christiano, L. J., & Fitzgerald, T. J. (2003). The band pass filter. *International economic review*, *44*(2), 435-465.
- Donayre, L. (2022). On the behavior of Okun's law across business cycles. *Economic Modelling*, *112*, 105858.
- Dutra, T. M., Dias, J. C., & Teixeira, J. C. (2022). Measuring financial cycles: Empirical evidence for Germany, United Kingdom and United States of America. *International Review of Economics & Finance*, *79*, 599-630.
- Fuller, W. A. (1996). "Introduction to Statistical Time Series", 2nd ed. New York: Wiley.
- Hamilton, J. D. (1994). "Time Series Analysis", Princeton: Princeton University Press.
- Hassler, J., Lundvik, P., Persson, T., & Söderlind, P. (1994). The Swedish business cycle: stylized facts over 130 years. *Oxford University Press, Clarendon Press*, *9*.
- Hessler, A. (2023). Unobserved components model estimates of credit cycles: Tests and predictions. *Journal of Financial Stability*, *66*, 101120.
- Hodrick, R. J., & Prescott, E. C. (1997). Postwar US business cycles: an empirical investigation. *Journal of Money, credit, and Banking*, 1-16.
- Klarl, T. (2020). The response of CO2 emissions to the business cycle: New evidence for the US. *Energy Economics*, *85*, 104560.
- Pollock, D. S. G. (2000). Trend estimation and de-trending via rational square-wave filters. *Journal of Econometrics*, *99*(2), 317-334.
- Priestley, M. B. (1981). "Spectral Analysis and Time Series", London: Academic Press.
- Yogo, M. (2008). Measuring business cycles: A wavelet analysis of economic time series. *Economics Letters*, *100*(2) 208.