

Moment Approximation of the First-Passage Time for the GNP Diffusion Process to a General Determined Value

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Abstract

Review Article

The development of a mathematical models for Economic growth of great importance in many fields. The growth and decline of real economical data can in many cases be well approximated by the solutions of a stochastic differential equations. However, there are many solutions in which the essentially random nature of economic growth should be taken into account. In this paper, we consider an accurate method done by approximating the differential equations by an equivalent difference equations for approximating the moments of the first – passage time for the Gross National Product GNP diffusion process with linear function drift coefficient to a general determined value.

Keywords: First Passage Time, GNP Diffusion Process, Difference Equations, General Determined Value, Linear Function Drift Coefficient.

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INTRODUCTION

The development of mathematical models in the area of applied probability especially in stochastic modelling is of great importance in many fields such as ecology, demography, genetics and economics. More specifically, First – passage time play an important rule in the area of applied probability theory especially in stochastic modeling. Several examples of such problems are the extinction time of a branching process, or the cycle lengths of a certain vehicle actuated traffic signals. Actually the the first – passage times to a moving barriers for diffusion and other markov processes arises in biological modeling [6], in statistics [4, 5].

Many important results related to the first – passage time have been studied from different points of view of different authors. For example [12], has derived the distribution of the integral functional

$$W_x = \int_0^{T_x} g\{X(t)\}dt, \text{ where } T_x \text{ is the first – passage}$$

time to the origin in a general birth – death process with $X(0) = x$ and $g(\cdot)$ is an arbitrary function. Also [9, 13], have been shown a number of classical birth and death processes upon taking diffusion limits to asymptotically approach the Ornstein – Uhlenbeck (O.U.).

Many properties such as a first – passage time to a barrier, absorbing or reflecting, located some

distance from an initial starting point of the O.U. process and the related diffusion process and the related diffusion process such as the case of the first passage time of a Wiener process to a linear barrier is a closed form expression for the density available is discussed in [3]. Also, others such as [10, 14, 7, 15, 2, 8, 1, 16], etc. have been discussed the first passage time from different points of view.

In particular [14], describes some mean first – passage time approximation for the Ornstein – Uhlenbeck process [15] have studied the first-passage time of a Markov process to a moving barriers as a first-exit time for a vector whose components include the process and the barrier.

Also, [2], has discussed the problem of finding the moments of the first passage time distribution for the birth-death diffusion and the Wright-Fisher diffusion processes to a moving linear barriers using the method of approximating the differential equations by difference equations.

In addition [16] describe the moments approximation of the first passage time for the birth and death gross national product (GNP) to a fixed determined value by approximating the differential equations by equivalent difference equation.

Furthermore [1] considered a stochastic diffusion process able to model the interest rate evolving with respect to time and propose a first passage time (FPT) approach through a boundary, defined as the “alert threshold”, in order to evaluate the risk of a proposed loan. Above this alert threshold, the rate is considered at the risk of usury, so new monetary policies have been adopted. Moreover, the mean FPT can be used as an indicator of the “goodness” of a loan; i.e., when an applicant is to choose between two loan offers, s/he will choose the one with a higher mean exit time from the alert boundary.

Finally [8] presented a methodology to build a log normal diffusion process with exogenous factors that models economic variables. They studied the problem of forecasting as first passage times and applied to the GNP of Spain.

In this paper, we consider the GNP diffusion process with linear function drift coefficient and study the first – passage time for such a process to a general determined value. More specifically, the moment approximations are derived using the method of difference equations.

THE GNP DIFFUSION FIRST – PASSAGE TIME MOMENT APPROXIMATIONS

Consider the GNP diffusion Process $\{X(t) : t \geq 0\}$ with infinitesimal mean $bx + \varepsilon$ and variance $2ax$ starting at some $x_0 > 0$, where b and a are the drift and the diffusion coefficients respectively and ε is the constant rate. Also, $\{X(t) : t \geq 0\}$ is a Markov process with state space $S = [0, \infty)$ and satisfies the Ito stochastic differential equation

$$dX(t) = (bX(t) + \varepsilon)dt + \sqrt{2aX(t)}dW(t) \tag{1}$$

Where $\{W(t) : t \geq 0\}$ is a standard Wiener process with zero mean and variance t . Assume that the existence and uniqueness conditions are satisfied. Let $\{Y(t) : t \geq 0\}$ be a general determined value equation of the GNP such that $Y(t) = h(t)$, with $Y(0) = h(0)$. Or equivalently

$$\frac{dY(t)}{dt} = h'(t)$$

Now, denote the first – passage time of a process $X(t)$ to a general determined value function $Y(t) = h(t)$ by the random variables

$$T_Y = \inf\{t \geq 0 : X(t) \geq h(t)\} \tag{2}$$

with probability density function

$$g(t; x_0) = - \frac{d}{dt} \int_{-\infty}^{h(t)} p(x_0, x; t) dx$$

Here $p(x_0, x; t)$ is the probability density function of $X(t)$ conditional on $X(0) = x_0$

Let $M_n(x_0, Y; t); n = 1, 2, 3, \dots$, be the n -th moment of the first – passage time T_Y , i.e.

$$M_n(x_0, Y; t) = E(T_Y^n) ; n = 1, 2, 3, \dots, \tag{3}$$

It follows from the forward Kolmogorov equation that the n -th moment of T_Y must satisfy the ordinary differential equation

$$axM_n''(x_0, Y; t) + (bx + \varepsilon)M_n'(x_0, Y; t) + h'(t)M_n'(x_0, Y; t) = -nM_{n-1}(x_0, Y; t) \tag{4}$$

Or equivalently

$$M_n''(x_0, Y; t) + \frac{bx + \varepsilon}{ax} M_n'(x_0, Y; t) + \frac{h'(t)}{ax} M_n'(x_0, Y; t) = -\frac{n}{ax} M_{n-1}(x_0, Y; t) \tag{5}$$

Where $M'_n(x_0, Y; t)$ and $M''_n(x_0, Y; t)$ are the first derivatives of $M_n(x_0, Y; t)$ with respect to x ($x_0 \leq x \leq Y$), with appropriate boundary conditions for $n=1,2,3,\dots$. Note that $M_0(x_0, Y; t) = 1$.

Now, rewrite the equation in (5), we obtain

$$M''_n(x_0, Y; t) = -\frac{n}{ax} M_{n-1}(x_0, Y; t) - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) M'_n(x_0, Y; t) \tag{6}$$

Let Δ be the difference operator. Then we defined the first order difference of $M_n(x_0, Y; t)$ as follows:

$$\Delta M_n(x_0, Y; t) = M_{n+1}(x_0, Y; t) - M_n(x_0, Y; t) \tag{7}$$

(Cf.) [9].

Note that equation (6) can be approximated by

$$M''_n(x_0, Y; t) = -\frac{n}{ax} M_{n-1}(x_0, Y; t) - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) \Delta M_n(x_0, Y; t) \tag{8}$$

By applying equation (7) to equation (8) we get :

$$M''_n(x_0, Y; t) = -\frac{n}{ax} M_{n-1}(x_0, Y; t) + \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) M_n(x_0, Y; t) - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) M_{n+1}(x_0, Y; t) \tag{9}$$

Now, we will use the matrix theory to solve the differential equation defined in equation (9). If we let

$$\vec{M}(x_0, Y; t) = [M_1(x_0, Y; t), M_2(x_0, Y; t), \dots]'$$

Then we get

$$\frac{d^2 \vec{M}(x_0, Y; t)}{dx^2} = A \vec{M}(x_0, Y; t) \tag{10}$$

Where

$$A = \begin{bmatrix} \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & -\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & 0 & 0 & \dots \\ -\frac{2}{ax} & \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & -\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & 0 & \dots \\ 0 & -\frac{3}{ax} & \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & -\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & \dots \\ 0 & 0 & -\frac{4}{ax} & \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right) & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Now let

$$\frac{d\vec{M}(x_0, Y; t)}{dx} = \vec{R}(x_0, Y; t) \tag{11}$$

This imply

$$\frac{d^2 \bar{M}(x_0, Y; t)}{dx^2} = \frac{d\bar{R}(x_0, Y; t)}{dx} \tag{12}$$

Apply to equation (10), we get

$$\frac{d}{dx} \begin{bmatrix} \bar{R}(x_0, Y; t) \\ \bar{M}(x_0, Y; t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \bar{R}(x_0, Y; t) \\ \bar{M}(x_0, Y; t) \end{bmatrix} \tag{13}$$

Where \mathbf{I} is the identity matrix and $\mathbf{0}$ is the zero matrix.

Thus, the solution of the system of equation in (13) is then given by

$$\begin{bmatrix} \bar{R}(x_0, Y; t) \\ \bar{M}(x_0, Y; t) \end{bmatrix} = e^{\begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{D} & \mathbf{0} \end{bmatrix} t} \cdot \begin{bmatrix} \bar{R}(x_0, Y; t_0) \\ \bar{M}(x_0, Y; t_0) \end{bmatrix} \tag{14}$$

Where $D = [d_{ij}]$; $i, j \geq 1$ is the diagonal matrix with entries

$$d_{ij} = \begin{cases} (h'(t) - x_0) & ; j = i \\ 0 & ; \text{Otherwise} \end{cases} \tag{15}$$

And $A^* = [a_{ij}^*]$; $i, j \geq 1$ is the matrix with entries

$$a_{ij}^* = \begin{cases} -\frac{i}{ax} \ln\left(\frac{h(t)}{x_0}\right) & ; j = i - 1 \\ \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right)(h(t) - x_0) & ; j = i \\ -\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax}\right)(h(t) - x_0) & ; j = i + 1 \\ 0 & ; \text{Otherwise} \end{cases} \tag{16}$$

Note that the matrix e^B where $B = \begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$ is defined by

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \dots$$

This series is convergent since it is a cauchy operator of equation (2.6) (Cf. [17]).

MEAN AND VARIANCE APPROXIMATION FOR THE GNP FIRST-PASSAGE TIME

Now for approximating the moments of the GNP first-passage time for such a process using the first and the second order difference operators to the differential equation in (9), we define the operators as follows:

Let Δ^2 be the second order difference operators. Then we defined the second order differences of $M_n(x_0, Y; t)$ and as follows:

$$\Delta^2 M_n(x_0, Y; t) = M_{n+2}(x_0, Y; t) - 2M_{n+1}(x_0, Y; t) + M_n(x_0, Y; t) \tag{17}$$

(Cf. [11]).

Note that equation (9) can be approximated by

$$\begin{aligned} \Delta^2 M_n(x_0, Y; t) &= -\frac{n}{ax} M_{n-1}(x_0, Y; t) \\ &+ \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) M_n(x_0, Y; t) \\ &- \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) M_{n+1}(x_0, Y; t) \end{aligned} \quad (18)$$

By applying equation (17) to equation (18) we get:

$$\begin{aligned} M_{n+2}(x_0, Y; t) - 2M_{n+1}(x_0, Y; t) + M_n(x_0, Y; t) &= \\ -\frac{n}{ax} M_{n-1}(x_0, Y; t) + \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) M_n(x_0, Y; t) & \\ - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) M_{n+1}(x_0, Y; t) & \end{aligned} \quad (19)$$

Now rewriting equation (19) we get:

$$\begin{aligned} M_{n+2}(x_0, Y; t) &= -\frac{n}{ax} M_{n-1}(x_0, Y; t) \\ &+ \left(\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) - 1 \right) M_n(x_0, Y; t) \\ &\left(2 - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) \right) M_{n+1}(x_0, Y; t) \end{aligned} \quad (20)$$

Through equation (20), the first moment $M_1(x_0, Y; t)$ and the second moment $M_2(x_0, Y; t)$ of the first –passage time can be approximated by

$$M_1(x_0, Y; t) \cong \left(2 - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) \right) \quad (21)$$

And

$$\begin{aligned} M_2(x_0, Y; t) &\cong \left(\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) - 1 \right) \\ &+ \left(2 - \left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) \right) M_1(x_0, Y; t) \end{aligned} \quad (22)$$

Therefore the variance $V(x_0, Y; t)$ can be approximated by

$$V(x_0, Y; t) \cong \left(\left(\frac{b}{a} + \frac{\varepsilon + h'(t)}{ax} \right) - 1 \right) \quad (23)$$

Note that these results are of great importance for the statistical inference problems.

CONCLUSION

In conclusion the advantage of this technique is to use the difference equation to approximate the ordinary differential equation since it is the discretization of the ODE. Also, the system of the solutions in equation (14) gives an explicit solution to the first – passage time moments for the GNP diffusion process with linear function drift coefficient to a general

determined value. Also, the mean and the variance of the GNP first-passage time for such a process are approximated which are useful for statistical inference problems. This increases the applicability of the diffusion process in stochastic modeling or in all area of applied probability theory especially in economics.

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