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Stability and HOPF Bifurcation Analysis of Periodic Solutions of a Duffing Equation

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Abstract **Original Research Article**

In this study, the stability and Hopf bifurcation analysis of periodic solutions of Duffing equations were considered. Also other types of bifurcation like the Saddle-node, Transcritical and Pitchkfork were also studied. The eigen value, Jacobian and Floquet theory were used to analyse both the stability and Hopf bifurcations of the periodic solutions of the equilibrium points. The results showed that equilibrium points have at most three -periodic solutions under a strong damped conditions due to the cubic nonlinearities. The bifurcation points showed; one critical, another subcritical and the third point showed that the homoclinic was present when the damping coefficient is zero. Furthermore, the presence of strange attractors varied with the driving force and damping. The MATCAD software was used to illustrate the numerical behaviour of the solution which extend some results in the literature.

Keywords: Periodic solution, Bifurcation, Stability, Duffing Equation.
Mathematics Subject Classifications (2010): 34F10, 34K13, 34K20

INTRODUCTION

Consider the Duffing Oscillator described by the Differential equation

$$\ddot{x} + \alpha\dot{x} - \beta x^3 = f\,sin\omega t \ldots\ldots f \geq 0 \qquad (1.1)$$

With initial condition

$$x(0)=2\pi\ \dot{x}(0) = 2\pi \qquad (1.2)$$

And the periodic solution

$$x(t)= x(t + T) \qquad (1.3)$$

Where x is the displacement,$\dot{x}$ is the velocity and $\ddot{x}$ is the acceleration.
F: $\mathcal{R}x\mathcal{R} \longrightarrow \mathcal{R}$., and α, β, f, ω, are constant (functions).

Duffing equation is a second order nonlinear differential equation, used to model damped and driven oscillators in [1, 2].

During the last decade many scholars investigated the qualitative behaviours of the solutions of this equation for instance Duta and Pragapatic [3] reported on symmetric investigation into the phase space of the double wall Duffing oscillator where bifurcation diagram was used to show the region characterized by the parameters for which one finds a periodic solution. They also observed that when the driving force is increased, there is a series of parallel "Islands" of parameters characterized by a periodic attractor. Furthermore, the study showed that if the model is perturbed by the linear term, it shows both periodic and chaotic behaviour. Again is that when the damping coefficient is taken to zero and the nonlinear stiffness parameter is taken sufficiently small, the model exhibits homoclinic nature for whatever value the force is.

In order to understand the global structure of periodic solutions and stability of each solutions under the cubic restoring force Chen and Li [4] used a different approach based on Grandale-Robinowttz bifucation theorem and contraction method but devoted their work to exact multiplicity of periodic solution under the cubic ono-linear restoring force with strong damped condition.

In a similar paper, Chen & Li [4] used the global bifurcation method based on maximum principle developed, [5-7] super sub-solutions methods are not applicable to degenerate situation and when the graph of the forcing term interact the critical level is missing where bifurcation occurs resulting in the periodic solution being at most three under strong damped condition and formed an "S" shaped smooth curve that is symmetric with the origin.

We have considered a linear and non-autonomous system with linear damping in the following set of equations.

We have modified the Duffing equation as

$$\ddot{x} + \alpha\dot{x} - \beta x^3 = f \sin \omega t \qquad (1.4)$$

Where $\dot{x}$ is the velocity, $\ddot{x}$ is the acceleration and we replaced $cos\omega t$ with $sin\ \omega t$ and assumed that $\beta < 0$. Hence we are studying about inverted Duffing oscillator and its chaotic properties. We find that our model undergoes chaotic behaviour as well as it also shows homo-clinic properties within a certain range of parameter values. Our study consists of the following:

1. First, we have fixed the parameters α, β, and studied the behaviour for different values of f. we obtained the Hopf bifurcation diagram for $0.1 < f < 15$.,
2. Secondly we have obtained the phase portrait and poincare section in this regard,
3. Thirdly we have shown x vs t graph for our model within same range of parameters, fourthly, we observed the behaviour of the strange attractors, how they vary with driving force and damping factor.
4. Lastly we will observe the Homoclinic behaviour of the Oscillator when damping coefficient is taken as zero.

2. PRELIMINARIES

Bifurcation Theory

A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameter) of a system causes a sudden "qualitative" or topological change in behaviour. Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other variant sets changes [8].

Theorem 1.1 (saddle-node bifurcation). Assume that the vector field f is of class $C^k, k \geq 2$, in a neighbourhood of (0,0) and satisfies:

$$\frac{\partial f}{\partial \mu}(0,0) =: a \neq 0, \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0 \qquad (2.1)$$

Bifurcation of Dimension 2: Hopf bifurcation

Here we consider Differential Equation in $\mathbb{R}^2$,

$$\ddot{x} + \alpha\dot{x} - \beta x^3 = f \sin \omega t F(u,\mu) \qquad (2.2)$$

Here the unknown u is given a real-valued function that takes values in $\mathbb{R}^2$, and the vector field F is real-valued depending, besides u, upon a parameter μ. The bifurcation parameter. We assume that the vector field is of class $C^k, k \geq 3$, in a neighbourhood of (0,0) satisfying:

$$F(0,0) = 0 \qquad (2.3)$$

This condition ensures that u=0 is an equilibrium of equation (2.1) at $\mu = 0$. The occurrence of a bifurcation is in this case determined by linearization of the vector field at (0,0):

$L = D_u F(0,0)$

Which is a linear operator acting in $\mathbb{R}^2$. When L has eigenvalues on the imaginary axes, bifurcation may occur at $\mu = 0$. We focus in this section on the case where L has a pair of complex conjugated purely imaginary eigenvalues. This is called the Hopf bifurcation (or Andronov-Hopf bifurcation).

Hypothesis 2.1:

Assume that the vector field is of class $C^k, k \geq 5$, in a neighbourhood of (0,0), that is satisfies (2.3) and the two eigenvalues of the linear operator L are $\pm i\omega$ for some $\omega > 0$.

We consider the eigenvector and associated to the eigenvalue iω of L,

$L\xi = i\omega\xi$

If L^* is the adjoint operator of L then we define ξ^* as the eigenvector of L^* satisfying:

$L^*\xi^* = i\omega\xi^*, \langle \xi, \xi^* \rangle = 1$

Where $\langle \cdot,\cdot \rangle$ denotes the Hermitian scalar product in $\mathbb{C}^2$. consider the Taylor extension of the vector field F in (2.8):

$F(U,\mu) =: \sum_{1 \leq r+q \leq} \mu^q F_{rq}(U^{(r)}) + o(|\mu| + \|U\|^k)$

Where F_{rq} is the r-linear symmetric operator from $(\mathbb{R}^2)^r$ to $\mathbb{R}^2$

$F_{rq} = \frac{1}{r!q!}\frac{\partial^q}{\partial \mu^q} D_u^r F(0,0)$

We define the coefficients

$$a = \langle F_{11}\xi + 2F_{20}(\xi, -L^{-1}F_{01}), \xi^* \rangle \qquad (2.4)$$

$$b = \langle 2F_{20}(\xi, (2i\omega - L)^{-1}F_{20}(\xi,\xi) + 2F_{20}(\xi, -2L^{-1}F_{20}(\xi,\bar{\xi}) + 3F_{30}(\xi, \overset{\infty}{\tilde{\bar{\xi}}}, \bar{\xi})), \xi^* \rangle \qquad (2.5)$$

Hypothesis 2.2

We assume that the complex coefficients a and b have non-zero real parts, $a_r \neq 0$ *and* $b_r = 0$. The coefficient $b_r = R_r(b)$ is called the Lyapunov coefficient.

Definition 1.2

1. A non-constant solution to the differential equation (2.8) is periodic if it exist $T > 0$ such that $U(t) = U(t+T)$. The image of the interval [O,T] under U in the state space $\mathbb{R}^2$ is called the periodic orbit.
2. A periodic orbit Γ on a plane is called a limit cycle if it is the$\alpha -$ limit set of $\omega -$limit set of some point z not on the periodic orbit, that is, the set of accumulation points of either forward or backward trajectory through z, is exactly Γ. Asymptotically stable and unstable periodic orbits are examples of limit cycles.

Theorem 2.1 (Hopf Bifurcation)

Assume that hypothesis 2.1 and 2.2 holds. Then, for the differential equation (2.1) a Superciritcal (respectively Subcritical) Hopf Bifurcation occurs at $\mu = 0$ when $b_r < 0$ $(respectively\ b_r > 0)$. More precisely, the following properties hold in a neighbourhood of O in $\mathbb{R}^2$ for small enough μ:

i. If $a_r b_r < 0$ $(respectively\ a_r b_r > 0)$ the differential equation has precisely one equilibrium u(μ) for $\mu < 0$ (respectively $\mu > 0$) with $\mu > 0$ with u(0)=0. This equilibrium is stable when $b_r < 0$ and unstable when $b_r > 0$.

ii. If $a_r b_r < 0$ $(respectively\ a_r b_r > 0)$ the differential equation possesses for u(μ) for $\mu < 0$ (respectively $\mu > 0$) and equilibrium u(μ) and a unique periodic orbit $U^*(\mu) = O(\sqrt{|U|})$, which surrounds this equilibrium. The periodic orbit is stable when $b_r < 0$ and unstable when $b_r > 0$, whereas the equilibrium has the opposite stability.

Remark 2.2

The number of equilibria of the differential equation stays constant upon varying μ in neighbourhood of O. The dynamics of the bifurcation change at the bifurcation point $\mu = 0$. Such bifurcation, are called dynamic bifurcations, whereas those in which the number of equilibria changes are also called steady bifurcation.

Hopf bifurcation theorem for vector fields

Let X_μ be a C^k $(k \geq 4)$ vector field on $\mathbb{R}^2$ such that $X_\mu(0) = 0$ for all μ and X=$(X_\mu, 0)$ is also C^k. Let $dX_\mu(0,0)$ have two distinct, simple complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ such that $\mu < 0$, Re $\lambda(\mu) < 0$, for $\mu = 0$, Re $\lambda(\mu) = 0$, and for $\mu > 0$, Re $\lambda(\mu) > 0$. Also assume $\frac{d\text{Re}\,\lambda(\mu)}{du}\Big|_{\mu=0} > 0.$ Then there is a C^{k-2} function $\mu\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that $(X_1, 0, \mu(X_1))$ is on a closed orbit of period $\approx \frac{2\pi}{|\lambda(0)|}$ and radius growing like $\sqrt{\mu}$, of the flow of X for $X_1 \neq 0$ and such that $\mu(0) = 0$. There is a neighbourhood u of (0,0,0) in $\mathbb{R}^3$ such that any closed orbit in u is one of the above.

Furthermore, if 0 is a 'vague attractor' (asymptotically stable) for $X_o, then\ \mu(X_1) > 0$ for all $X_1 = 0$ and the orbit is attracting.

If, instead of a pair of conjugate eigenvalues crossing the imaginary axis, a real eigenvalue crosses the imaginary axis, two stable fixed point will branch off instead of a closed orbit.

Center manifold theorem

The center manifold theorem is one of the important bifurcation theorem and the key job is that it enables one to reduce to a finite dimensional problem. In the case of a Hopf bifurcation theorem, it enables a reduction to two dimensions without losing any information concerning stability.

Theorem 2.2 (Center manifold theorem): Let ψ be a mapping of a neighbourhood of zero in a Banach space Z into Z. We assume that ψ is C^{k+1}, k≥ 1 and that $\psi(0) = 0$. We further assume that $D\psi(0)$ has spectral radius 1 and that the spectrum of $D\psi(0)$ splits into a part on the unit circle and the remainder which is at a non-zero distance form the unit circle. Let Y denote the generalized eigenspace of $D\psi(0)$ belonging to the part of the spectrum on the unit circle; assume that Y has dimension d< ∞ then there exist a neighbourhood v of O in Z and a C^k submanifold M of v of dimension d, passing through O and tangent to Y at O. such that

a. **Local invariance:** If x∈ M and ψ(x) ∈ V then ψ(x) ∈ M

b. **Local attractively:** If $\psi^n(x) \in V$ for every n=0,1,2,…, then as n→ ∞, the distance from $\psi^n(x)$ to M→ 0. This holds automatically if Z is finite dimensional or, more generally, if $D\psi(0)$ is compact.

Existence and Uniqueness

1. Lipschitz conditions

Consider $\frac{dy}{dt} = f(t, y)$

Y(t_0)=y_0

Where f is a differentiable function. We would like to know when we have existence of a unique solution for given initial date. One condition on f which guarantees this in the following:

Given a subset S of the (t,y)-plane, we say that f is lipschitz with respect to y on the domain s if there exist some constant k such that;

$|f(t, y_2) - f(t, y_1)| \leq k|y_2 - y_1|$ for every point (t,y_1) and (t,y_2) in S. The constant K is called the Lypschitz constant.

Example 2.1 Let f(t,y)=ty^2

then since $|f(t, y_2) - f(t, y_1)| \leq f|y_2 + y_1||y_2 - y_1|$ is not bounded by any constant times $|y_2 - y_1|$, f is not Lipschitz continuous with respect to y on the domain ℝXℝ. However f is Lipschitz on any rectangle $\mathbb{R} = [a, b]x[cxd]$ since we have

$t|y_2 + y_1| \leq 2\max\{|a|, |b|\}.\max\{|c|, |d|\}$ on ℝ.

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) = d(x, y) + d(y)$

Floquet theory:

The fundamental matrix x(t) of

$\dot{x} = \dot{A}(t)x$ (2.6)

With $x(t) = 1$, has a Floquet normal form

$x(t) = Q(t)e^{Bt}$ (2.7)

where $Q \in C^1(\mathbb{R})$ is T periodic and the matrix B, $B \in \mathbb{C}^{nxn}$ satisfies the equation

$C = X(T) = e^{BT}$ (2.8)
$Q(0) = 1$ and $Q(t)$ is an invertible matrix for all t.

PROOF:
By lemma 2.5, there exist a non-singular constant matrix C with x(t+T)= X(t)C using
X(t+T) = X(t)X(T) = x(t)C and Lemma 2.6 gives $C=X(T)=-e^{BT}$
For some matrix B, if $Q(t) = X(t)e^{BT}$, then for all t,
$Q(t+T) = x(t+T)e^{-B(t+T)}$
$= x(t)Ce^{-Bt}e^{-BT}$
$= x(t)e^{BT}e^{-Bt}e^{-BT}$
$= x(t)e^{-Bt}$
$= Q(t)$
This means that
$x(t) = Q(t)e^{Bt}$ where
$Q \in C^1(\mathbb{R})$ is T-periodic and $x(0) = X(0)e^0 = 1$
The matrix e^{-Bt} is invertible for all t, because exponential of square matrices are invertible and x(t) is invertible Hence, Q(t) is invertible.

Lemma 2.3

If x(t) is a fundamental matrix of (1), then so is Y(t) = x(t)B for non-singular constant matrix B.

Lemma 2.4

If x(t) is a fundamental matrix of (1), then so is Y(t) = X(t + T).

Lemma 2.5

If x(t) is a fundamental matrix of $\dot{x} = A(t)x\ by\ lemma$ 2.4, Y(t) = X(t + T) is a fundamental matrix of (1), then there exist a non-singular constant matrix C with
$X(t+T) = X(t)C$ (2.9)

Floquet Multiplier

We continue using the fundamental matrix X(t) for (1) in Lemma 2.5, we proved that $X(t+T) = X(t)C$
Where C is a non-singular constant matrix. Recall in (5)
$C = C(0) = x^{-1}(0)Y(0) = x^{-1}(0)x(T)$ (2.10)
This C is known as the monodromy matrix.

Definition 2.3

The eigenvalues of the monodromy matrix are called the Floquet multiplier of (1).

Definition 2.4

The eigenvalues of the matrix B of the Floquet form $x(t) = Q(t)e^{Bt}$, are called the Floquet exponents of (1). Since the monodromy matrix is non singular, its eigenvalues are non zero, therefore, we can state the following:

Corollary 2.5

Let λ_1, $\lambda_2, \ldots, \lambda_n$ be the Floquet multipliers and μ_1, $\mu_2, \ldots, \mu_n$ be the Floquet exponents for (1), we can write
$\lambda_j = e^{\mu_j t}$ for all j= 1,…,n

Stability of the Floquet System

Floquet multipliers are very useful in stability analysis of periodic system. Recall the following definitions.

Definition 3.5:

An eigenvalue λ of A is simple if its algebraic multiplicity equals 1.

Definition 3.6:

Let λ be an eigenvalue of a matrix A the geometric multiplicity of λ is dim(Null (A-λI)) in order words, the number of linearly independent eigenvector associated with λ.

Definition 3.7:

An eigenvalue λ of A is semi simple if its geometric multiplicity equals its algebraic. A simple eigenvalue is always semi-simple. But the converse is not true.

Definition 3.8

Consider the system $\dot{x} = A(t)x\ in\ v = [t_0, \infty]$
(2.11)

And assume A(t) is T periodic and continuous in V. The solution ψ(t) to system (2.6) is

1. Stable on v if for every $\epsilon > 0$, there exist a $\delta > 0$, such that $|\Psi(t_0) - x(t_0)| < \delta$ implies that $|\Psi(t) - x(t)| < \epsilon, for\ every$ t≥ 0 And the solution x(t) is defined for all t ∈ v
2. Asymptotically stable on v If it is stable and if in addition $\lim_{t\to\infty}|\psi(t) - x(t)| \longrightarrow 0$
3. Unstable if it is not stable on v. it can be proven that the following stability condition hold for the Floquet system.

Theorem 2.10

Assume $\lambda_1, \lambda_2, \ldots, \lambda_n$ are Floquet multipliers of system
1. Then the zero solution of (1) is
   i. Asymptotically stable on [0,∞) if and only if $|\lambda_i| < 1$ for i=1,…,n
   ii. Stable on $[0, \infty)$ if $|\lambda_j| \leq 1$ for all I=1,…,n and whenever $|\lambda_j| = 1, \lambda_j$ is a semi-simple eigen value
   iii. Unstable in all other cases.

It should be noted that for the Floquet exponents, the conditions $|\lambda_j| < 1, |\lambda_j| \leq 1, |\lambda_j| > 1$ is equivalent to Re $\mu_j < 0$, Re $\mu_j \leq 0$, and Re $\mu_j > 0$

Eigenvalue

Eigenvalue are a special set of scalars associated with a linear system of equation (ie a matrix equation) that are sometimes known as characteristic roots, characteristic value [6], proper value, or latent roots [13].

Theorem 2.1

The following gives the link between the characteristic polynomial of a matrix A and its eigenvalues. If A is an nxn matrix and λ is a complex number the the following are equivalent

a. λ is an eigenvalue of A
b. The system of equation (A-λI=0) has a trivial solution
c. There is a non-zero vector X in $\mathbb{C}^n$ such that Ax=λx
d. λ is a solution of the characteristic equation $\det(A - \lambda I)$. Some coefficient of the characteristic polynomial of A have a specific shape. The following theorem gives the information about it.

Theorem 2.1.2

If A is an n x n matrix, then the characteristic polynomial P(λ) of A has degree n, the coefficient of $\lambda^n \ is \ (-1)^n$, the coefficient of λ^{n+1} is $(-1)^{n-1}$ trace (A) and the constant term is det(a), where trace (A)=$a_{11} + a_{22} + \ldots + a_{nn}$. In some structured matrices, eigenvalues can be read as shown in theorem 2.1.3.

Theorem 2.1.3

If a is an nxn triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are entries of the main diagonal of A.

Cayley-Hamilton's theorem is one of the most important statements in linear algebra. The Theorem states that.

Theorem 2.1.4

Substituting the matrix A for λ in characteristic polynomial of A, we get the result of zero matrix ie, P(A)=0

Jacobian Theorem

If u and v are functions of the two independent variables of x and y,

the determinant $\begin{pmatrix} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \end{pmatrix}$ is called the Jacobian of u,v with respect to x,y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ $or \ J(\frac{u,v}{x,y})$

Properties of Jacobian

First property

If U and V are the function of x and y then $\frac{\partial(u,v)}{\partial(x,y)}$ X $\frac{\partial(x,y)}{\partial(u,v)} = 1$

Second Property

If U,V are the functions of r,s where r and s are function of x,y, the

$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}$ X $\frac{\partial(r,s)}{\partial(x,y)}$

Third Property

If function U,V,W of three independent variables x,y,z are not independent then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Recently, Rachunkova in [6] and Torres in [9] studied periodic boundary problems by using signed Green's function combining Kransnoselskii's fixed point theorem on compression and expansion of cones, and they obtained the new existence and multiplicity result concerning one signed periodic solution of the equation as well as equations with singularity. But the method mentioned above is difficult to be applied for estimate of the sharp number of solutions of (1.1), because it is impossible to determine the sharp norm for the Green's function. For the case indefinite weight, even the existence of T- periodic solution is not known. It seems that infinite dimensional singularity theory established in [11] provides a natural platform to deal with such kind of problems, and has been already successfully applied to non- homogeneous non-linear elliptic equations with both Dirchlet and Neumann boundary values respectively. To know more about this approach, one can refer to the approach, one can refer to comprehensive survey articles [10, 13].

In order to understand the global structure of periodic solutions and stability of each solutions under the cubic restoring force, Chen and Li in [4] used the different approach based on Crandall-Robinowitz bifurcation theorem and contraction method but devoted their work to the exact multiplicity and stability of periodic solution under cubic nonlinear restoring force with a strong damped condition.

$a(t) \leq \frac{(\pi)^2}{T^2} + \frac{C^2}{4}$, and $\bar{a} > 0$ where $\bar{a}$ denotes the average of $a(t)$ over a period.

In this paper they used global bifurcation method to cover the situation that the method to cover the situation that the method based on maximum principle developed in [5-7] and super-sub solutions methods are not applicable to degenerate situation and the interesting case when the graph of forcing term h(t) intersect the critical level is missing where bifurcation occurs. There result confirmed the first issue of the number of periodic solution of (1.1) is at most three under strong damped condition. Generally exactly one or three. The periodic solution of (1.1) forms an "S" - shape smooth curve, symmetric with respect to the origin.

Dutta and Prajapatic in [3] reported a symmetric investigation in the phase space of the double well Duffing Oscillator, they used bifurcation diagram to show the region characterized by the parameters for which one finds periodic solutions, a periodic solution. They also observed that when driving force is increased, there is a series of parallel “islands” of parameters characterized by a periodic attractors. They found that even the model is perturbed by linear term, it shows periodic and chaotic behaviour and that when damping coefficient is taken as zero and the non-linear stiffness parameter is taken sufficiently small, the model shows homoclinic nature for whatever the value of force.

3. RESULT AND DISCUSSION

Our modified Duffing equation is

$$\ddot{x} + \alpha\dot{x} + \tau^2 x - \beta x^3 = F sin\omega t \qquad (3.1)$$

With initial condition

$$x(0)=2\pi\ \dot{x}(0) = 2\pi \qquad (3.2)$$

And the periodic solution

$$x(t)= x(t+T) \qquad (3.3)$$

Going back to the duffing equation, we have tried different values of α and ω and observed where the periodic doubling route to chaos occurs.

We have fixed $\alpha = 0.1, \tau^2 = 2, \beta = -2, \omega = 1.2$.

$$\ddot{x} + 0.1\dot{x} - 2x^3 = F sin1.2t$$

with initial condition

$x(0)=1$ $\dot{x}(0) = 0$

and the solution of the periodic condition

$$x(t)= x(t+T) \qquad (3.4)$$

And F is taken within 0.1 to 15 to obtain the bifurcation diagram below

SIMULATION OF THE ODE

$$\ddot{x}_1 + \alpha\dot{x}_1 + \tau^2 x_1 - \beta x_1^3 = F\sin\omega t$$

Vectorizing the ode; let $x_1 = x$, the equation becomes

$$\ddot{x}_1 + \alpha\dot{x}_1 + \tau^2 x_1 - \beta x_1^3 = F\sin\omega t$$

Now if we let $\dot{x}_1 = x_2$ so that $\ddot{x}_1 = \dot{x}_2$ then we have a system of first order odes written

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\alpha x_2 - \tau^2 x_1 + \beta x_1^3 + F\sin\omega t$$

MathCAD 14 Solution

Numerical values of the parameters

$\alpha := 0.1$ $\beta := -2$ $\tau := \sqrt{2}$ $\omega := 1.2$ $F := 1$

Define a function that determines a vector of derivative values at any solution point (t,Y):

$$D(t,X) := \begin{bmatrix} X_1 \\ F\cdot\sin(\omega\cdot t) - \alpha\cdot X_1 - \tau^2\cdot X_0 + \beta\cdot(X_0)^3 \end{bmatrix}$$ Other definitions

Define additional arguments for the ODE solver:

$t0 := 0$ Initial value of independent variable

$t1 := 50$ final value of independent variable

$X0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Vector of initial function values

$num := 1\times10^3$ Number of solution values on [t0, t1]

S1 =

	0	1	2
0	0	0	1
1	0.01	$9.995\cdot10^{-3}$	0.999
2	0.02	0.02	0.998
3	0.03	0.03	0.997
4	0.04	0.04	0.995
5	0.05	0.05	0.994
6	0.06	0.06	0.993
7	0.07	0.07	0.991
8	0.08	0.08	0.989
9	0.09	0.089	0.988
10	0.1	0.099	0.986
11	0.11	0.109	0.984
12	0.12	0.119	0.982
13	0.13	0.129	0.98
14	0.14	0.139	0.978
15	0.15	0.148	...

Solution matrix

Graphical profiles

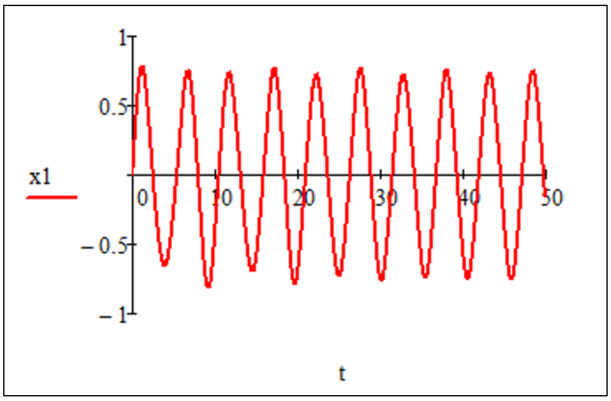

Fig-1: Position as a function of time

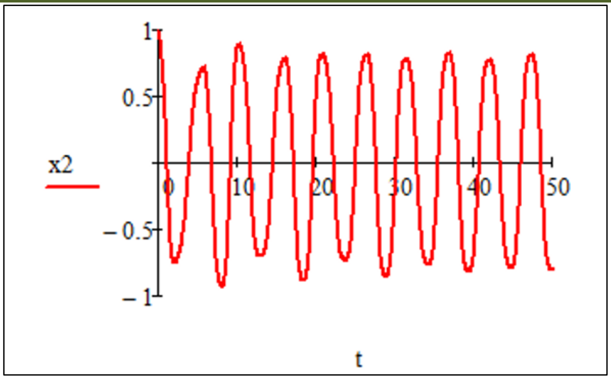

Fig-2: Velocity as a function of time

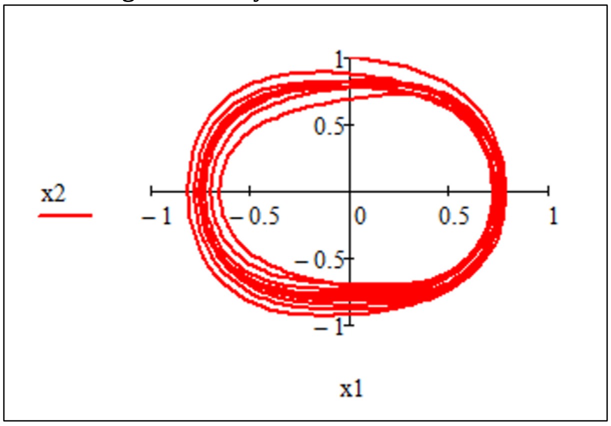

Fig-3: Phase portrait

Discussion

For the given values of the values of the parameters, figure 1 depicts the variation of the position with time. The trajectory is clearly oscillatory and fairly regular but with varying amplitude.

Figure 2 represents a profile of the velocity with time, and this is also oscillatory, regular with varying amplitude.

Figure 3 depicts the phase portrait which is the plot of the velocity against the position. The closed curve in the phase plane tells us that the system under study is conservative.

Clearly we see period doubling route to chaos. With parameters chosen in the region of limit cycles the system is either in the well of positive x or in the well of negative x, depending on the precise value of F, but does not hop between the wells, we see that there is a repetition of period one, period two behaviour which ultimately leads to chaotic behaviour.

Bifurcation using Stability Formular Developed in Marsden and Mccraven

Let $\dot{x} = y, \dot{y} = \ddot{x}$

$\dot{y} = -\propto y - \tau^2 x + \beta x^3 + Fsinwt$

Let $X_\alpha(x,y) = (y, -\propto y - \tau^2 x + \beta x^3 + Fsinwt)$

Now $X_\alpha(0,0) = 0$ for every α and

$dx_\alpha(0,0) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & -\alpha \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -\tau^2 & -\alpha \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$

$\begin{pmatrix} 0 & 1 \\ -\tau^2 & -\alpha \end{pmatrix} - \lambda\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$

$\begin{pmatrix} -\lambda & 1 \\ -\tau^2 & -\alpha-\lambda \end{pmatrix} = 0 \Rightarrow (-\lambda)(-\alpha-\lambda) - -\tau^2 = 0$

$\alpha\lambda + \lambda^2 + \tau^2 = 0$

$\lambda^2 + \alpha\lambda + \tau^2 = 0$

$\lambda = \frac{-\alpha \pm \sqrt{b^2-4ac}}{2a}$

$\lambda(\alpha) = \frac{-\alpha \pm \sqrt{\alpha^2-4\tau^2}}{2}$

Consider $\propto$ such that $|\propto| < 2$

In this case $\frac{-\propto}{2} + \frac{i\sqrt{\propto^2-4\tau^2}}{2}$

$im\lambda(\alpha) \neq 0$,

Where $\lambda(\alpha) = \frac{-\alpha \pm \sqrt{\alpha^2-4\tau^2}}{2} = \frac{-\propto}{2} + \frac{i\sqrt{\alpha^2-4\tau^2}}{2}$

Furthermore, for $-2 <\propto< 0$, Re $\lambda(\alpha) < 0$ and for $\propto = 0$, Re $\lambda(\alpha) = 0$ and for $2 >\propto> 0$, Re $\lambda(\alpha) > 0$ and

$\left.\frac{d(R_e\,\lambda(\alpha))}{d\propto}\right|_{\propto=0} = -\frac{1}{2}$

Therefore the Hopf bifurcation theorem applies and we conclude that there is one parameter family of closed orbits of x=$(x_\alpha, 0)$ in a neighbourhood of (0,0,0)

To find out if these orbits are stable and if they occur for $\alpha > 0$, we look at

$X_0(x,y) = (y, -\tau^2 x + \beta x^3 + Fsinwt)$.

$dx_0(0,0) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}$ and $\lambda(0) = \tau i$

Recall that to use the stability formular in we must choose coordinate so that

$dx_\alpha(0,0) = \begin{pmatrix} 0 & Im(\lambda_0) \\ -Im(\lambda_0) & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}$

Which is not in the required form. We must make a change of coordinates so that

$dx_0(0,0)$ becomes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ that is we must find vectors $\hat{e_1}$ and $\hat{e_2}$ so that $dx_0(0,0)\hat{e_1} = -\hat{e_2}$ and $dx_0(0,0)\hat{e_2} = \hat{e_1}$

The vectors $\hat{e_1} = (1,-1)$ and $\hat{e_2} = (0,1)$ will do.

A procedure for finding $\hat{e_1}$ and $\hat{e_2}$ is to find $\propto$ and $\overline{\propto}$ the complex eigenvectors. We may then take $\hat{e_1} = \propto + \overline{\propto}$ and $\hat{e_2} = i(\propto - \overline{\propto})$

$X_0(x\hat{e_1} + y\hat{e_1}) = X_0(x, y-x)$

$= (y, -\propto(y-x) - \tau^2 x + \beta x^3 + Fsin\omega t)$

$= (y\hat{e_1}, -\propto(y-x) + \tau^2 x + \beta x^3 + Fsin\omega t)\hat{e_2}$

$\therefore\therefore\therefore X_0(x,y) = (y, -\propto(y-x) + \tau^2 x + \beta x^3 + Fsin\omega t)$

$\frac{\partial^n x_1}{\partial x^j \partial y^{n-j}}$ (0,0)=0 for every n> 1

$\therefore\ x_1(x,y) = y$

$X_2(x,y) = -\propto(y-x) + \tau^2 x + \beta x^3 + Fsin\omega t$

$\therefore\ \frac{\partial^2 x_2}{\partial y^2}(0,0)=0,\ \frac{\partial^2 x_2}{\partial x \partial y}(0,0)=0$

$\frac{\partial^3 x_2}{\partial x^3}(0,0)=6\beta,\ \frac{\partial^3 x_2}{\partial x^2 \partial y}(0,0)=0$

$\frac{\partial^3 x_2}{\partial x \partial y^2}(0,0)=0,\ \frac{\partial^3 x_2}{\partial y^3}(0,0)=0$

$\therefore\ \ddot{v}(0) = \frac{3\pi}{4|\lambda(0)|}(6\beta)$

The orbits are unstable and bifurcation takes place below criticality. The orbits occur for $\mu < \mu_0$ and are repelling on the center manifold, and so are unstable by general.

CONCLUSION

We have fixed the parameters α, β, τ^2 and studied the behaviour for different values of we obtained the Hopf bifurcation diagram for $0.1 < f < 15$., secondly we have obtained the phase portrait and poincare section in this regard, thirdly we have shown x vs t graph for our model within same range of parameters, fourthly, we observed the behaviour of the strange attractors, how they vary with driving force and damping factor. Lastly we observed the Homoclinic behaviour of the Oscillator when damping coefficient is taken as zero.

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