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An Improved XShanker Distribution with Applications to Rainfall and Vinyl Chloride Data

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Abstract **Review Article**

This article is an improvement on the XShanker distribution having a single parameter which is the scale parameter and also in the class of Lindley distributions. It is named Double XShanker following the approach that generated it. The distributional properties which include the non-central moment with the associated statistics, the moment generating function, characteristic function, mean residual life function, stress-strength reliability function, Bonferroni and Lorenz curve functions, odd function, stochastic ordering, distribution of order statistics, and Reny entropy. The parameter was estimated using the method of maximum likelihood. Some visualizations based on theoretical values were presented. Some statistics were computed from theoretical values with convergence behavior recorded. A simulation study was conducted with 1000 samples of different sizes and the behavior observed is that as sample size increases the estimates decrease indicating precision. Data on rainfall and Vinyl chloride were used to validate the usefulness of the suggested distribution.

Keywords: Double XShanker distribution, Exponential distribution, Flexibility, Performance metrics, Shanker distribution

1 INTRODUCTION

Continuous innovation in distributions is to attain flexibility and achieve improved model scenarios above the parent distributions. [1] modified the Shanker distribution and was able to achieve a better fit and model performance compared to both parent distribution and some competing distributions. The same applied to the several extensions of [2], namely [3, 4, 5, 6] and [7]. In the literature, some sophisticated models have been proposed with more than one parameter where some parameter(s) account for scale, and others account for shape and location hence presenting a better visualization and robust inference. Interesting among those complex distributions include [8, 9, 10, 11, 12, 13, 14, 15]. However, the trade-off in the use of complex models is in the estimation of parameters (both in the tractability and economy of numerical iteration when closed-form expressions are not feasible). Hence, the smaller the number of parameters, the better the estimation tasks.

On the above basis, many researchers have focused on single-parameter distributions to model real situations and there are many instances where one-parameter distribution outperforms multi-parameter counterpart. Essentially, whether it is a one-parameter or more than one-parameter, the motivation is to develop distributions that will be able to fit data sets without many ambiguities. This article therefore is aimed at modifying the XShanker distribution suggested by [1] with improved goodness of fit and better parameter estimate yet having a single-parameter and guarantees parsimony.

Suppose $X_1 \sim$ XShanker (θ), with p.d.f given as

$$g(x_1) = \frac{\theta^2}{(\theta^2+1)^2}\left[\theta^3 + 2\theta + x\right]e^{-\theta x}; \quad x > 0, \quad \theta > 0 \tag{1}$$

and $X_2 \sim$ Exponential(θ), with pdf given as

$$g(x_2) = \theta e^{-\theta x}; \quad x > 0, \quad \theta > 0 \tag{2}$$

using a mixing proportion of $p = \frac{\theta^2}{(\theta^2+1)}$ and the mixture of the form $pg(x_2) + (1-p)g(x_1)$, one obtains an improvement on the XShanker distribution.

Definition 1.1. Let $X \sim$ Doulble XShanker (θ), then the pdf and cdf are respectively

$$f(x) = \frac{\theta^2}{(\theta^2+1)^3}\left[\theta^5 + 3\theta^3 + 3\theta + x\right]e^{-\theta x}; \quad x > 0, \quad \theta > 0 \tag{3}$$

and

$$F(x) = 1 - \left[1 + \frac{\theta x}{(\theta^2+1)^3}\right]e^{-\theta x} \tag{4}$$

The survival and hazard rate functions are respectively

$$S(x) = \left[1 + \frac{\theta x}{(\theta^2+1)^3}\right]e^{-\theta x} \tag{5}$$

and

$$hrf(x) = \frac{\theta^2\left(x + 3\theta + 3\theta^3 + \theta^5\right)}{\theta x + \left(1+\theta^2\right)^3} \tag{6}$$

The limiting values of the Double XShanker hazard function are

$$\lim_{x\to 0} hrf(x) = \frac{\left(3\theta^3 + 3\theta^5 + \theta^7\right)}{\left(1+\theta^2\right)^3} \quad \text{and} \quad \lim_{x\to\infty} hrf(x) = 0$$

The plots are displayed in the figures 1, 2, 3, and 4 below

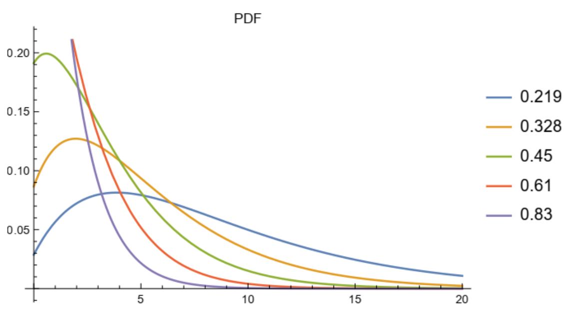

Figure 1: pdf of Double XShanker distribution

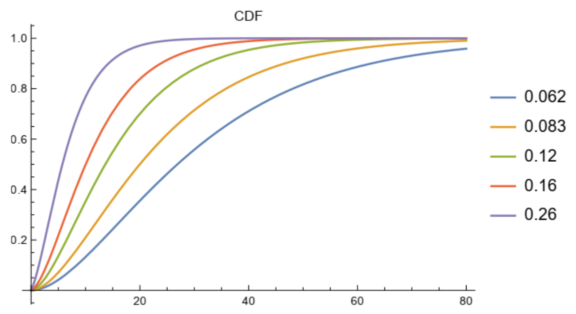

Figure 2: cdf of Double XShanker distribution

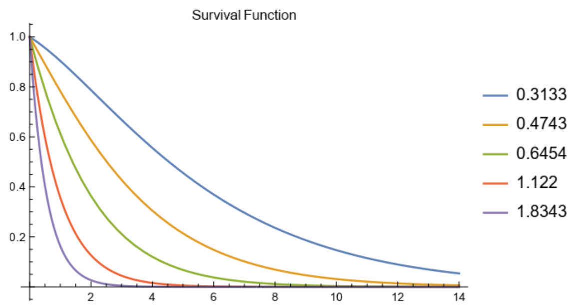

Figure 3: survival function of Double XShanker distribution

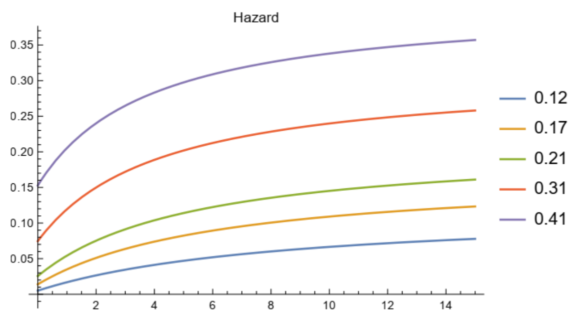

Figure 4: hazard function of Double XShanker distribution

The remaining sections of this article are in the following order: In section 2, the distributional properties of the proposed Double XShanker distribution are obtained. Section 3 is on the estimation of the model parameter. In section 4, we conduct a simulation study and apply the distribution to real-life situations using two data sets. The article is concluded in section 5 with remarks.

2 Distributional Properties of Double XShanker distribution

In this section, we derive the basic distributional properties of the proposed model.

Definition 2.1. Let X ~ Double Shanker (θ), then the s^{th} non-central moment is expressed as

$$\mu_s^{'} = \frac{s\theta^{-s}\left(s+\left(1+\theta^2\right)^3\right)\Gamma[s]}{\left(1+\theta^2\right)^3} \tag{7}$$

The first, second, third and fourth moments are respectively

$$\mu = \frac{\theta^2\left(\frac{2}{\theta^3}+\frac{3}{\theta}+3\theta+\theta^3\right)}{\left(1+\theta^2\right)^3}; \quad \mu_2^{'} = \frac{\theta^2\left(6+\frac{6}{\theta^4}+\frac{6}{\theta^2}+2\theta^2\right)}{\left(1+\theta^2\right)^3}; \quad \mu_3^{'} = \frac{6\left(4+3\theta^2+3\theta^4+\theta^6\right)}{\theta^3\left(1+\theta^2\right)^3}; \quad \text{and} \quad \mu_4^{'} = \frac{24\left(5+3\theta^2+3\theta^4+\theta^6\right)}{\theta^4\left(1+\theta^2\right)^3} \tag{8}$$

Variance is used to measure thee spread in the data and for Double XShanker distribution, it can be expressed as

$$\sigma^2 = \frac{2+12\theta^2+21\theta^4+22\theta^6+15\theta^8+6\theta^{10}+\theta^{12}}{\theta^2\left(1+\theta^2\right)^6} \tag{9}$$

The skewness ζ, kurtosis η and coefficient of variation ξ are respectively

$$\zeta = \frac{6\left(4+3\theta^2+3\theta^4+\theta^6\right)\left(1+\theta^2\right)^6}{\left(2+12\theta^2+21\theta^4+22\theta^6+15\theta^8+6\theta^{10}+\theta^{12}\right)^{\frac{3}{2}}}; \quad \eta = \frac{24\left(1+\theta^2\right)^9\left(5+3\theta^2+3theta^4+\theta^6\right)}{\left(2+12\theta^2+21\theta^4+22\theta^6+15\theta^8+6\theta^{10}+\theta^{12}\right)^2}; \quad \text{and}$$
$$\xi = \frac{\sqrt{2+12\theta^2+21\theta^4+22\theta^6+15\theta^8+6\theta^{10}+\theta^{12}}}{2+3\theta^2+3\theta^4+\theta^6} \tag{10}$$

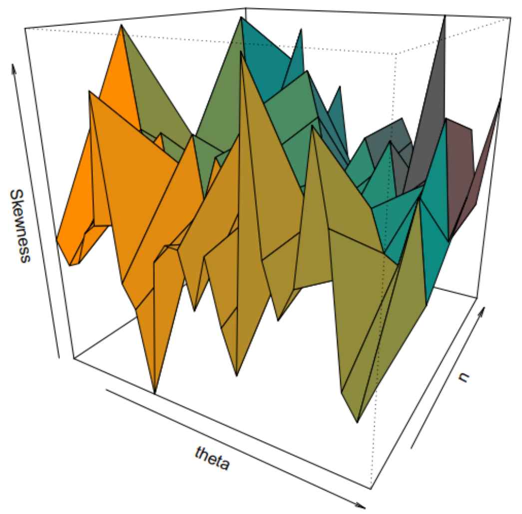

Figure 5: skewness surface plot from simulated samples of Double XShanker distribution

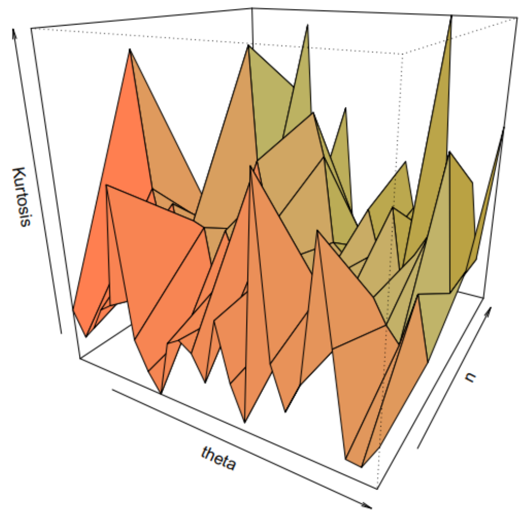

Figure 6: kurtosis surface plot from simulated samples of Double XShanker distribution

Table 1: Some theoretical statistics of the Double XShanker distribution

θ	μ	μ_2'	μ_3'	μ_4'	σ^2	ζ	η	ξ
0.10000	19.70590	588.23606	23470.62266	1171766.54206	199.91351	1.41511	-16.63614	0.71750
0.30714	5.74013	53.55452	681.09790	10927.88126	20.60544	1.46611	-9.17091	0.79080
0.51429	2.90617	15.04179	109.56061	1021.82460	6.59599	1.62386	-3.02308	0.88373
0.72143	1.78049	6.02926	29.61835	189.42709	2.85912	1.80001	0.33274	0.94968
0.92857	1.24368	3.03785	10.97495	52.27522	1.49112	1.91557	1.95152	0.98186
1.13571	0.95384	1.80886	5.11925	19.23154	0.89905	1.96934	2.62816	0.99407
1.34286	0.77849	1.20979	2.81521	8.72077	0.60375	1.98960	2.87478	0.99811
1.55000	0.66154	0.87473	1.73394	4.58023	0.43710	1.99650	2.95792	0.99939
1.75714	0.57744	0.66673	1.15452	2.66503	0.33330	1.99879	2.98542	0.99979
1.96429	0.51353	0.52739	0.81237	1.66833	0.26367	1.99956	2.99471	0.99993
2.17143	0.46299	0.42871	0.59544	1.10265	0.21435	1.99983	2.99798	0.99997
2.37857	0.42185	0.35590	0.45040	0.75997	0.17795	1.99993	2.99918	0.99999
2.58571	0.38759	0.30045	0.34936	0.54162	0.15023	1.99997	2.99965	1.00000
2.79286	0.35858	0.25716	0.27664	0.39679	0.12858	1.99999	2.99985	1.00000
3.00000	0.33367	0.22267	0.22289	0.29748	0.11133	1.99999	2.99993	1.00000

From table 1, the estimates decrease as the parameter value is increased showing convergence behavior.

For a Double XShanker distributed random variable *X*, the moment generating function *MX* (*t*) is

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \tag{11}$$

$$M_X(t) = \int_0^\infty e^{tx}\left[\frac{\theta^2}{(\theta^2+1)^3}\left[\theta^5+3\theta^3+3\theta+x\right]e^{-\theta x}\right]dx \tag{12}$$

hence

$$M_X(t) = \frac{\theta^2\left(1-(t-\theta)\theta\left(3+3\theta^2+\theta^4\right)\right)}{(t-\theta)^2\left(1+\theta^2\right)^3} \tag{13}$$

Similarly, for a Double XShanker distributed random variable X, the characteristic function $\Phi_X(it)$ is

$$\Phi_X(it) = E(e^{itx}) = \int_0^\infty e^{itx}\left[\frac{\theta^2}{(\theta^2+1)^3}\left[\theta^5+3\theta^3+3\theta+x\right]e^{-\theta x}\right]dx \tag{14}$$

hence

$$\Phi_X(it) = \frac{\theta^2\left(1-(it-\theta)\theta\left(3+3\theta^2+\theta^4\right)\right)}{(it-\theta)^2\left(1+\theta^2\right)^3} \tag{15}$$

If X is a nonnegative random variable representing the life of a component having distribution function F(.) represented in eq, then the mean residual life is defined by

$$MRL = \frac{1}{1-F(x)}\int_x^\infty [1-F(t)]dt = \frac{2+x\theta+3\theta^2+3\theta^4+\theta^6}{\theta\left(x\theta+\left(1+\theta^2\right)^3\right)} \tag{16}$$

Another important indexes are the Bonferroni and Lorenz curve functions which have applications not only in Economics for studying income and poverty distribution, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curve functions for Double XShanker distributed random variable X can be expressed as

$$B(p) = \frac{1}{p\mu}\int_0^q f(x)dx = \frac{1}{p\left(\frac{2}{\theta^3}+\frac{3}{\theta}+3\theta+\theta^3\right)}\left[\theta^3\gamma(2,q)+3\theta\gamma(2,q)+\frac{3\gamma(2,q)}{\theta}+\frac{\gamma(3,q)}{\theta^3}\right]$$
$$L(p) = \frac{1}{\mu}\int_0^q f(x)dx = \frac{1}{\left(\frac{2}{\theta^3}+\frac{3}{\theta}+3\theta+\theta^3\right)}\left[\theta^3\gamma(2,q)+3\theta\gamma(2,q)+\frac{3\gamma(2,q)}{\theta}+\frac{\gamma(3,q)}{\theta^3}\right] \quad (17)$$

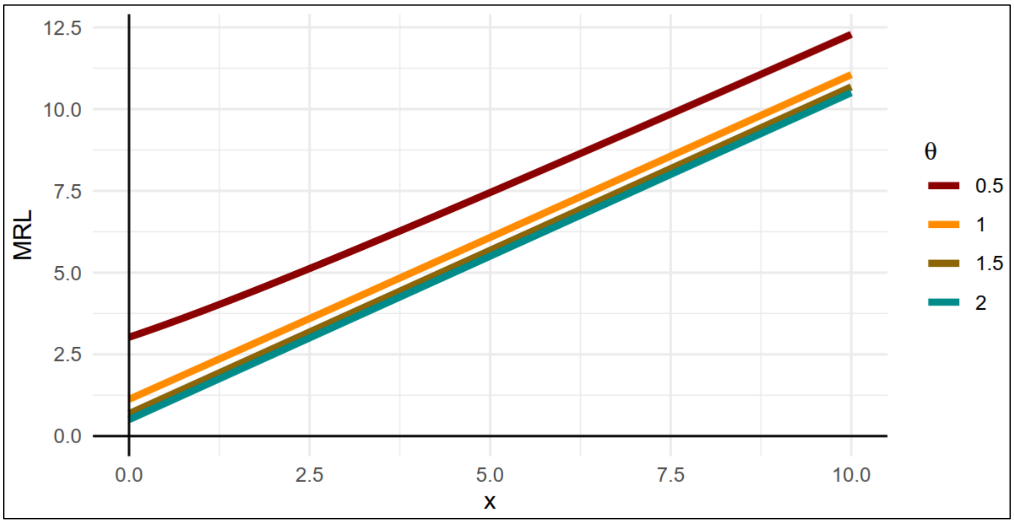

Figure 7: Mean Residual Life function plots

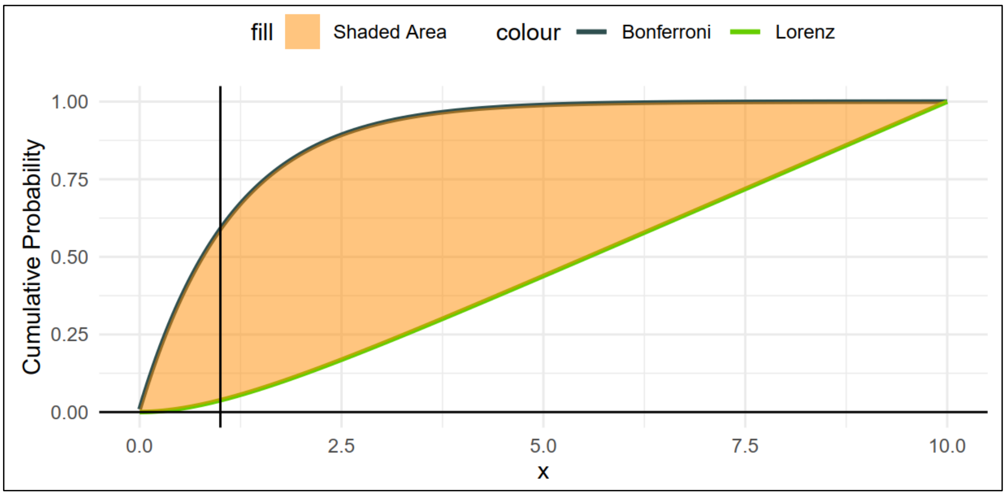

Figure 8: Bonferroni and Lorenz curves for Double XShanker distribution

The odd function is another reliability measure. With the Double XShanker (DXS) distributed randome variable X, the odd function is given as ratio of its c.d.f to the survival function, which is

$$O_{DXS}(x;\theta) = \frac{x\theta+\left(1+\theta^2\right)^3-e^{-x\theta}\left(1+\theta^2\right)^3}{\left(x\theta+\left(1+\theta^2\right)^3\right)} \quad (18)$$

Definition 2.2. A random variable X is said to be smaller than another random variable Y in the stochastic order ($X \leq_{st} Y$)if $F_Y(x) \geq F_Y(x)\ \forall_x$ Hazard order ($X \leq_{hr} Y$) if $h_X(x) \geq h_y(x)\ \forall_x$. Mean residual life order ($X \leq_{mrl} Y$)$if\, m_X(x) \geq m_Y(x)\ \forall_x$ Likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(y)}$ decreases in x. This implies that $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y$

Theorem 1. Let $X \sim DoubleXShanker(\theta_1)$ and $Y \sim DoubleXShanker(\theta_2)$. if $\theta_1 \geq \theta_2$ then $X \leq_{lr} Y$ hence $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$

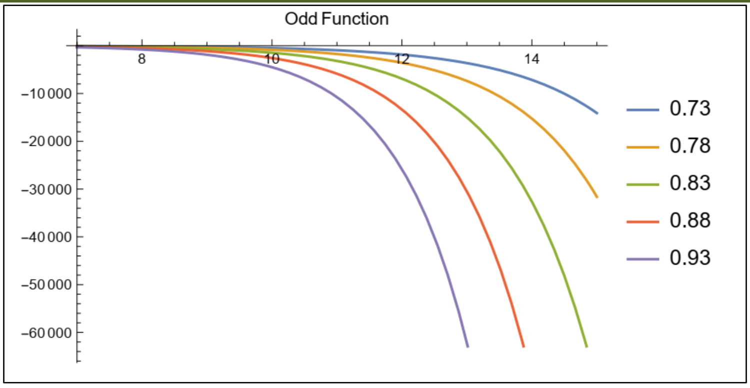

Figure 9: Odd Function of Double XShanker distribution

$$\frac{f_x(x)}{f_y(x)} = \frac{\frac{\theta_1^2}{(\theta_1^2+1)^3}\left[\theta_1^5+3\theta_1^3+3\theta_1+x\right]e^{-\theta_1 x}}{\frac{\theta_2^2}{(\theta_2^2+1)^3}\left[\theta_2^5+3\theta_2^3+3\theta_2+x\right]e^{-\theta_2 x}} = \frac{\theta_1^2(\theta_2^2+1)^3(\theta_1^5+3\theta_1^3+3\theta_1+x)}{\theta_2^2(\theta_1^2+1)^3(\theta_2^5+3\theta_2^3+3\theta_2+x)}e^{(\theta_2-\theta_1)x} \tag{19}$$

Taking a natural log of the ratio will yield

$$ln\frac{f_x(x)}{f_y(x)} = ln\frac{\theta_1^2(\theta_2^2+1)^3}{\theta_2^2(\theta_1^2+1)^3} + ln\frac{(\theta_1^5+3\theta_1^3+3\theta_1+x)}{(\theta_2^5+3\theta_2^3+3\theta_2+x)} + (\theta_2-\theta_1)x \tag{20}$$

Differentiating the natural log of the ratio w.r.t x will result

$$= -\theta_1 + \frac{1}{\theta_1^5+3\theta_1^3+3\theta_1+x} + \theta_2 - \frac{1}{\theta_2^5+3\theta_2^3+3\theta_2+x} \tag{21}$$

If $\theta_2 \geq \theta_1$, $\frac{d}{dx}ln\frac{f_x(x)}{f_y(x)} \leq 0$, and $\frac{f_x(x,\theta_1)}{f_y(x,\theta_2)}$ is decreasing in x.

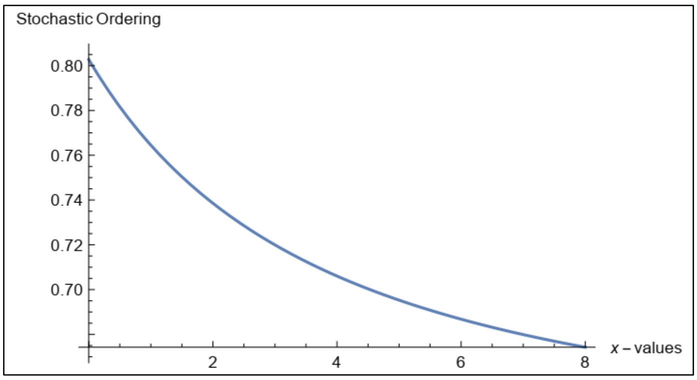

Figure 10: Stochastic Ordering of Double XShanker distribution

Definition 2.3. Stress-strength reliability is the probability that the strength of a system exceeds its stress. The inferences of the stress-strength reliability $R = P(X > Y)$ where Y is the stress and X is the strength.

$$\begin{aligned} R &= P(Y<X) = \int_0^\infty P(Y<X|X=x)f_x(x)dx = \int_0^\infty f(x;\theta_1)F(x;\theta_2)dx \\ &= \frac{1}{(\theta_1^2+1)^3(\theta_1+\theta_2)^3(\theta_2^2+1)^3}\theta_2^2\left[3\theta_1(\theta_2^2+1)^3+\theta_2(\theta_1^2+1)^3+\theta_1^8\theta_2(3+3\theta_2^2+\theta_2^4)\right. \\ &+3\theta_2^2\theta_2(4+6\theta_2^2+4\theta_2^4+\theta_2^6)+\theta_1^6\theta_2(10+12\theta_2^2+6\theta_2^4+\theta_2^6)+3\theta_1^3(1+6\theta_2^2+6\theta_2^4+3\theta_2^6) \\ &+3\theta_1^5(1+6\theta_2^2+6\theta_2^4+2\theta_2^6)+\theta_1^7((1+6\theta_2^2+6\theta_2^4+2\theta_2^6)+\theta_1^2\theta_2(9+15\theta_2^2+11\theta_2^4+3\theta_2^6) \end{aligned} \tag{22}$$

Definition 2.4. Suppose X_1, X_2,....X_n is a random sample of X_r; r=(1,2....,n) are the rth order statistics obtained by arranging X_r in ascending order of magnitude $\ni X_1 \leq X_2 \leq ....X_r$ where X_1 is the smallest of all variable and X_r is the largest of all variable, then the pdf of the rth order statistics is given by pdf of the rth order statistics is given by

$$\begin{aligned} f_{r:n}(x;\theta) &= \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} \frac{\theta^2}{(\theta^2+1)^3}(\theta^5+3\theta^3+3\theta+x)e^{-\theta x}\left[1-\left(1+\frac{\theta x}{(\theta^2+1)^3}\right)e^{-\theta x}\right]^{r-1}\left(1+\frac{\theta x}{(\theta^2+1)^3}e^{-\theta x}\right)^{n-r} \end{aligned} \tag{23}$$

The pdf of highest order statistics is obtained by setting $r = n$

$$\begin{aligned} f_{n:n}(x;\theta) &= \frac{n\theta^2}{(\theta^2+1)^3}(\theta^5+3\theta^3+3\theta+x)e^{-\theta x}\left(1-\left(1+\frac{\theta x}{(\theta^2+1)^3}\right)e^{-\theta x}\right)^{n-1} \\ &= n(\theta^5+3\theta^3+3\theta+x)\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^j\binom{n-1}{j}\binom{j}{i}\theta^{i+2}x^i(\theta^2+1)^{-3(1+i)}e^{-(1+j)\theta x} \end{aligned} \tag{24}$$

pdf of minimum order statistics is obtained by setting $r = 1$

$$\begin{aligned} f_{1:n}(x;\theta) &= \frac{n\theta^2}{(\theta^2+1)^3}(\theta^5+3\theta^3+3\theta+x)e^{-\theta x}\left(\left(1+\frac{\theta x}{(\theta^2+1)^3}\right)e^{-\theta x}\right)^{n-1} \\ &= n(\theta^5+3\theta^3+3\theta+x)\sum_{j=0}^{\infty}\binom{n-1}{j}\theta^{j+2}x^j(\theta^2+1)^{-3(1+j)}e^{-\theta n x} \end{aligned} \tag{25}$$

Definition 2.5. Rényi entropy was originally introduced in the field of information theory as a parametric relaxation of Shannon (in physics, Boltzmann–Gibbs) entropy. [16]. The Renyi Rntropy of Double XShanker can be expressed as

$$\begin{aligned} R_{\omega}(x) &= \frac{1}{1-w} log \int_0^{\infty} f(x)^{\omega} dx \\ &= \frac{1}{1-\omega} log \int_0^{\infty}\left(\frac{\theta^2}{(\theta^2+1)^3}\left[\theta^5+3\theta^3+3\theta+x\right]e^{-\theta x}\right)^{\omega} dx \\ &= \frac{1}{1-\omega} log \frac{\theta^{2\omega}}{(\theta^2+1)^{3\omega}}\int_0^{\infty}\left[\theta^5+3\theta^3+3\theta+x\right]^{\omega} e^{-\theta\omega x} dx \end{aligned} \tag{26}$$

let;

$\theta^5+3\theta^3+3\theta = a$, $x = b$, $\omega = n$. Then $(a+b)^n = \sum_{j=0}^{\omega} a^j b^{\omega-j}$

$$\begin{aligned} R_{\omega}(x) &= \frac{1}{1-\omega} log \frac{\theta^{2\omega}}{(\theta^2+1)^{3\omega}}\int_0^{\infty}\sum_{j=0}^{\omega}\binom{\omega}{j}(\theta^5+3\theta^3+3\theta)^j x^{\omega-j}e^{-\theta\omega x} dx \\ &= \frac{1}{1-\omega} log \frac{\theta^{2\omega}}{(\theta^2+1)^{3\omega}}\sum_{j=0}^{\omega}\binom{\omega}{j}(\theta^5+3\theta^3+3\theta)^j \int_0^{\infty} x^{\omega-j}e^{-\theta\omega x} dx \\ &= \frac{1}{1-\omega} log \frac{\theta^{2\omega}}{(\theta^2+1)^{3\omega}}\sum_{j=0}^{\omega}\binom{\omega}{j}(\theta^5+3\theta^3+3\theta)^j \frac{\Gamma_{\omega-j+1}}{(\theta\omega)^{\omega-j+1}} \\ \therefore R_{\omega}(x) &= \frac{1}{1-\omega} log \frac{\theta^{\omega+j-1}}{(\theta^2+1)^{3\omega}}\sum_{j=0}^{\omega}\binom{\omega}{j}(\theta^5+3\theta^3+3\theta)^j \frac{(\omega-j)!}{(\omega)^{\omega-j+1}} \end{aligned} \tag{27}$$

3 Maximum Likelihood Function

Let (X_1, X_2,.., X_n) be random variables of double Double XShanker. then the Maximum Likelihood Function is given as

$$\begin{aligned} \ell(f_{(x;\theta)}) &= \prod_{i=1}^{n}\frac{\theta^2}{(\theta^2+1)^3}\left[\theta^5+3\theta^3+3\theta+x\right]e^{-\theta x} \\ &= \frac{\theta^{2n}}{(\theta^2+1)^{3n}}\prod_{i=1}^{n}\left[\theta^5+3\theta^3+3\theta+x\right]e^{-\theta\sum_{i=1}^{n}x_i} \end{aligned} \tag{28}$$

Taking the log of the function

$$\psi = 2nln\theta - 3nln(\theta^2+1) - \theta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\left[\theta^5+3\theta^3+3\theta+x\right]$$

Differentiate ψ w.r.t θ i.e $\frac{d\psi}{d\theta}$ and equate to zero yields the result below

$$\frac{2n}{\theta} + \frac{6\theta n}{(\theta^2+1)} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{5\theta^4+9\theta^2+3}{\theta^5+3\theta^3+3\theta+x} = 0 \tag{29}$$

4 Applications

In this section, we demonstrate the usefulness of the proposed distribution through simulation study and practical examples from real-life data.

5 Simulation

We draw $N = 1000$ samples of sizes (n = 25,50,75,100,200,500 and 1000) assuming the parameter values (θ = 0.1,0.25,1.5, and 1.75) from Double XShanker distribution. The following performance indices (Mean, Bias, Mean Square Error (MSE), Root Mean Square Error (RMSE), Lower CI, Upper CI and Average length of the confidence intervals (AL.CI)) are computed including the estimated mean of the parameter.

$$\begin{aligned} Bias_\theta(n) &= \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i-\theta) \\ RMSE_\theta(n) &= \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i-\theta)^2} \\ MSE_\theta(n) &= \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i-\theta)^2 \\ AL_\theta(n) &= \frac{3.919928}{1000}\sum_{i=1}^{1000}S_{\hat{\theta}_i} \end{aligned} \quad (30)$$

Table 2: Mean, Bias, MSE, RMSE, Lower CI, Upper CI and Average Length of CI for simulated study

n	$\theta_{initial}$	$\bar{\theta}$	Bias	MSE	RMSE	LowerCI	UpperCI	AL.CI
	0.10000	0.10040	0.00040	0.00393	0.06271	0.00000	0.24000	0.24000
	0.25000	0.25148	0.00148	0.00920	0.09590	0.08000	0.44000	0.36000
25	1.50000	1.51080	0.01080	0.05863	0.24214	1.08000	2.04000	0.96000
	1.75000	1.74624	-0.00376	0.06904	0.26276	1.24000	2.28000	1.04000
	0.10000	0.09832	-0.00168	0.00181	0.04258	0.02000	0.18000	0.16000
	0.25000	0.25484	0.00484	0.00498	0.07057	0.12000	0.40000	0.28000
50	1.50000	1.49770	-0.00230	0.03154	0.17761	1.14000	1.86000	0.72000
	1.75000	1.75214	0.00214	0.03247	0.18020	1.42000	2.10000	0.68000
	0.10000	0.09831	-0.00169	0.00133	0.03646	0.04000	0.17333	0.13333
	0.25000	0.24872	-0.00128	0.00331	0.05750	0.14667	0.37333	0.22667
75	1.50000	1.50288	0.00288	0.01905	0.13803	1.22667	1.78700	0.56033
	1.75000	1.75041	0.00041	0.02298	0.15158	1.45333	2.05367	0.60033
	0.10000	0.09930	-0.00070	0.00095	0.03078	0.05000	0.16000	0.11000
	0.25000	0.24980	-0.00020	0.00262	0.05116	0.15000	0.35000	0.20000
100	1.50000	1.50323	0.00323	0.01547	0.12439	1.28000	1.76000	0.48000
	1.75000	1.75059	0.00059	0.01797	0.13406	1.50000	2.03000	0.53000
	0.10000	0.10022	0.00022	0.00050	0.02237	0.06000	0.15000	0.09000
	0.25000	0.24940	-0.00061	0.00127	0.03569	0.18500	0.32500	0.14000
200	1.50000	1.49928	-0.00073	0.00739	0.08595	1.33000	1.67000	0.34000
	1.75000	1.74764	-0.00236	0.00823	0.09070	1.57975	1.94500	0.36525
	0.10000	0.09968	-0.00032	0.00020	0.01407	0.07400	0.12800	0.05400
	0.25000	0.25140	0.00140	0.00055	0.02337	0.20600	0.30000	0.09400
500	1.50000	1.50177	0.00177	0.00298	0.05464	1.39995	1.60800	0.20805
	1.75000	1.74949	-0.00051	0.00342	0.05851	1.63400	1.86805	0.23405

From the simulation results in table 2, the statistics decrease as sample size increases showing precision.

5.1 Application to Lifetime Data

In this subsection, we will use real-life data sets to illustrate the usefulness of the proposed Double XShanker distribution and compare it with the following known distributions which are in the class of Lindley distribution.

Table 3: List of one-parameter distributions in the class of Lindley

Distribution	$f(x)$	$F(x)$
proposed Double XShanker	$\frac{\theta^2}{(\theta^2+1)^3}\left(\theta^5+3\theta^3+3\theta+x\right)e^{-\theta x}$	$1-\left\{1+\frac{\theta x}{(\theta^2+1)^3}\right\}e^{-\theta x}$
Rani [17]	$\frac{\theta^5}{\theta^5+24}\left(\theta+x^4\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta^3x^3+4\theta^2x^2+12\theta x+24)}{\theta^5+24}\right]e^{-\theta x}$
Shanker [18]	$\frac{\theta^5}{\theta^2+1}(\theta+x)e^{-\theta x}$	$1-\left[1+\frac{\theta x}{\theta^2+1}\right]e^{-\theta x}$
XShanker [1]	$\frac{\theta^2}{(\theta^2+1)^2}\left(\theta^3+2\theta x+x\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x}{(\theta^2+1)^2}\right]e^{-\theta x}$
Rama [19]	$\frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}$	$1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}$
Lindley [20]	$\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	$1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}$
XLindley [21]	$\frac{\theta^2(2+\theta+x)}{(1+\theta)^2}e^{-\theta x}$	$1-\left[1+\frac{\theta x}{(1+\theta)^2}\right]e^{-\theta x}$
Chris-Jerry [2]	$\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta x+2)}{\theta+2}\right]e^{-\theta x}$

The application is on rainfall reported at the Los Angeles Civic Center from 1943 to 2018 and studied by [22]. The data is in table 4

Table 4: The rainfall reported at the Los Angeles Civic Center from 1943 to 2018 in the month of March

4.55	2.47	3.43	3.66	0.79	3.07	1.40	0.87	0.44	6.14	0.4 8	2.99	0.56	1.02
5.30	0.31	0.57	1.10	2.78	1.79	2.49	0.53	2.5	3.34	1.49	2.36	0.53	2.70
3.78	4.83	1.81	1.89	8.02	5.85	4.79	4.10	3.54	8.37	0.28	1.29	5.2 7	0.95
0.26	0.81	0.17	5.92	7.12	2.74	1.86	6.98	2.16	4.06	1.24	2.82	1.17	0.32
4.32	1.47	2.14	2.87	0.05	0.01	0.35	0.48	3.96	1.75	0.54	1.18	0.87	1.60
0.09	2.69												

The measures of model performance for the distributions are the negative Log-Likelihood (NLL), Akaike Information Criterion (AIC), Corrected AIC (CAIC), Bayesian Information Criterion (BIC), Hannan–Quinn information criterion (HQIC), Cramer von Mises ($W*$), Anderson Darling ($A*$), while the Kolmogorov-Smirnov (K-S) statistic and the p-value determine the fitness of the distribution to the data.

Table 5: MLEs, measures of fitness and performance of the models for March Rainfall data

Distr	NLL	AIC	CAIC	BIC	HQIC	W*	A*	θ	K-S	P-value
Double XShanker	135.68	273.362	273.419	275.639	274.268	0.026	0.174	0.578	0.060	0.9555
Lindley	135.82	273.630	273.687	275.907	274.537	0.026	0.170	0.655	0.080	0.7469
Chris-Jerry	136.3	274.596	274.653	276.872	275.502	0.30	0.204	0.946	0.090	0.5999
Shanker	136.18	274.355	274.413	276.632	275.262	0.027	0.176	0.691	0.090	0.5977
Rama	138.63	279.261	279.319	281.538	280.168	0.077	0.483	1.303	0.104	0.4206
XLindley	135.7	273.408	273.465	275.685	274.314	0.027	0.176	0.573	0.062	0.942
Rani	140	281.990	282.048	284.267	282.897	0.140	0.830	1.584	0.105	0.4052
XShanker	135.65	273.295	273.295	275.571	274.201	0.025	0.168	0.621	0.067	0.9014

From table 5, the p-value for Double XShanker distribution is 0.9555. That of the XShanker distribution comes second with 0.9014 while that of the Shanker distribution is a distant 0.5977. With this evidence, the Double XShanker no doubt is an improvement in XShanker distribution. The model also performs better than the rest compared given that its performance metrics are the least among others.

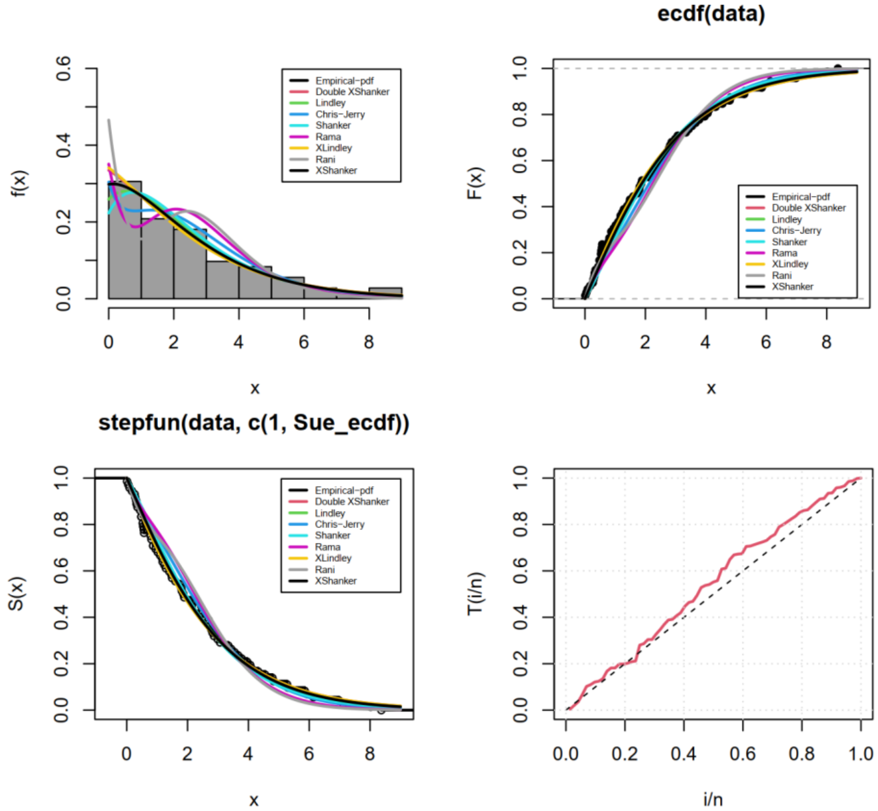

Figure 11: Density, cdf, survival function, and TTT plot for the March rainfall data

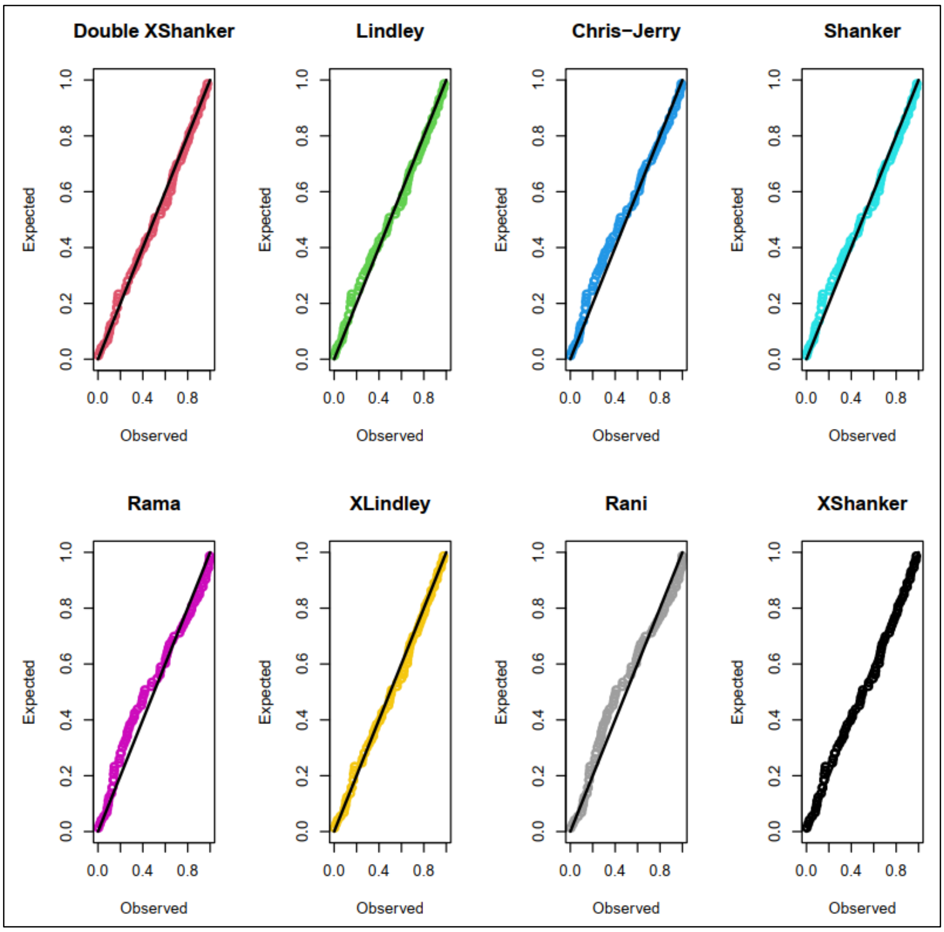

Figure 12: Pp plots for the March rainfall data

Fig 12, shows how well the competing distributions individually fit the rainfall data and it is obvious that the proposed distribution best fits the data.

The second application is on Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L) studied by [13] in table 1.

Table 6: Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L)

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3
3.2	2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2

Table 7: MLEs, metrics for fitness and model performance for the Vinyl Chloride data

Distr	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	θ	K-S	P-value
Double XShanker	55.64	113.281	113.406	114.808	113.802	0.053	0.341	0.700	0.102	0.872
Lindley	56.3	114.607	114.732	116.134	115.128	0.063	0.404	0.824	0.133	0.5878
Chris-Jerry	57.93	117.854	117.979	119.38	118.374	0.103	0.655	1.164	0.178	0.2303
Shanker	56.46	114.913	115.038	116.439	115.433	0.064	0.413	0.853	0.131	0.607
Rama	59.34	120.683	120.808	122.210	121.204	0.154	0.952	1.531	0.177	0.2383
XLindley	55.7	113.401	113.526	114.927	113.921	0.054	0.350	0.714	0.108	0.8218
Rani	59.88	121.752	121.877	123.278	122.272	0.194	1.165	1.784	0.152	0.4151
XShanker	55.85	113.693	113.818	115.219	114.213	0.057	0.369	0.757	0.111	0.7979

From table 7, the p-value for Double XShanker distribution is 0.872. That of the XShanker distribution comes second with 0.7979 while that of the Shanker distribution is a distant 0.607. With this evidence, the Double XShanker no doubt is an improvement in XShanker distribution. The model also performs better than the rest compared given that its performance metrics are the least among others.

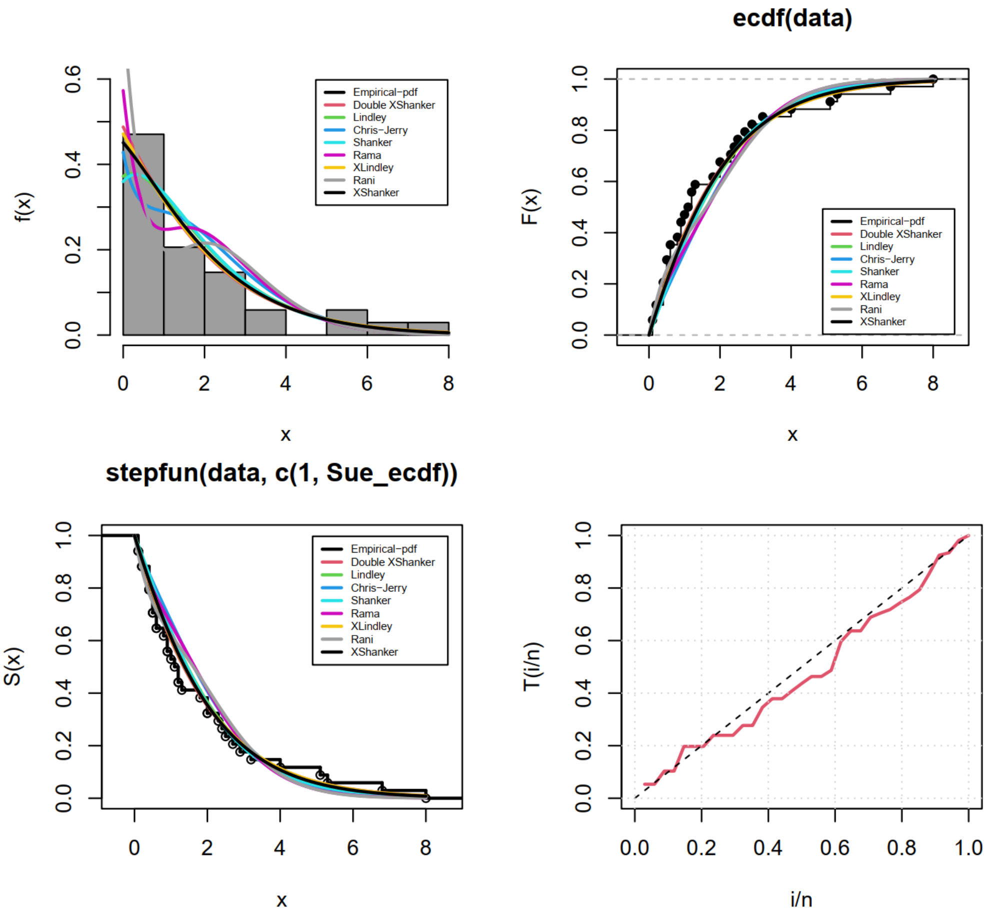

Figure 13: Density, cdf, survival function and TTT plots for Vinyl Chloride data

Fig 14, shows how well the competing distributions individually fit the vinyl chloride data and it is obvious that the proposed distribution best fits the data.

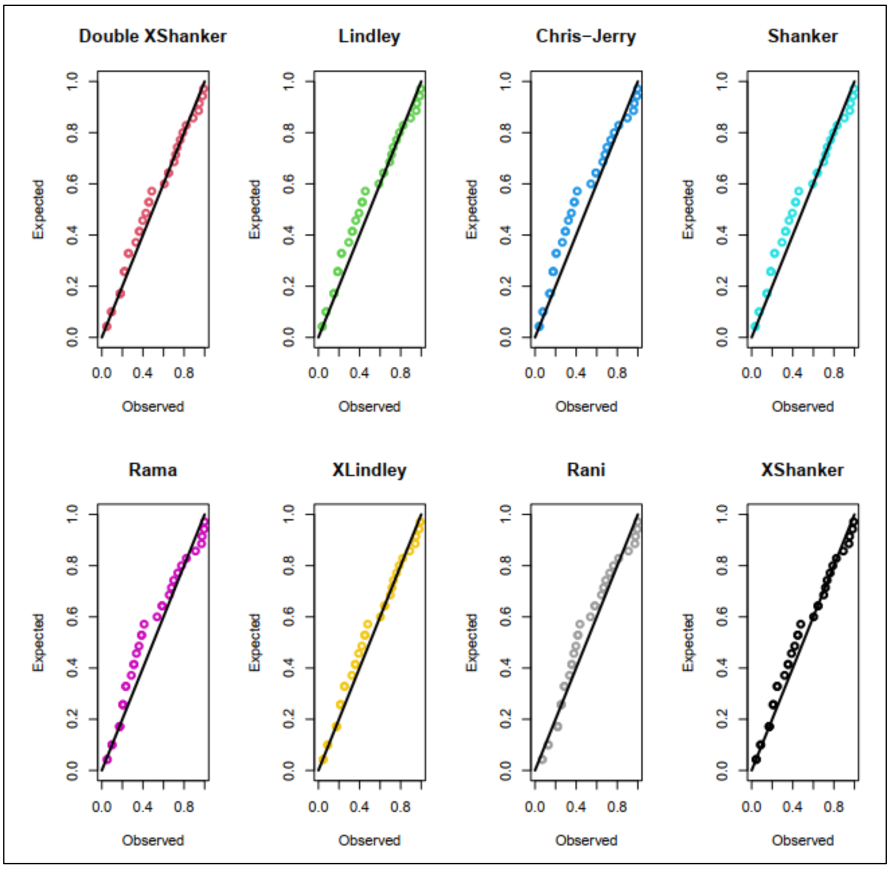

Figure 14: Pp plots for Vinyl Chloride data

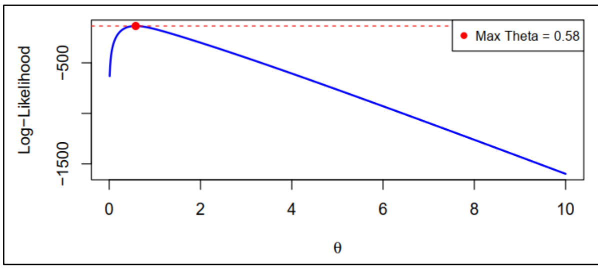

Figure 15: Log-likelihood profile for the rainfall data

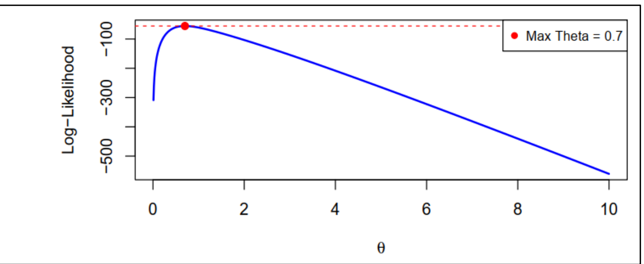

Figure 16: Log-likelihood profile for the vinyl chloride data

From the log-likelihood profiles for the rainfall and vinyl chloride data sets shown in fig 15 and fig 16, we see that the maximum values of the parameter are exactly what was obtained in tables 4 and 6 respectively

6 CONCLUSION REMARKS

This piece of research is an improvement on the XShanker distribution. It is also a one-parameter distribution just as the XShanker distribution but with a better goodness of fit and model performance than the parent distribution. In this article, the properties of the proposed distribution were derived with some visualization for better appreciation. Some theoretical values of the properties were obtained and a convergence behaviour was observed. The parameter of the model was estimated using maximum likelihood estimation. A simulation study was conducted with 1000 samples of sizes (n = 25,50,75,100,200,500), and the behavior observed is that as sample size increases the estimates decrease indicating precision. The proposed distribution is better appreciated with the rainfall and vinyl chloride data having p-values very close to 1.

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Conflict of Interest

The authors declare that there is no conflict of interest

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