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The Double XLindley Distribution: Properties and Applications

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Abstract **Review Article**

In this article, we have modified the XLindley distribution and called it the Double XLindley distribution. We studied the basic distributional properties and presented some theoretical results for the properties. The unknown parameter of the suggested distribution was estimated using the maximum likelihood estimation and two real data sets were deployed to illustrate its usefulness. Overall, the new distributions perform better than its competitors.

Keywords: Double XLindley distribution, Exponential distribution, Gamma distribution, XLindley distribution, Lindley Distribution.

1. INTRODUCTION

The quest to develop new probability distributions has increased in the past decades. This is attributed to researchers finding a good distribution that adequately fits some data, particularly on failure rate. This increasing interest can be seen in the literature since more and more distributions are developed daily. In 1958, [16] combined an exponential distribution with parameter θ and gamma distributions (2, θ). Ghitany *et al.*, (2008) [13] studied the properties of the Lindley distribution and showed that it is more flexible than the exponential distribution. The Lindley distribution has undergone many transformations since it was developed. There are several literatures on modification and extension of the Lindley and related distribution amongst which are [3] who work on the extension of the Lindley distribution with increasing interest, [1] in 2015 developed the Inverse Lindley distribution as well as [32] developed the Gamma Lindley distribution [2, 5, 8-12, 14, 17-20, 22-24]. Some extensions of Lindley distribution can be found in the following articles [4, 21, 33]. This article is motivated by the need to obtain more flexible model for lifetime data. Further modification of Lindley distribution by [7] was obtained by a mixture of exponential distribution with parameter θ given as $\theta e^{-\theta x}$ and Lindley distribution with parameter θ given as $\frac{\theta^2}{(1+\theta)}(1+\theta)e^{-\theta x}$ having a mixing proportion $p1$ = $p_1 = \frac{\theta}{(1+\theta)}$ and $p_2 = \frac{1}{(1+\theta)}$. This mixture generated a new distribution called the XLindley with a pdf and cdf given as $\frac{\theta^2(2+\theta+x)}{(1+\theta)^2}e^{-\theta x}$ and $1-(1+\frac{\theta}{(1+\theta)^2})e^{-\theta x}$ respectively. Having stated the aim of this article, The remaining part of this article will be divided into sections. In section 2, The Mathematical properties of the Double XLindley (DXL) distribution were studied. In section 3, we applied two lifetime data sets and demonstrated that the proposed distribution is useful among the Lindley class of distributions. The article is concluded in section 5.

Let $X \sim$ DXL (θ), then the pdf and cdf are respectively

$$f(x) = \frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}; \quad x>0, \quad \theta>0 \tag{1}$$

and

$$F(x) = 1 - \left\{1 + \frac{x\theta}{(\theta+1)^3}\right\} e^{-\theta x} \tag{2}$$

The Double XLindley was generated from the mixture of Exponential with a parameter θ and the XLindley with a parameter θ having a mixing proportion $p_1 = \frac{\theta}{(1+\theta)}$ and $p_2 = \frac{1}{(1+\theta)}$.

The survival and hazard rate functions are respectively

$$S(x) = \left\{1 + \frac{x\theta}{(\theta+1)^3}\right\} e^{-\theta x} \tag{3}$$

and

$$hrf(x) = \frac{\theta^2\left(1+(3+x)\theta+3\theta^2+\theta^3\right)(3+x+\theta(3+\theta))}{(1+\theta)^6} \tag{4}$$

The limiting values of the DXL hazard function are

$$\lim_{x\to 0} hrf(x) = \frac{\theta^2\left(1+3\theta+3\theta^2+\theta^3\right)(3+\theta(3+\theta))}{(1+\theta)^6} \quad \text{and} \quad \lim_{x\to\infty} hrf(x) = \infty$$

The plots are displayed in the figures 1, 2, 3, and 4 below

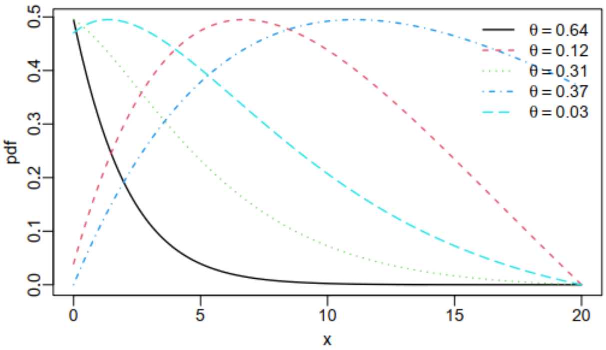


Figure 1: Pdf of DXL distribution

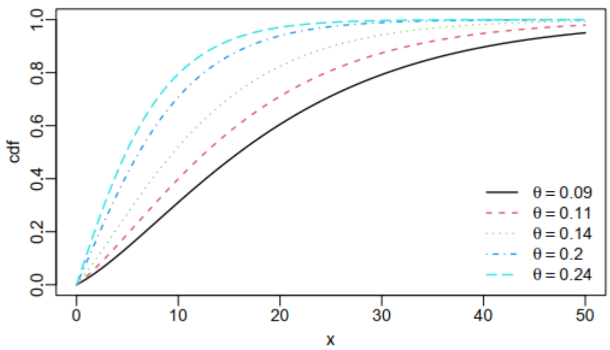


Figure 2: Cdf of DXL distribution

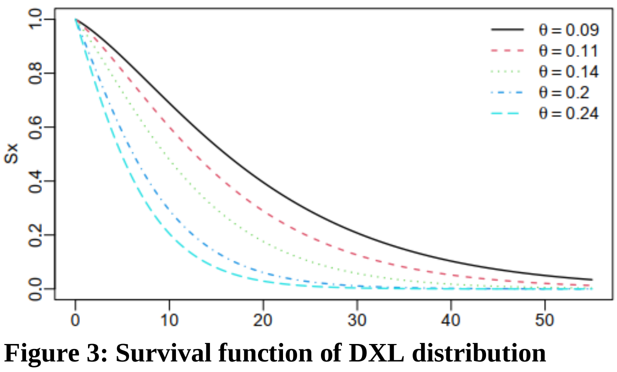


Figure 3: Survival function of DXL distribution

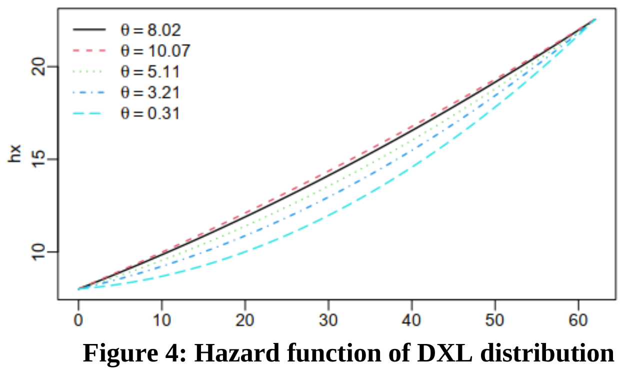


Figure 4: Hazard function of DXL distribution

2. Distributional Properties of DXL distribution

In this section, we derive the basic distributional properties of the proposed model.

Definition 2.1 (Moment). Let $X \sim \text{DXL}(\theta)$, r^{th} crude moment is given as

$$\mu_r' = \frac{\theta^{-r}\left(r+(1+\theta)^3\right)\Gamma[1+r]}{(1+\theta)^3} \tag{5}$$

The mean, second, third, and fourth crude moments are respectively

$$\mu = \frac{(2+\theta)\left(1+\theta+\theta^2\right)}{\theta(1+\theta)^3}; \quad \mu_2^{'} = \frac{6+2\theta(3+\theta(3+\theta))}{\theta^2(1+\theta)^3}; \quad \mu_3^{'} = \frac{6(4+\theta(3+\theta(3+\theta)))}{\theta^3(1+\theta)^3}; \quad \text{and} \quad \mu_4^{'} = \frac{24(5+\theta(3+\theta(3+\theta)))}{\theta^4(1+\theta)^3} \tag{6}$$

Definition 2.2 (The variance of DXL). To understand the dispersion in the observations, the variance is often used though the standard deviation is often preferred due to the measurement unit. Generally, the variance of a random variable X is defined thus $\sigma^2 = EX^2 - (EX)^2$

$$\sigma^2 = \frac{2+12\theta+21\theta^2+22\theta^3+15\theta^4+6\theta^5+\theta^6}{\theta^2(1+\theta)^6} \tag{7}$$

The variance of DXL is well-defined for $\theta > 1$

Definition 2.3. The measures of the Skewness (SK), Kurtosis (Ku) and coefficient of variation (CV) are respectively

$$SK = \frac{6(1+\theta)^6(4+\theta(3+\theta(3+\theta)))}{\left(2+12\theta+21\theta^2+22\theta^3+15\theta^4+6\theta^5+\theta^6\right)^{\frac{3}{2}}}; \quad Ku = \frac{24(1+\theta)^9(5+\theta(3+\theta(3+\theta)))}{\left(2+12\theta+21\theta^2+22\theta^3+15\theta^4+6\theta^5+\theta^6\right)^2};$$
$$\text{and} \quad CV = \frac{\sqrt{2+12\theta+21\theta^2+22\theta^3+15\theta^4+6\theta^5+\theta^6}}{(2+\theta)^3\left(1+\theta+\theta^2\right)} \tag{8}$$

Definition 2.4. Let $X \sim$ DXL (θ), then the moment generating function can be expressed as;

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx}\left\{\frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}\right\}dx = \frac{\theta^2\left((1+\theta)^3 - x(3+\theta(3+\theta))\right)}{(x-\theta)^2(1+\theta)^3} \tag{9}$$

Definition 2.5. Let $X \sim$ DXL (θ), then the characteristic function can be expressed as

$$E(e^{itx}) = \int_0^\infty e^{itx}\left\{\frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}\right\}dx = \frac{\theta^2\left((1+\theta)^3 - ix(3+\theta(3+\theta))\right)}{(ix-\theta)^2(1+\theta)^3} \tag{10}$$

Definition 2.6. The Odds function is a measure of reliability that provides relative information about a characteristic of an event. Let $X \sim$ DXL (θ), the Odd function can be expressed as

$$O_{DXL} = \frac{1+(3+x)\theta+3\theta^2+\theta^3 - e^{x\theta}(1+\theta)^3}{1+(3+x)\theta+3\theta^2+\theta^3} \tag{11}$$

Which is a function of both θ and x.

Definition 2.7 (Stress-Strength Reliability Analysis). Stress strength is a tool used in reliability engineering for measuring the interference of the component's strength and the stresses placed on the component. Let X be the stress pressured on a system and let Y be the strength, then the stress-strength of the component can be expressed as

$$R = P(Y < X) = \frac{\theta_2(1+\theta_2)^3\left(\theta_1(2+\theta_1)\left(1+\theta_1+\theta_1^2\right)+(1+\theta_1)^3\theta_2\right) - \theta_2\theta_1^2\left((1+\theta_1)^3+(3+\theta_1(3+\theta_1))\theta_2\right)}{(1+\theta_1)^3(1+\theta_2)^3(\theta_1+\theta_2)^2} \tag{12}$$

2.1 Stochastic ordering of DXL distribution

A random variable X is said to be smaller than another random variable Y in the stochastic order $(X \leq_{st} Y)$if $F_Y(x) \geq F_Y(x)\ \forall_x$ Hazard order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_y(x)\ \forall_x$. Mean residual life order $(X \leq_{mrl} Y) if\, m_X(x) \geq m_Y(x)\ \forall_x$ Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(y)}$ decreases in x.

This implies that $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y$

Theorem 1. Let $X \sim$ DXL (θ_1) and $Y \sim$ DXL (θ_2). if $\theta_1 \geq \theta_2$ then $X \leq_{lr} Y$ hence $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$

$$\begin{aligned}\frac{f_x(x)}{f_y(x)} &= \frac{\frac{\theta_1^2}{(\theta_1+1)^3}\left[3+3\theta_1+\theta_1^2+x\right]e^{-\theta_1 x}}{\frac{\theta_2^2}{(\theta_2+1)^3}\left[3+3\theta_2+\theta_2^2+x\right]e^{-\theta_2 x}} \\ &= \frac{\theta_1^2(3+x+\theta_1(3+\theta_1))(1+\theta_2)^3}{\theta_2^2(3+x+\theta_2(3+\theta_2))(1+\theta_1)^3}e^{(\theta_2-\theta_1)x}\end{aligned} \tag{13}$$

Table 1: Some theoretical statistics of the proposed DXL distribution

θ	n	Mean	Variance	Skewness	Kurtosis	Coef_of_Variation
0.1	20	0.024289	2.43E-05	-0.01951	2.19054	0.202776
0.25	20	0.116072	0.000103	-2.7513	9.819726	0.087419
0.75	20	0.713218	0.142651	0.663403	2.509077	0.52956
1.5	20	2.957938	15.98445	1.640762	4.430654	1.351636
2	20	17.72641	2037.038	2.776434	9.246187	2.54612
0.1	50	0.025932	1.96E-05	-0.64482	3.411136	0.170847
0.25	50	0.116278	0.000102	-3.71119	19.8305	0.087036
0.75	50	0.722453	0.173432	1.073874	3.844995	0.576442
1.5	50	2.00631	7.236173	3.251948	15.32836	1.340777
2	50	12.27604	418.9275	2.605879	9.324192	1.66729
0.1	100	0.023418	3.23E-05	-0.58484	2.881073	0.242516
0.25	100	0.116498	0.000119	-4.17456	23.97722	0.093605
0.75	100	0.618744	0.094013	0.812617	3.342225	0.495544
1.5	100	3.031273	22.9144	3.296725	16.5969	1.579171
2	100	11.51081	1184.675	5.015983	29.71649	2.990155
0.1	200	0.023977	3.01E-05	-0.5615	3.163804	0.228855
0.25	200	0.117837	3.76E-05	-2.847	13.98406	0.052018
0.75	200	0.703413	0.174947	1.667402	8.630457	0.594624
1.5	200	3.374808	40.42739	4.901618	33.73182	1.884034
2	200	7.305934	806.6444	11.6071	149.3261	3.887454
0.1	500	0.024018	3.22E-05	-1.06009	4.947523	0.236307
0.25	500	0.117738	6.33E-05	-4.23826	25.85988	0.067588
0.75	500	0.70216	0.200075	1.556225	5.836694	0.63703
1.5	500	3.575282	49.71561	5.602991	47.92676	1.972132
2	500	10.53034	1307.157	12.73594	215.8108	3.433378

Taking a natural log of the ratio will yield

$$ln\frac{f_x(x)}{f_y(x)} = ln\frac{\theta_1^2(1+\theta_2)^3}{\theta_2^2(1+\theta_1)^3} + ln\frac{(3+x+\theta_1(3+\theta_1))}{(3+x+\theta_2(3+\theta_2))} + (\theta_2-\theta_1)x \quad (14)$$

Differentiating the natural log of the ratio w.r.t x will result

$$= \theta_2 - \theta_1 - \frac{3\theta_1 - \theta_1^2 + \theta_2(3+\theta_2)}{\left(3+x+3\theta_1+\theta_1^2\right)\left(3+x+3\theta_2+\theta_2^2\right)} \quad (15)$$

If $\theta_2 \geq \theta_1$, $\frac{d}{dx}ln\frac{f_x(x)}{f_y(x)} \leq 0$, and $\frac{f_x(x,\theta_1)}{f_y(x,\theta_2)}$ is decreasing in x.

2.2 Mean Residual Function for DXL

Given that a component has survived until time t, the expected additional lifetime is called the mean residual lifetime.

$$MRL = \frac{1}{1-F(x)}\int_x^\infty 1-F(t)dt = \frac{(\theta+1)^3}{((\theta+1)^3+x\theta)e^{-\theta x}}\int_x^\infty \frac{((\theta+1)^3+t\theta)e^{-\theta t}}{(\theta+1)^3}dt = \frac{1}{((\theta+1)^3+x\theta)}\left\{\frac{(\theta+1)^3e^{-\theta x}}{\theta} + \frac{\gamma(2,x)}{\theta}\right\} \quad (16)$$

2.3 Distribution of the Order Statistics

Suppose X_1, X_2,....X_n is a random sample of X_r; r=(1,2....,n) are the rth order statistics obtained by arranging X_r in ascending order of magnitude $\ni X_1 \leq X_2 \leq ....X_r$ where X_1 is the smallest of all variable and X_r is the largest of all variable, then the pdf of the rth order statistics is given by

$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}f(x)(F(x))^{r-1}(1-F(x))^{n-r}$ the pdf of rth order statistics of DXL is given as:

$$f_{r:n}(X) = \frac{n!}{(r-1)!(n-r)!}\frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}\left(1-\left\{1+\frac{x\theta}{(\theta+1)^3}\right\}e^{-\theta x}\right)^{r-1}\left(\left\{1+\frac{x\theta}{(\theta+1)^3}\right\}e^{-\theta x}\right)^{n-r} \tag{17}$$

The pdf of minimum order statistics is obtained by setting $r = 1$

$$f_{1:n}(x) = \frac{n\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}\left(\left\{1+\frac{x\theta}{(\theta+1)^3}\right\}e^{-\theta x}\right)^{n-1} \tag{18}$$

pdf of maximum order is obtained by setting $r = n$

$$f_{n:n}(x) = \frac{n\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}\left(1-\left\{1+\frac{x\theta}{(\theta+1)^3}\right\}e^{-\theta x}\right)^{n-1} \tag{19}$$

Definition 2.8. *Bonferroni and Lorenzi Curve*

$$\begin{aligned} B_p &= \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{\theta}{p(2+\theta)(1+\theta+\theta^2)}\left[\frac{3\gamma_{2,q}}{\theta^2}+\frac{3\theta\gamma_{2,q}}{\theta^2}+\frac{\theta^2\gamma_{2,q}}{\theta^2}+\frac{\gamma_{3,q}}{\theta^3}\right] \\ L_p &= \frac{1}{\mu}\int_0^q xf(x)dx = \frac{\theta}{(2+\theta)(1+\theta+\theta^2)}\left[\frac{3\gamma_{2,q}}{\theta^2}+\frac{3\theta\gamma_{2,q}}{\theta^2}+\frac{\theta^2\gamma_{2,q}}{\theta^2}+\frac{\gamma_{3,q}}{\theta^3}\right] \end{aligned} \tag{20}$$

Definition 2.9. Rényi entropy was originally introduced in the field of information theory as a parametric relaxation of Shannon (in physics, Boltzmann–Gibbs) entropy. [**fuentes2022renyi**]. The Renyi Rntropy of DXL can be expressed as

$$R_\omega(x) = \frac{1}{1-w}log\int_0^\infty f(x)^\omega dx = \frac{1}{1-\omega}log\int_0^\infty\left(\frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}\right)^\omega dx = \frac{1}{1-\omega}log\frac{\theta^{2\omega}}{(\theta^+1)^{3\omega}}\int_0^\infty\left[3+3\theta+\theta^2+x\right]^\omega e^{-\theta\omega x}dx \tag{21}$$

let;

$\theta^5+3\theta^3+3\theta = a$, $x = b$, $\omega = n$. Then $(a+b)^n = \sum_{j=0}^{\omega} a^j b^{\omega-j}$

$$\begin{aligned} R_\omega(x) &= \frac{1}{1-\omega}log\frac{\theta^{2\omega}}{(\theta+1)^{3\omega}}\int_0^\infty\sum_{j=0}^{\omega}\binom{\omega}{j}(3+3\theta+\theta^2)^j x^{\omega-j}e^{-\theta\omega x}dx = \frac{1}{1-\omega}log\frac{\theta^{2\omega}}{(\theta+1)^{3\omega}}\sum_{j=0}^{\omega}\binom{\omega}{j}(3+3\theta+\theta^2)^j\int_0^\infty x^{\omega-j}e^{-\theta\omega x}dx \\ &= \frac{1}{1-\omega}log\frac{\theta^{2\omega}}{(\theta+1)^{3\omega}}\sum_{j=0}^{\omega}\binom{\omega}{j}(3+3\theta+\theta^2)^j\frac{\Gamma_{\omega-j+1}}{(\theta\omega)^{\omega-j+1}} \therefore R_\omega(x) = \frac{1}{1-\omega}log\frac{\theta^{\omega+j-1}}{(\theta+1)^{3\omega}}\sum_{j=0}^{\omega}\binom{\omega}{j}(3+3\theta+\theta^2)^j\frac{(\omega-j)!}{(\omega)^{\omega-j+1}} \end{aligned} \tag{22}$$

3 Maximum Likelihood Function

Let $(X_1, X_2, .., X_n)$ be random samples of size n which follow the DXL distribution, then the likelihood function is given as

$$\begin{aligned} \ell(f_{(x;\theta)}) &= \prod_{i=1}^{n}\frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x} \\ &= \frac{\theta^{2n}}{(\theta+1)^{3n}}\prod_{i=1}^{n}\left[3+3\theta+\theta^2+x\right]e^{-\theta\sum_{i=1}^n x_i} \end{aligned} \tag{23}$$

Taking the log of the function

$$\psi = 2nln\theta - 3nln(\theta+1) - \theta\sum_{i=1}^{n}x_i + \log\sum_{i=1}^{n}\left[3+3\theta+\theta^2+x\right]$$

Differentiate ψ w.r.t θ i.e $\frac{d\psi}{d\theta}$ and equate to zero yields the result below

$$\frac{2n}{\theta} - \frac{3n}{(\theta+1)} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{2\theta+3}{3+3\theta+\theta^2+x} = 0 \tag{24}$$

Eq 24 is not a compact function, therefore its convergence will be guaranteed by numerical iteration implemented in R using **optim()** function.

4. Applications

In this section, some real data analysis will be carried out to check the performance of the proposed distribution among the following competing distributions namely, XRama, Lindley, Ishita, Akash, Prenav, Chris-Jerry, Shanker, Rama, XLindley, Rani, and XShanker.

Table 2: List of Competing one-parameter distributions in the class of Lindley

Distribution	$f(x)$	$F(x)$
Proposed DXL	$\frac{\theta^2}{(\theta+1)^3}\left[3+3\theta+\theta^2+x\right]e^{-\theta x}$	$1-\left\{1+\frac{x\theta}{(\theta+1)^3}\right\}e^{-\theta x}$
XRani [11]	$\frac{\theta^5}{(\theta^5+24)^2}\left[\theta^6+48\theta+24x^4\right]e^{-\theta x}$	$1-\left\{1+\frac{24\theta^4x^4+96\theta^3x^3+288\theta^2x^2+576\theta x}{(\theta^5+24)^2}\right\}e^{-\theta x}$
Rani [28]	$\frac{\theta^5}{\theta^5+24}\left(\theta+x^4\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta^3x^3+4\theta^2x^2+12\theta x+24)}{\theta^5+24}\right]e^{-\theta x}$
Shanker [26]	$\frac{\theta^5}{\theta^2+1}(\theta+x)e^{-\theta x}$	$1-\left[1+\frac{\theta x}{\theta^2+1}\right]e^{-\theta x}$
XShanker [9]	$\frac{\theta^2}{(\theta^2+1)^2}\left(\theta^3+2\theta x+x\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x}{(\theta^2+1)^2}\right]e^{-\theta x}$
XGamma [25]	$\frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x+\frac{\theta^2x^2}{2}}{1+\theta}\right]e^{-\theta x}$
Rama [27]	$\frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}$	$1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}$
Lindley [16]	$\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	$1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}$
XLindley [6]	$\frac{\theta^2(2+\theta+x)}{(1+\theta)^2}e^{-\theta x}$	$1-\left[1+\frac{\theta x}{(1+\theta)^2}\right]e^{-\theta x}$
Ishita [29]	$\frac{\theta^3}{\theta^3+2}\left(\theta+x^2\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}$
Akash [30]	$\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}$
Pranav [15]	$\frac{\theta^4}{\theta^4+6}\left(\theta+x^3\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta^2x^2+3\theta x+16)}{\theta^4+6}\right]e^{-\theta x}$
Chris-Jerry [22]	$\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)e^{-\theta x}$	$1-\left[1+\frac{\theta x(\theta x+2)}{\theta+2}\right]e^{-\theta x}$

The first application is on Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L) studied by [31] in table 1.

Table 3: Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L)

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3
3.2	2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2

The following model adequacy metrics are deployed; Akaike information criterion (BIC), Corrected Akaike Information criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan–Quinn information criterion (HQIC), the model performance is proved if the proposed distribution has minimum value for each of the criteria. The K-S, Cramer von Misses W^*, Anderson Darling statistics A^*, and p-value for goodness of fit.

Table 4: Analytical Measures for the Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L)

Distr	NLL	AIC	CAIC	BIC	W*	A*	K-S	p-value	θ	std.err
DXL	55.54	113.0757	113.2007	114.6021	0.4956	0.3215	0.0977	0.902	0.6514	0.0888
Xgama	78.55	159.0958	159.2208	160.6222	0.0729	0.4725	0.2478	0.3809	0.8082	0.1011
Lindley	56.3	114.6073	114.7323	116.1336	0.0627	0.4046	0.1327	0.5878	0.8234	0.1054
Ishitta	57.3	116.6058	116.7308	118.1322	0.0947	0.604	0.1404	0.5142	1.1571	0.0962
Akash	57.57	117.1473	117.2743	118.6756	0.099	0.6296	0.1565	0.3758	1.1656	0.1126
Prenav	58.34	118.6715	118.7965	120.1979	0.1361	0.8471	0.1463	0.4606	1.4844	0.0977
Chris-Jerry	57.93	117.8536	117.9786	119.38	0.1032	0.6549	0.1782	0.2303	1.1644	0.1321
Shanker	56.46	114.9129	115.0379	116.4393	0.0634	0.4129	0.1307	0.6065	0.8534	0.9478
Rama	59.34	120.6832	120.8082	122.2096	0.1543	0.9518	0.1768	0.2383	1.531	0.1176
Xlindley	55.7	113.4008	113.5258	114.9271	0.0598	0.3498	0.1081	0.8218	0.7144	0.0943
Rani	59.88	121.7515	121.8765	123.2771	0.1936	1.1646	0.1516	0.4151	1.7844	0.0997
XShanker	59.88	113.6927	113.8177	115.2191	0.0569	0.3685	0.1108	0.7979	0.7567	0.0828

A good look at the results in Table 4 showed that the proposed distribution is the best fit for the data with a p-value of 0.902, followed by XLindley with a fitting probability of 0.8218. The K-S statistic is also the minimum among the thirteen competing distributions compared. Viewing the performance metrics namely NLL, AIC, CAIC, and BIC, the values are minimum for the proposed distribution compared to those of other competing distributions except XRama.

Figures 5 and 6 display the goodness of fit of the selected distributions to the data on the data. The figures show that the DXL distribution fits the data.

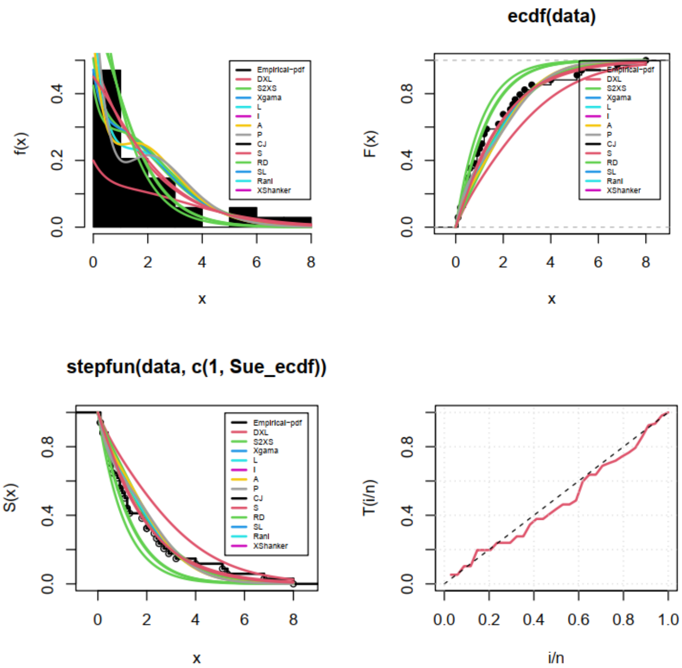


Figure 5: Density, cdf, survival, and TTT plots for the Vinyl Chloride Data

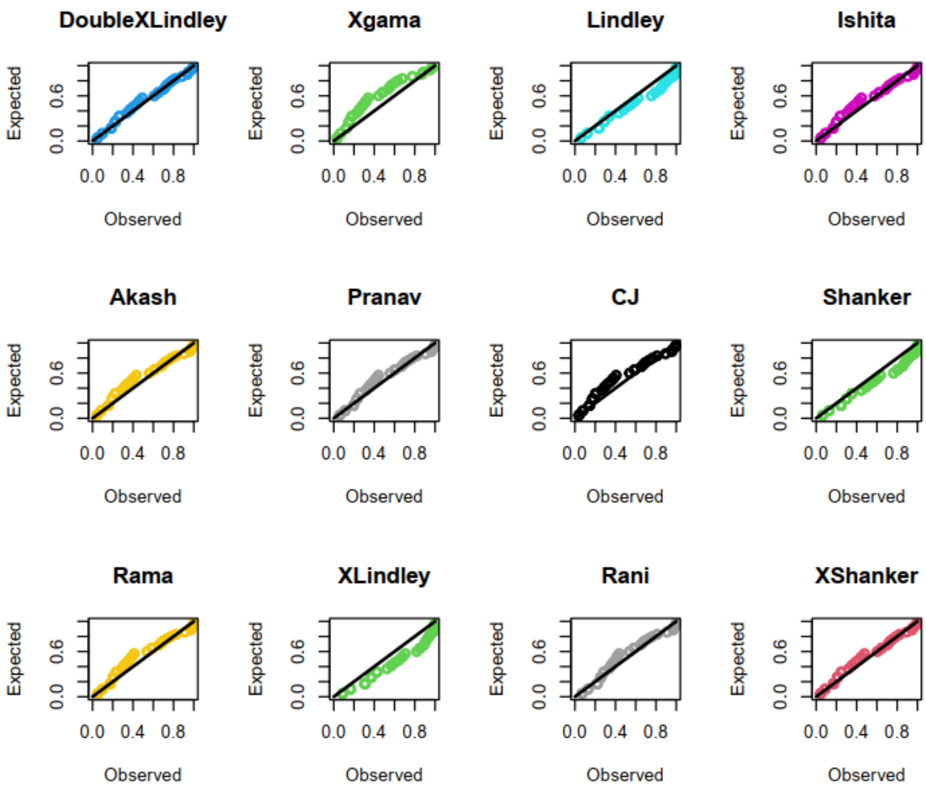


Figure 6: PP plots for the Vinyl Chloride data

The next application is on rainfall reported at the Los Angeles Civic Center in the Months of February from 1943 to 2018 and studied by [10] The data is in table?

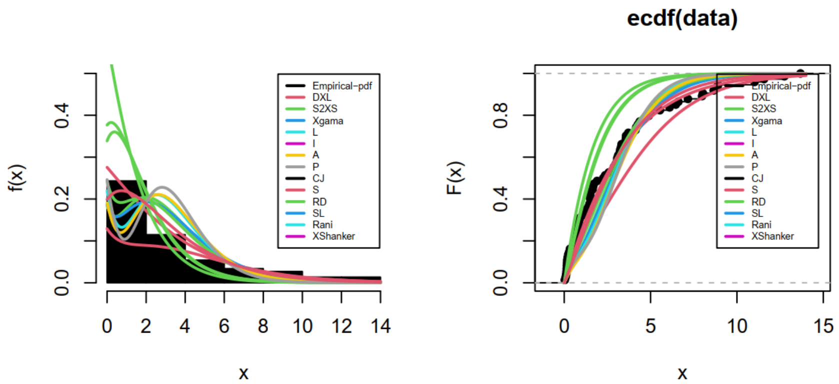
Table 5: rainfall data in the months of February

3.07	8.65	3.24	1.52	0.86	1.29	1.41	1.6	1.48	0.63
0.33	2.98	0.68	0.59	1.47	6.46	3.32	2.26	0.15	11.57
2.88	0.23	1.51	0.11	0.49	8.03	2.58	0.67	0.13	7.98
0.14	3.54	3.71	0.17	8.91	3.06	12.75	1.48	0.70	4.37
2.84	6.10	1.22	1.72	1.90	3.12	4.13	7.16	6.61	3.21
1.3	4.94	0.08	13.68	0.56	5.54	8.87	0.29	4.64	4.89
11.02	2.37	0.92	1.64	3.57	4.27	3.29	0.16	0.20	3.58
0.83	0.79	4.17	0.03						

Table 6: Analytical Measures of Fitness and Performance for the Rainfall Data in the Months of February

Distr	NLL	AIC	CAIC	BIC	W*	A*	K-S	p-value	θ	std.err
DXL	160	323.2067	323.2622	325.5107	0.0377	0.268	0.0804	0.7248	0.4227	0.0372
Xgama	198.06	398.128	398.1833	400.4318	0.05359	0.3829	0.2032	0.0044	0.5609	0.0447
Lindley	163.94	329.8854	329.9409	332.1894	0.0421	0.3085	0.1264	0.1876	0.5185	0.0439
Ishitta	170.78	343.5547	343.6103	345.8588	0.0864	0.6064	0.1691	0.2091	0.8303	0.0476
Akash	169.18	340.3678	340.4233	342.6718	0.0836	0.5867	0.1773	0.0191	0.7894	0.0509
Prenav	176.6	355.2028	355.2583	357.5068	0.1673	1.0913	0.2035	0.0043	1.1046	0.0524
Chris-Jerry	167.1	336.2053	336.2608	338.5093	0.0779	0.5462	0.1805	0.0161	0.737	0.0548
Shanker	166.41	334.8205	334.476	337.1245	0.0445	0.3274	0.1386	0.1167	0.5671	0.0425
Rama	177.54	357.0729	357.1284	359.377	0.1695	1.1081	0.2257	0.0011	1.0855	0.0577
Xlindley	161.41	324.828	324.884	327.1328	0.0383	0.2762	0.0954	0.511	0.4606	0.0398
Rani	184.52	371.036	371.0915	373.34	0.2696	1.6789	0.2341	0.0006	1.3891	0.05696
XShanker	184.52	328.7697	328.8253	331.0738	0.0414	0.3032	0.1134	0.2972	0.5213	0.0379

The results in Table 6 showed that the proposed distribution is the best fit for the data with a p-value of 0.7248, followed by XShanker with a fitting probability of 0.8218. The K-S statistic is also the minimum among the thirteen competing distributions compared. Viewing the performance metrics namely NLL, AIC, CAIC, BIC, and HQIC the values are minimum for the proposed distribution compared to those of other competing distributions.



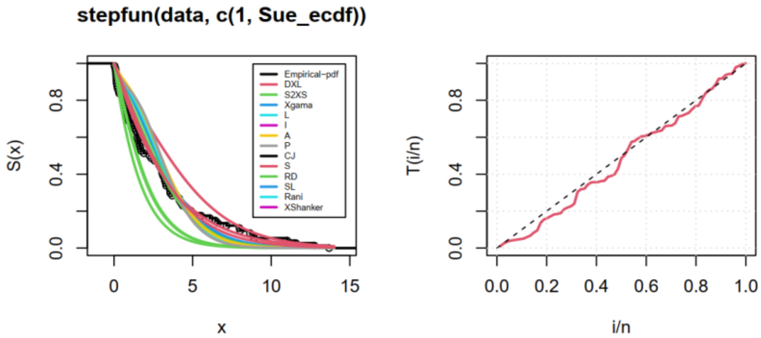


Figure 7: Density, CDF, survival and TTT plots for the data

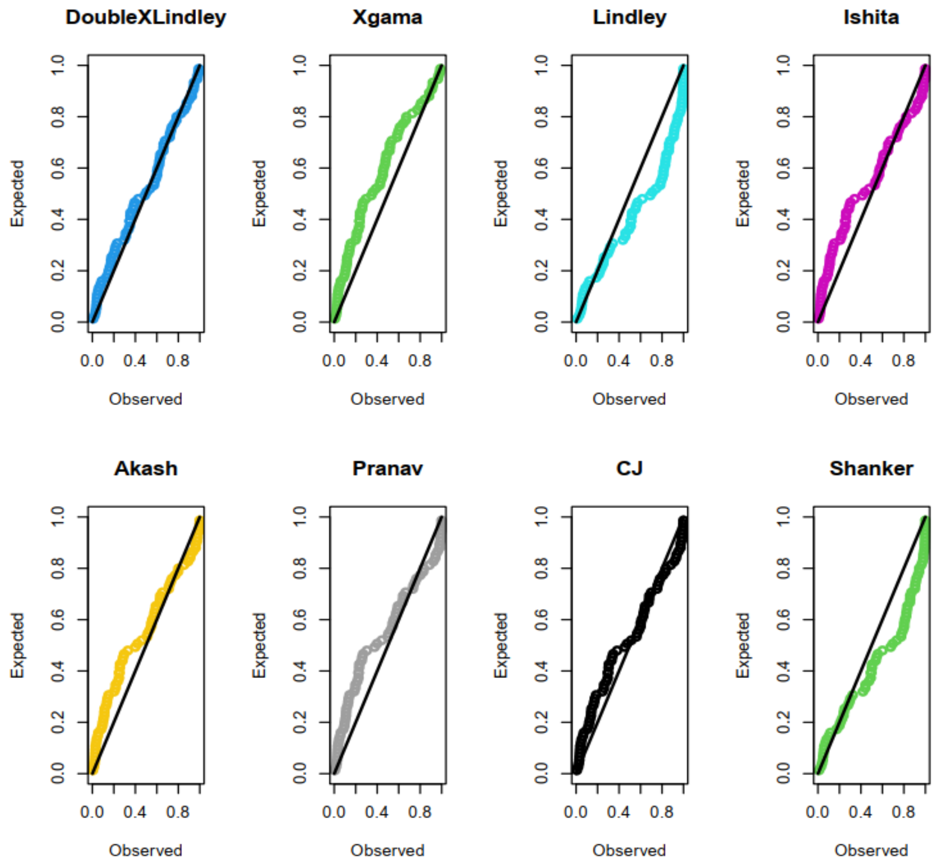


Figure 8: PP plots for the Data

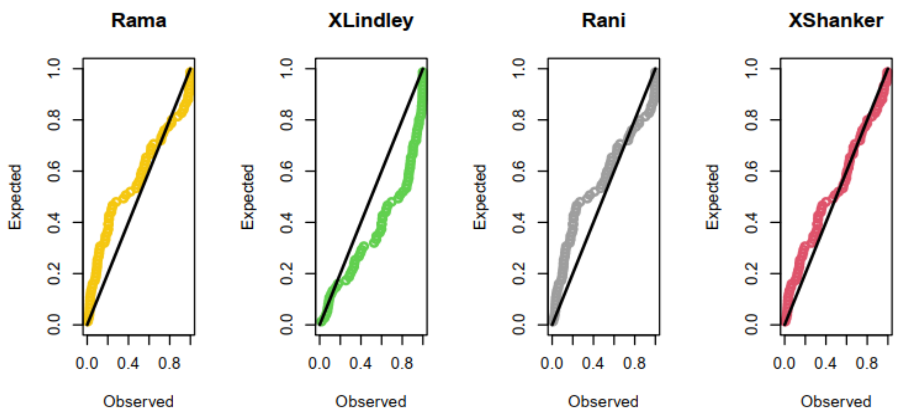


Figure 9: PP plots for the Data (continue)

Figures 7, 8, and 9 display the goodness of fit of the selected distributions to the data on the data. The figures show that the DXL distribution fits the data better than the competing distributions.

5. CONCLUSION

DXL was proposed for modeling increasing failure rate data in this paper. In the study, the properties included moment and its associated measures such as the mean which is the first crude moment, the second, third, and fourth crude moments, the variance, skewness, kurtosis, and coefficient of variation measures with some theoretical statistics presented for better understanding. The reader's attention was beamed at the moment generating function, characteristic function, odd function, stress-strength reliability, stochastic ordering, mean residual life function, and distribution of order statistics. The parameter of the distribution was estimated using the popular maximum likelihood estimation. The proposed distribution was justified by deploying two real data sets on vinyl chloride and rainfall.

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