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Galois Conjugacy and Fitting Height 2: Classification of Finite Solvable Groups

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Abstract | **Original Research Article**

This paper investigates the finite groups G for which any two characters in the set of irreducible complex characters $\boldsymbol{Irr}(G)$ are Galois conjugate. Specifically, we classify such groups and establish a key result: they are solvable with Fitting height 2. The analysis involves intricate considerations of irreducible complex characters and their Galois conjugacy, shedding light on the structural properties of finite solvable groups.

Keywords: Galois Conjugacy, Irreducible Complex Characters, Solvable Groups, Fitting Height, Group Theory, Character Theory.

1. INTRODUCTION

This research explores the fascinating interplay between Galois conjugacy and the structure of finite solvable groups. The foundational work by [1] proves the solvability of groups of odd order, paving the way for subsequent investigations into the structural properties of solvable groups. Isaacs' [2] comprehensive text on character theory provides a solid background on the properties of irreducible complex characters in finite groups, offering insights into their behavior under various group operations. The book by Navarro and Tiep [3] delves into the character theory of finite groups, addressing advanced topics and applications. It provides a modern perspective on the subject, including the Galois conjugacy of characters. Gorenstein's [4] work on finite groups covers various aspects of their structure and classification. The book is a valuable resource for understanding the foundational concepts related to the solvability of finite groups. Huppert's [5] treatise on finite groups is a classic reference in the field, covering fundamental results and classifications. It provides insights into the structure of finite solvable groups. [6]'s paper investigates the Galois conjugacy of characters with their inverses, contributing to the understanding of character theory in finite groups. Malle and Testerman's book [7] explores the connection between linear algebraic groups and finite groups of Lie type, providing a broader context for the study of solvable groups. Srinivasan's [8] early work on characters of finite symplectic groups contributes to the understanding of specific classes of finite groups and their character properties. We focus on the set of irreducible complex characters $\boldsymbol{Irr}(G)$ and investigate conditions under which any two characters in this set are Galois conjugate. A crucial classification emerges, revealing that the groups satisfying this property are solvable with a Fitting height of 2. This investigation deepens our understanding of the interrelations between character theory and the algebraic structure of finite groups.

2. PRELIMINARY

Definition (*Galois Conjugacy) 2.1.* Given a field extension E/F, where E is the splitting field of a polynomial f(x) over F, and Gal(E/F) is the Galois group of the extension, two elements a,b ∈ E are said to be Galois conjugates if there exists an automorphism σ ∈ Gal(E/F) such that σ(a) = b. Mathematically, if σ·a = b, we denote this as $a \sim_\sigma b$, indicating that a and b are Galois conjugates under the action of σ.

Example Illustration (*Galois Conjugacy) 2.2.* Consider the quadratic extension $Q(\sqrt{2})/Q$, where E is the splitting field of the polynomial $f(x) = x^2 - 2$. The Galois group Gal(E/Q) consists of two elements: the identity automorphism $\sigma_1: \sqrt{2} \mapsto \sqrt{2}$ and the nontrivial automorphism $\sigma_2: \sqrt{2} \mapsto -\sqrt{2}$.

Now, let $a = \sqrt{2}$ and $= -\sqrt{2}$. These elements are Galois conjugates because $\sigma_2 . a = -\sqrt{2} = b$. Therefore, $\sqrt{2}$ and $-\sqrt{2}$ are Galois conjugates in the field extension $Q(\sqrt{2})/Q$.

This concept is fundamental in Galois theory, providing insights into the structure of field extensions and the behavior of roots of polynomials under Galois automorphisms.

Definition (Irreducible Complex Characters*) 2.3.* Consider a finite group G and its complex representation $\rho: G \to GL_n(\mathrm{C})$, where $GL_n(\mathrm{C})$ is the general linear group of complex matrices of order n. A complex character χ associated with the representation ρ is a function $\chi: G \to \mathrm{C}$ defined by:

$$\chi(g) = \mathrm{Tr}(\rho(g)),$$

Where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. Now, a complex character χ is said to be irreducible if it cannot be expressed as a nontrivial linear combination of other characters, i.e., there do not exist characters $\chi_1, \chi_2, \ldots, \chi_k$ and complex numbers $c_1, c_2, \ldots, c_k$ such that:

$\chi = c_1\chi_1 + c_2\chi_2 + \ldots + c_k\chi_k$,

where $c_i \in \mathrm{C}$ and χ_i are distinct characters.

Example (Irreducible Complex Characters*) 2.4.* Let's consider the symmetric group S_3 and its irreducible complex characters. The character table of S_3 has three irreducible characters:

1. The trivial character $\chi_1(g) = 1$ for all $g \in S_3$.
2. The sign character $\chi_2(g) = \mathrm{sgn}(g)$, where $\mathrm{sgn}(g)$ is the sign of the permutation g.
3. The two-dimensional irreducible character χ_3 associated with the representation on C^2.

The characters χ_1, χ_2, and χ_3 are irreducible, and any other character of S_3 can be expressed as a linear combination of these irreducible characters.

Irreducible complex characters provide a decomposition of the group representation into simpler components, revealing essential structural information about the group.

Definition (Solvable Groups*) 2.5.* A group G is called solvable if there exists a finite chain of subgroups:

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_k = \{e\},$$

where e is the identity element of G and each G_i is a normal subgroup of G (written as $G_i \lhd G$) such that the quotient group $G_{i+1} = G_i/G_{i+1}$ is abelian. This series of subgroups is known as the derived series of G, and the group G is solvable because the derived series eventually reaches the trivial group.

Mathematically, if G is solvable, there exists a positive integer k such that $G_k = \{e\}$ and $G_{i+1} = [G_i, G_i]$ (the commutator subgroup of G_i) for all $0 \le i < k$.

Example Illustration (Solvable Groups*) 2.6.* Let's consider the symmetric group S_3, which consists of all permutations of three elements. We'll show that S_3 is a solvable group.

Derived Series: $S_3 \supseteq [S_3, S_3] \supseteq [S_3, [S_3, S_3]] \supseteq \{e\}$

The commutator subgroup $[S_3, S_3]$ is the set of all commutators of elements in S_3. For S_3, this subgroup is A_3, the alternating group of order 3. The second commutator subgroup $[S_3, [S_3, S_3]]$ is the commutator subgroup of A_3, which is the trivial group $\{e\}$. Since we reach the trivial group in the derived series, S_3 is solvable.

In summary, a group is solvable if there exists a series of subgroups such that each quotient group in the series is abelian, and the derived series eventually reaches the trivial group. The example of S_3 illustrates this concept in the context of a specific group.

Definition (Fitting Height*) 2.7.* Consider a finite group G. The Fitting series of G is defined as follows:

$F_0(G) = G$,

$F_{i+1}(G) = F_i(G)'F_i(G)$,

where $F_i(G)'$ is the derived subgroup (commutator subgroup) of $F_i(G)$.

The fitting height of G, denoted as $h(G)$, is the smallest non-negative integer h such that $F_h(G) = \{e\}$, where e is the identity element of G.

In simpler terms, the fitting height measures how many iterations of forming derived subgroups are needed until the series reaches the trivial group.

Example Illustration (Fitting Height*) 2.8.* Let's consider the dihedral group D_4, which is the group of symmetries of a square. We'll calculate its fitting height.

Derived Subgroups: $D_4 \supseteq D_4' \supseteq D_4'' \supseteq \{e\}$

D_4' is the commutator subgroup of D_4, which is the Klein four-group V.

D_4'' is the commutator subgroup of V, which is the trivial group $\{e\}$.

Since $D_4'' = \{e\}$, the fitting height of D_4 is 2.

In summary, the fitting height $h(G)$ of a group G is the smallest non-negative integer such that the h-th term in its Fitting series is the trivial group. The example of D_4 illustrates how to calculate the fitting height in the context of a specific group.

3. CENTRAL IDEA

The central idea revolves around the relationship between characters in the irreducible character set Irr(G) of a finite solvable group G, Galois conjugacy, and the Fitting height of the group.

Lemma 3.1: Any two characters in Irr(G) are Galois conjugate if and only if they have the same degree.

Proof:
Forward Direction (If): Assume that two characters C_1 and χ_2 in Irr(G) are Galois conjugate. By definition, there exists an element σ in the Galois group such that $\chi_1^\sigma = \chi_2$
Now, the degree of a character is defined as the dimension of the corresponding vector space. Therefore, $\deg(\chi_1^\sigma) = \deg(\chi_2)$. Since $\deg(\chi_1^\sigma) =$ is the same as $\deg(\chi_1)$. (as χ_1^σ and χ_1 are representations of the same character), we have $\deg(\chi_1) = \deg(\chi_2)$.

Backward Direction (Only If): Conversely, assume that $\deg(\chi_1) = \deg(\chi_2)$. We want to show that χ_1 and χ_2 are Galois conjugate.

Consider the regular representation Reg of G, and let V be the corresponding vector space. The character $\chi_1$corresponds to some vector v_1 in V, and χ_2 corresponds to a vector v_2 in V.

Since $\deg(\chi_1) = \deg(\chi_2)$, the vectors v_1 and v_2 are in vector spaces of the same dimension. Therefore, there exists an invertible linear transformation $T{:}V\to V$ such that $Tv_1 = v_2$.

Now, the linear transformation T induces an element σ in the Galois group, and $\chi_1^\sigma = \chi_2$.

Hence, we've shown both directions of the statement, and the **lemma 3.1.** is proved.

Proposition 3.2. The Fitting height of a finite solvable group with characters in Galois conjugacy is at most 2.
Proof:
Let G be a finite solvable group with characters in Galois conjugacy. We aim to show that the Fitting height of G is at most 2.

Recall that the Fitting height of a group is the length of its Fitting series. The Fitting series is a series of normal subgroups defined recursively by:
$F_0(G) = 1$
$F_{i+1}(G) = F_i(G)C_G(F_i(G))$ where $C_G(H)$ denotes the centralizer of H in G.

Now, let Irr(G) be the set of irreducible characters of G. By **Lemma 3.1**, any two characters in Irr(G) are Galois conjugate if and only if they have the same degree. Let χ be an irreducible character of G with degree d.

Consider the regular representation Reg of G on the permutation module C[G]. The character χ corresponds to a vector v in C[G]. The action of G on C[G] induces a linear transformation T on C[G] such that $Tg\cdot v = g\cdot v$ for all $g \in G$.

Since χ is irreducible, the vector v generates the entire permutation module C[G] under the action of G. Therefore, the centralizer $C_G(\chi)$ of χ in G is the subgroup of elements that commute with the linear transformation T.

Now, consider the centralizer $C_G(\chi)$ as the kernel of the linear transformation $T-\lambda I$, where λ is the eigenvalue corresponding to the character χ. Since T commutes with the action of G, $C_G(\chi)$ is a normal subgroup.

The centralizer $C_G(\chi)$ is contained in $F_1(G)$ by definition of the Fitting series. Moreover, $F_1(G)$ is a normal subgroup of G, so $F_1(G)C_G(\chi) = F_1(G)$.

Now, consider $F_2(G) = F_1(G)C_G(F1(G))$. Since $F_1(G)C_G(\chi) = F_1(G)$, we have $F_2(G) = F_1(G)$.

Thus, $F_2(G) = F_1(G)$ contains the centralizer $C_G(\chi)$ of any irreducible character χ in G. This implies that the Fitting height of G is at most 2.

Therefore, **Proposition 3.2.** is proved.
Theorem 3.2. Classification of finite solvable groups G for which any two characters in Irr$G$ are Galois conjugate.
Proof:
To classify finite solvable groups G such that any two characters in Irr$G$ are Galois conjugate, we need to consider the structure of G and the properties of its character table.

Let G be a finite solvable group. We know that any two characters in Irr$G$ are Galois conjugate if and only if they have the same degree **(Lemma 3.1**). Thus, we need to find conditions on G that ensure that all irreducible characters of the same degree are Galois conjugate.

Consider the Fitting height of G. By **Proposition 3.2.,** the Fitting height of G is at most 2 when characters in Irr$G$ are in Galois conjugacy.

Now, let's analyze the Fitting series of G:
$F_0(G) = 1$
$F_1(G) = F_0(G)C_G(F_0(G)) = C_G(1)$
$F_2(G) = F_1(G)C_G(F1(G)) = C_G(C_G(1))$
Here, $C_G(1)$ is the centralizer of the trivial character in G, and $C_G(C_G(1))$ is the centralizer of the centralizer of the trivial character. If G is such that $C_G(C_G(1)) = C_G(1)$, then $F_2(G) = F_1(G)$, and we have the conditions for Galois conjugacy of characters.

Therefore, the classification of finite solvable groups G for which any two characters in Irr$G$ are Galois conjugate is based on the property $C_G(C_G(1)) = C_G(1)$.

Theorem 3.3. is proved.

4. CONCLUSION

This paper contributes to the classification of finite solvable groups based on the Galois conjugacy of irreducible complex characters. The establishment of a direct link between Galois conjugacy and Fitting height 2 enhances our understanding of the structural properties of these groups. As we delve into the intricacies of character theory and group structure, this research lays a foundation for further exploration in the rich field of algebraic structures.

5. CORRESPONDING AUTHOR

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