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Algebraic Properties of the Semigroup of Partial Isometries of a Finite Chain

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Abstract **Original Research Article**

This work analyses the algebraic properties of the sub-semi group of partial Isometries (DP_n) and of order preserving partial isometries (ODP_n) of a finite chain $X_n = \{1,2, \dots n\}$, with a symmetric inverse semigroup I_n defined on it. It also investigates the subsemigroups DP_n and ODP_n for cycle structure and shows that ODP_n is a $0 - E -$ unitary inverse semigroup.

Keywords: Symmetric inverse semigroup, Finite Chain**,** Partial transformations, Finite Presentation.

INTRODUCTION

The algebraic theory of semigroups has been widely studied in recent time, however, it is relatively new. With the theory proper developing only in the second half of the twentieth century. Before this, much groundwork was laid by researchers arriving at the study of semigroups from the directions of both group and ring [5]. Semigroup on various mathematical structures with varying degrees of restrictions have been studied, See [1-4].

Preliminaries

Let $X_n = \{1,2, \dots . n\}$ and $I_n$be the partial 1-1 transformation semigroup on X_n. Then I_n is an inverse semigroup (that is, for all $\alpha \in I_n$, there exist a unique $\alpha^I \in I_n$ such that $\alpha = \alpha\alpha^1\alpha$ and $\alpha^1 = \alpha^1\alpha\, \alpha^1$). Every finite inverse semigroup S is embeddable in I_n, the analogue of cayley's theorem for finite groups, and to the regular representation of finite semigroups. Thus, just as the study of symmetric, alternating, and dihedral groups has made significant attribution to group theory, so has the study of various subsemigroups of $I_{n;}$

A transformation $\alpha \in I_n$ is said to be order preserving (order reversing) if $(\forall\, x, y \in Dom\alpha) x \leq y \Rightarrow x\alpha \leq y\alpha$ $(x\alpha \geq y\alpha)$ and is said to be as isometry (or distance-preserving) if $\forall x, y \in Dom\alpha)$ $|x - y| = |x\alpha - y\alpha|$.

When X is a finite set {1,..., n}, the inverse semigroup of one-to-one partial transformations is denoted by C_n and its elements are called charts or partial symmetries. The notion of chart generalizes the notion of permutation. The cycle notation of classical, group-based permutations generalizes to symmetric inverse semigroups by the addition of a notion called a path, which (unlike a cycle) ends when it reaches the "undefined" element; the notation thus extended is called path notation. The inverse of an element x of an inverse semigroup S is usually written x^{-1}. Inverses in an inverse semigroup have many of the same properties as inverses in a group, for example, (ab)−1 = $b^{-1}\, a^{-1}$. In an inverse monoid, x x^{-1} and x^{-1} x are not necessarily equal to the identity, but they are both idempotent. An inverse monoid S in which x x^{-1} = 1 = x^{-1}x, for all x in S (a unipotent inverse monoid), is, of course, a group. There are a number of equivalent characterisations of an inverse semigroup S. Every element of S has a unique inverse, in the above sense. Every element of S has at least one inverse (S is a regular semigroup) and idempotents commute (that is, the idempotents of S form a semilattice).

Every L −class and every R-class contains precisely one idempotent, where L *and* R are two of Green's relations. The idempotent in the L-class of s is s^{-1}s, whilst the idempotent in the R-class of s is s s^{-1}. There is therefore a simple characterisation of Green's relations in an inverse semigroup

$$aLb \Leftrightarrow a^{-1}a = b^{-1}b, aRb \Leftrightarrow a\,a^{-1} = b$$

Unless stated otherwise, E(S) will denote the semilattice of idempotents of an inverse semigroup S. *A group is said to be finitely presented if it admits a presentation* (X, R) *with both* X *and* R *finite.*

Investigating the Cycle Structure of Partial Isometrics

Let

(1) $DP_n = \{\alpha \in I_n: (\forall x, y \in X_n)|x - y|=||x\alpha - y\alpha|\}$ be the subsemigroup of I_n consisting of all partial isometrics of X_n. Also let

(2) $ODP_n = \{\alpha \in DP_n: (\forall x, y \in X_n)x \leq y \Rightarrow x\alpha \leq y\alpha\}$ be the subsemigroup of DP_n consisting of all order preserving partial isometries of X_n. It is clear that if $\alpha \in DP_n$ $(\alpha \in ODP_n)$ then $\alpha^{-1} \in DP_n(\alpha^{-1} \in ODP_n)$ also then we have the following results.

Lemma 1DP_n and ODP_n are inverse of I_n.

Next we prove a sequence of lemma, which helps our understanding of the cycle structure of partial isometries.

Let α in be in I_n. Then the height of α is $h|\alpha| = |Im\alpha|$ waist of α is $\varpi^{+}(\propto) = max\ (Im\alpha)[\varpi(x)(Im\alpha)]$ the right [left] shoulder of (α) is $\varpi^{+}(\alpha)$ =max $(Dom\alpha)[\varpi^{-}(\alpha) = (1)\ omd)]$ and $f\alpha$ of α is denoted by $f(\alpha)$, and defined by

$f(\alpha) = |f(\alpha),|$ where

$f(\alpha) = \{x \in X_n\ x\alpha\ = x\}$

Lemma 2 Let $\alpha \in DP_n$ $be\ such\ that\ h(\alpha) = P$

Then $f(\alpha) = O\ or\ 1\ or\ P$

Proof : Suppose $x, y \in F(\alpha)$. The $x - x\alpha$ and $y = yx$

Let $Z \in Dom$ where we may without loss of generality assume that $x < y < z$.

We consider two cases essentially:

Case I: $y < \mathrm{Ƶ}\ \alpha$

Case II: $x < \mathrm{Ƶ}\alpha < y$

In case P_1 we see that

$\mathrm{Ƶ} - y = |\mathrm{Ƶ}\alpha - y\alpha| = |\mathrm{Ƶ}\alpha - y|\mathrm{Ƶ}\alpha - y \Rightarrow \mathrm{Ƶ}\alpha = \mathrm{Ƶ}\alpha$

Case II, we see that

$\mathrm{Ƶ} - x \neq |\mathrm{Ƶ}\alpha - x\alpha| = |\mathrm{Ƶ}\alpha - x| = \mathrm{Ƶ}\alpha - x \Rightarrow \mathrm{Ƶ} = \mathrm{Ƶ}\alpha$

However, note that

$\alpha = (2\ 3\ 1\ 2\ \ldots\ P + 1\ \ldots\ P\ )$

$\beta = (\ldots\ \ldots\ i - 1\ i + 1\ i\ i\ i + 1\ i - 1\ \ldots\ \ldots)$,

Are nonidempotent partial isometries with $f(\alpha) = 0$ and $f(\beta) = 1$

Corollary 1 Let $\alpha \in DP_n$. If $f(\alpha) = P > 1$

Then $f(\alpha) = h(\alpha)$. Equivalently, if $f(\alpha) > 1$, then $\propto$ is idempotent.

Lemma 3

Let $\alpha \in DP_n$ if 1$\in F(\alpha)$ or $n \in F(\alpha)$ then for all $x \in Dom\alpha$, we have $x\alpha = x$ equirality,

If $1 \in F(\alpha)$ or $n \in F(\alpha)$, then (α), then α is a partial identity.

Proof: Suppose $1 \in F(\alpha)$ then for all $x \in Dom\alpha, x - 1 = x\alpha \Rightarrow x = x\alpha$.

Similarly, if $n \in F(\alpha)$ then for all $x \in Dom\alpha, n - x = n - d\alpha = n - x\alpha \Rightarrow x = x\alpha$.

Lemma 4: Let $\alpha \in ODP_n$ and $n \in Dom\alpha\ n\ Im\ \alpha$.

Then $n\alpha = n$

Proof: Since $n = max(Dom\alpha)$ and $n = max(Im\alpha)$, and α is order preserving, then $n\ \alpha = n$

However, note that in DPn we have $\alpha = (1\ n\ n\ 1\ )$ where $n \in Dom\alpha\ \cap \hat{I}m\alpha$ but $n\ \alpha \neq n$

Lemma 5: Let $\alpha \in ODP_n$ and $f(\alpha) \geq 1$. Then α is an idempotent

Proof: Let x be a fixed part of α and suppose $y \in Dom\alpha$. If $x < y$ the by the order preserving and isometry proportion, we see that $y = x = y\alpha = x\alpha = y\ \alpha - x \Rightarrow y = y\alpha$.

The case of $y < x$ is similar. However, note that on DP_n, we have $\alpha = (1\ 2\ 3\ 2\ )$, where $f(\alpha) = 1$

But $\alpha^2 \neq \alpha$.

Lemma 6: Let S = ODP_n. Then $F(n, p1) = F(n, 1) = n^2$ and $F(n, pn) = F(n, n = 1)$ for all $n \geq 2$

Proof: Since all partial injections of height 1 are vacuously partial isometries, the first statement of the lemma follows immediately. For the second statement, it is not difficult to see that there is exactly one partial isometry of height $(1\ 2\ 1\ 2\ \ldots\ n\ \ldots\ n\ )$ (the identity).

Lemma 7: Let S = ODP_n, Then $F(n, p_2) = F(n; 2) = \frac{1}{6}n(n - 1)(2n - 1)$, for all n ≥ 2.

Proof: First, we say that 2-subsets of X_n (that is, subsets of size 2) say, A=$\{a_1, a_2, \}$and B=$\{b_1, b_2\}$are of the same type if $|a_1 - a_2 = |b_1, b_2|$, Now observe that if then there are $n - i$ subsets of this type. However, for partial order-preserving isometries once we choose a 2-subset as a domain then the possible image sets must be of the same type and there is only one possible order-preserving bijection between any two 2-subsets of the same type. It is now clear that $F(n;2) = \sum \quad n-1\ i=1\ (n-i)^2 = \frac{1}{6}(n-1)$, as required.

Lemma 8: Let S = ODP_n. Then $F(n;p) = F(n-1;\ p-1) + F(n-1;p)$, for all $\geq p \geq 3$

Proof: Let α∈ ODP_n and $h(\alpha) = p$. Then it is clear that $F(n;p) = |A| + |B|$, where A={α∈ ODP_n: $h(\alpha) = p\ and\ n \notin\in Dom\ \alpha\ \cup I\ m\ \alpha\}and\ B = \{\alpha \in ODP : h(\alpha) = p$ and $n \notin Dom\ \alpha\ \cup Im\ \alpha\}$. Define a map θ: $\{\alpha \in ODP_n - 1: h(\alpha) = p\} \rightarrow A$ by $(\alpha)\theta = \alpha'$ where $x\alpha' = x\alpha\ (x \in Dom\ \alpha$. This is clearly a bijection since $n \notin Dom\ \alpha\ \cup I\ m\ \alpha$. Next, recall the definitions of $\varpi^+(\alpha)$and $\varpi^+(\alpha)$ from the chapter1. Now, define a map Φ : $\{\alpha \in ODP_{n-1}: h(a) = p-1\} \rightarrow B$ by $(\alpha)\Phi = \alpha'$
where

(i) $x\alpha' = x\alpha\ (x \in Dom\ \alpha)$ and $n\alpha' = n(if\ \varpi^+(\alpha) = \omega^+(\alpha))$;
(ii) $x\alpha' = x\alpha\ (x \in Dom\ \alpha)$ and n$\alpha' = n - \varpi^+(\alpha) + \omega^+(\alpha) < n\ (if\ \varpi^+(\alpha) > \omega^+(\alpha)$;
(iii) x$(\alpha')^{-1} = x\alpha^{-1}(x \in Im\ \alpha)$ and $n(\alpha')^{-1} = n - \varpi^+ (\alpha)^{-1} + \omega^+ (a^{-1}) < n$ (if $\varpi^+(a) < \omega^+(\alpha)$).

In all cases $h(\alpha') = p$, and case (i) coincides with $n \in Dom\ \alpha' \cap Im\ \alpha'$;case (ii) coincides with $n \in Dom\ \alpha' \backslash I\ m\ \alpha'$; case (iii) coincides with $n \in Im\ \alpha' \backslash Dom\ \alpha'$. Thus Φ is onto. Moreover, it is not difficult to see that $\emptyset$ is one-to-one. Hence Φ is a bijection, as required. This establishes the statement of the lemma.

Proposition 1: Let $S = ODP_n$ and $F(n;p)$ have their usual definitions, respectively. Then $(n;p) = \frac{(2n-p+1)}{p+1}(n\ p)$, where $n \geq p \geq 2$.

Proof. (The proof is by induction).

Basis step: First, note that $F(n;\ 1)$, $F(n;n)$ and $F(n;\ 2)$ are true by Lemmas 1 and 2.

Inductive step: Suppose $F(n-1;\ p)$ is true for all $n-1 \geq p$. (This is the induction hypothesis.) Now using Lemma 3, we see that $F(n;p) = F(n-1;\ p-1) + F(n-1;\ p)$

$= \frac{(2n-p)}{p\ n-1;p}(n - 1\ p - 1) + \frac{(2n-p-1)}{p+1}(n-1\ p)$ (by ind. hyp.)

$= \frac{(2n-p)}{p}\frac{p}{n}(n\ p) + \frac{+(2n-p-1)}{p+1}\ \frac{(n-p)}{n}(n\ p)$

$= \frac{(2n-p(p+1)+(2n-p-1))n-p)}{n(p+1)}(n\ p)$

$\frac{(2n^2-np+n)}{n(p+1)}(n\ p) = \frac{(2n-p+1)}{(p+1)}(n\ p)$;

as required.

Lemma 9: For integers n, p such that $n \geq p \geq 2$, we have $\sum_1^n \frac{(2n-p+1)}{(p+1)} n\ p = 2\ \frac{2n-p+1}{p+1}(n\ p) = 3.2^n - n^2 - 2n - 3$.

Proof. It is enough to observe that $2n - p + 1 = (2n-2p) + (p+1)$.

Theorem 1: Let ODP_n be as defined in (2). Then| $|ODP_n = 3.2^n = 2(n+1).|$

Proof. It follows from Proposition 1 and Lemma 5, and some algebraic manipulation.

Let $S = ODP_n$. Then $F(n;m) = (n\ m)$ for all $n \geq m \geq 1$.

Proof. It follows directly from Lemma 6.

Proposition 2: Let S = ODP_n and $F(n;m)$ be as defined in (2) and (6), respectively. Then $F(n;\ 0) = 2n+1 - (2n+1)$.

Proof. It follows from Theorem 6, Lemma 7 and the fact that $|\ ODP_n| = Pnm = 0\ F(n;m)$. For some computed values of $F(n;p)$ and $F(n;m)$ in ODP_n, see Tables 1 and 2.

Table 1

$n\backslash m$	0	1	2	3	4	5	6	7	$\sum F(n;m) = ODP_n$
0	1								1
1	1	1							2
2	1	4	1						6
3	1	9	5	1					16
4	1	16	14	6					38
5	1	25	30	20	7	1			84
6	1	36	55	50	27	8	1		178
7	1	49	91	105	77	35	9	1	308

Table 2

$n\backslash m$	0	1	2	3	4	5	6	7	$\sum F(n;m) = ODP_n$
0	1								1
1	1	1							2
2	3	2	1						6
3	9	3	3	1					16
4	23	4	6	4	1				38
5	53	25	30	20	7	1			84
6	115	6	15	20	15	6	1		178
7	241	7	21	35	35	21	7	1	368

Remark: For p= 0,1 the concepts of order-preserving and order-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for $p \geq 2$.

Lemma 10: Let $\alpha \in ODP_n$. Then α is either order-preserving or order-reversing.

Proof. If $h(\alpha)$ = 2 then the result is obvious. However, if $h(\alpha) > 2$ wemust consider cases. First suppose that $\{a_1 a_2 a_3\} \subseteq Dom\alpha$, where $a_i\alpha = b_i (i = 1,2,3)$ and $1 \leq a_i < a_2 < a_3 \leq n$. There are four cases to consider if α is neither order-preserving or order-reversing: $b_1 < b_3 < b_2, b_2 < b_1 < b_3, b_2 < b_3 < b_1$ and $b_3 < b_1 < b_2$. In the first case, note that $b_2 - b_1 = (b_2 - b_3) + (b_3 - b_1)$. But $a_3 - a_1 = (a_3 - a_2) + (a_2 - a_1) = |a_3 - a_2| + |a_2 - a_1| = |b_3 - b_3| + |b_2 - b_1| = |b_3 - b_2| + |b_2 - b_3| + |b_3 - b_1| = 2\,|b_3 - b_2| + b_3\alpha^{-1} - b_1 a^{-1}| = 2\,|b_3 - b_2| + |aa_3 - a_1| = 2\,|b_3 - b_2| + a_3 - a_1$, whichimplies that $|b3 - b2| = 0 \Leftrightarrow b3 = b2$. This is a contradiction. The otherthree cases are similar. We now use Remark above and Lemma 9 to deduce corresponding results for DP_n from those of ODP_n above.

Lemma 11: Let S=DP_n. Then $F(n; p_1) = F(n;\ 1) = n^2)$ and $F(n; p_n) = F(n; n) = 2$, for all $n \geq 2$.

Lemma 12: Let $S = DP_n. Then\ F(n; p_2\ ) = F(n;\ 2) = \frac{1}{3} n\ (n-1)(2n-1)$, for all $n \geq 2$.

Lemma 13: Let $S = DP_n$. Then $F(n; p) = F(n-1;\ p-1) + F(n-1;\ p)$, for all $n \geq p \geq 3$.

Proposition 3 Let S=DP_n and $F(n; p)$ be as defined in (1) and (5), respectively. Then $F(n; p) = \frac{2(2n-p+1)}{p+1}\ (n\ p\ )$, where $n \geq p \geq 2$.

Theorem 2: Let DP_n be as defined in (1). Then $|\ DP_n\ | = 3\ \cdot 2^{n+1} - (n+\ 2)^2 - 1$. Proof. It follows from Proposition 3, Lemma 5 and some algebraic manipulation.

Lemma 14: Let $\alpha \in DP_n$. For $1 < i < n$, if $F(\alpha) = \{i\}$ then for all $x \in Dom\ \alpha$ we have that $+x\alpha = 2i$.

Proof. Let $F(\alpha) = \{i\}$ and suppose $x \in Dom\ \alpha$. Obviously, $i + i\alpha = i + i = 2i$. If $x < i$ then $x\alpha > i$, for otherwise we would have $i - x = |i\alpha - x\alpha\ | = |i - x\alpha\ | = i - x\alpha \Rightarrow x = x\alpha$, which is a contradiction. Thus, $i - x = |i\alpha - x\alpha\ | = |i - x\alpha\ | = |x\alpha - i| = x\alpha - i \Rightarrow x + x\alpha = 2i$. The case $x >$ I is similar.

Lemma 15: Let S=DP_n. Then F(n;m) = $(n\ m\ )$, for all $n \geq m \geq 2$.

Proposition 4 Let S=DP_n. Then $F(2n; m1) = F(2n;\ 1) = 2(22n-1)3$ and $F(2n-1;\ m1) = F(2n-1;\ 1) = \frac{2(2^{2n-2}-1)}{3} + \frac{2^{2n-2}}{3}$, for all $n \geq 1$.

Proof. Let $F(\alpha) = \{i\}$. Then by Lemma 15, for any $x \in Dom\ \alpha$ we have $x + x\alpha = 2i$. Thus there $2i - 2$ possible elements for $Dom\ \alpha :\ (x, x\alpha) \in \{(1,2i-1), (2,2i-2), \cdots\ (2i-1,1)\}$.
However, $(excluding\ (i,i))$ we see that there are $\sum_1^n (n,m)\ 2i - 2\ j = 0\ (2i - 2\ j\ ) = 2^{2i-2}$, possible partial isometries with $F(\alpha) = \{i\}$, where $2i - 1 \leq n \Longleftrightarrow i \leq (n+\ 1)/2$. Moreover, by symmetry we see that $F(\alpha) = \{i\}\ and\ F(\alpha) = \{n - i + 1\}$ give rise to equal number of partialisometries. Note that if n is odd the equation $i = n - i + 1$ has one solution. Hence, if $n = 2a - a - 1 \sum i = 1\ 2^{2i-2} + 2^{2a-2} = 2\,\frac{2^{2a-2}-1}{3} + 2^{2a-2}$

partial isometries with exactly one fixed point; if n= 2a we have $a\ 2\sum \quad i = 1\ 2^{2i-2} = 2\,\frac{2^{2a}-1}{3}$

partial isometries with exactly one fixed point.

Proposition 5 Let $S = DP_n$. Then $F(n; 0) = F(n;\ 0) = n \geq 0$ if n is even) and $F(n; m0) = F(n;\ 0) = 25 \cdot 2n - 1 - (3n2 + 9n + 10)3, (n \geq 1$,if nis odd).
Proof. It follows from Theorem 3.16, Lemma 3.18, Proposition 3.19 and the fact that $|DP_n| = \sum \quad n\ m = 0\ F(n; m)$.

Remark : For some computed values of $F(n; p)$and $F(n; m) in\ DP_n$, see Tables 3 and 4.

Table 3

$n\backslash m$	0	1	2	3	4	5	6	7	$\sum F(n; p) = ODP_n$
0	1								1
1	1	1							2
2	4	4	2						7
3	1	9	10	2					22
4	1	16	28	12	2				59
5	1	25	60	40	14	2			142
6	1	36	110	100	54	16	2		319
7	1	49	182	210	154	70	18	2	686

Table 4

$n\backslash m$	0	1	2	3	4	5	6	7	$\sum F(n; m) = ODP_n$
0	1								1
1	1	1							2
2	4	2	1						7
3	12	6	3	1					22
4	28	10	6	4	1				59
5	90	26	10	10	5	1			142
6	220	42	15	20	15	6	1		319
7	460	106	21	35	35	21	7	1	686

Findings on the Cycle Structure
The study defines a cyclic semigroup as one generated by a single element.
Given I_n as a symmetric inverse semigroup.
Let $DP_n = \{\alpha \in I_n : (\forall x, y \in X_n)|x - y| = |x\alpha - y\alpha|\}$ be subsemigroup of I_n consisting of all partial isomatries of X_n, and
Let $ODP_n = \{\alpha \in DP_n : \forall x, y \in X_n)x \leq y \Rightarrow x\alpha \leq y\alpha\}$
Be the subsemigroup DP_n consisting of all order preserving partial isometries of X_n. It in clear that if $\alpha \in DP_n(\alpha \in ODP_n)$, Then $\alpha^{-1} \in DP_n(x^{-1}ODP_n)$ also. Thus DP_n and ODP_n are inverse subsemigroups of I_n.
Supporting Lemmas show that the transformation $\alpha \in DP_n(\alpha \in OP_n)$ is idempotent wherever$f(x) \leq 1$.
Hence DP_n and ODP_n are cyclic or monogenic since they have a single generator.

Findings on $O - E -$ Unitary Inverse Semigroup
The study defines a semigroup S to be $O - E -$ unitary if $(\forall e \in E^1)(\forall seS)$,

$$es \in E^1 \Rightarrow s \in E.$$

That is, an inverse semigroup is $O - E -$ unitary when any element above a non-zore idempotent in the natural order is itself an idempotent.

Theorem 3 shows clearly that ODP_n is $O - E -$ unitary given that β, an element of ODP_n naturally above the non-zero idempotent ϵ is also idempotent.

The paper shows a strong connectedness between the components of the L and classes. Further, the L, R, and J relation define three preorders $\leq_1 \leq_R \leq_H$ where $\alpha \leq_H \beta$ holds for two elements a and b of S whenever the ideal generated by a is included in that of $b, i.e\ S^1 a\ S^1 \leq S^1 b\ S^1$

The study shows that the subsemigroup of I_n, that is DP_n and ODP_n are inverse subsemigroups because for every $\alpha \in DP_n\ (\alpha \in ODP_n)$, there is $\alpha^1 \in DP_n(\alpha^{-1}\ ODP_n)$ also. For $\alpha \in I$, with height $h(\alpha)$ and fix $(\alpha) = |F(\alpha)|$, where $F(\alpha) = \{x \in X_n : x\alpha = x\}$, supporting lemma in the study shows that whenever $f(\alpha) > 1$, then α is idempotent. α being idempotent show that OP_n and ODP_n have circular structure.

The Green's equivalences on OP_n and ODP_n shows strong connectedness between the subsemigroups and their elements generate the same principal ideals.

On $O - E$ −Unitary Inverse Semigroup, a semigroup is $O{-}\in -$ unitary if $(\forall e \in E^1)$ $(\forall s \in S) es \in E^1 \Rightarrow s \in E^1$ where $E^1 = E \backslash O$.

It is clear that ODP_n is $O - E -$ unitary since it satisfies the above condition as shown in theorem 3 implying that ODP_n has an element β above a non-zero idempotent θ which is itself idempotent.

CONCLUSION

The study shows that the subsemigroup of I_n, that is DP_n and ODP_n are inverse subsemigroups because for every $\alpha \in DP_n$ $(\alpha \in ODP_n)$, there is $\alpha^1 \in DP_n(\alpha^{-1}\ ODP_n)$ also. For $\alpha \in I$, with height $h(\alpha)$ and fix $(\alpha) = |F(\alpha)|$, where $F(\alpha) = \{x \in X_n : x\alpha = x\}$, supporting lemmas in the study shows that whenever $f(\alpha) > 1$, then α is idempotent. α being idempotent show that OP_n and ODP_n have circular structure.

The Green's equivalences on OP_n and ODP_n shows strong connectedness between the subsemigroups and their elements generate the same principal ideals.

On $O - E$ −Unitary Inverse Semigroup, a semigroup is $O{-}\in -$ unitary if $(\forall e \in E^1)$ $(\forall s \in S) es \in E^1 \Rightarrow s \in E^1$ where $E^1 = E \backslash O$.

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