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Discontinuous Galerkin Method of the First Order Parabolic Differential Equation

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Abstract **Original Research Article**

This paper is concerned with the mathematical study of the discontinuous Galerkin (DG) finite element method for the parabolic differential equation. The DG method is a vital numerical method with much mass compensation and more flexible meshing than other methods. In this study, we give a general introduction and discuss about the discontinuous Galerkin Method of first order parabolic problem. The parabolic problem satisfies the condition of the existence and uniqueness of DG solution. The error analysis of this problem is also established. The main goal of this study is to theoretically explore the convergence of the solution of the above methods and show the validity of the results.

Keywords: Galerkin Method, Parabolic Differential Equation, finite element method.

1. INTRODUCTION

This study provides a theoretical concept to approximate the error of the solutions of a parabolic differential equation. We focus on the weak formulation of discontinuous Galerkin (DG) method of the first order parabolic problem. Finite element methods (FEM) have been proven extremely useful in the numerical approximation of solutions to self-adjoint or ‘‘nearly’’ self-adjoint parabolic partial differential equation (PDE) problems and related indefinite PDE systems or to their parabolic counterparts. Possible reasons for the success of FEM are their applicability in very general computational geometries of interest and the availability of tools for their rigorous error analysis. The error analysis is usually based on the variational interpretation of the FEM as a minimization problem over finite-dimensional sets. The variational structure is inherited by the corresponding variational interpretation of the underlaying PDE problems, thereby facilitating the use of tools from PDE theory for the error analysis of the FEM. In 1971, Reed and Hill proposed a new class of FEM, namely the discontinuous Galerkin finite element method for the numerical solution of the nuclear transport PDE problem, which involves a linear first-order hyperbolic PDE. DG methods were first proposed and analysed in the early 1970s as a technique to numerically solve partial differential equations. The origin of the DG method for parabolic problems cannot be traced back to a single publication as features such as jump penalization in the modern sense were developed gradually. The discontinuous Galerkin (DG) method has been extensively studied and applied to a wide range of parabolic problems. In this literature review, we will highlight some key research works in the field. The comprehensive book by Beatrice Riviere, Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations; covered theory, implementation and other information [1]. The paper by Jan S. Hesthaven and Tim Warburton has been described Nodal Discontinuous Galerkin Methods; Algorithms, Analysis and also described applications [2]. The paper by P. E. Lewis and J.P. Ward provides The Finite Element Method; Principles and Application [3]. The paper by D. N. Arnoldis represented ‘‘An interior penalty finite element method with discontinuous elements’’ [4]. This paper has been provided ‘‘Energy norm a posteriori error estimation for discontinuous Galerkin methods.’’ by R. Becker, P. Hansbo, and M. G. Larson [5]. The paper ‘‘A unifying theory of a posteriori error control for discontinuous Galerkin FEM.’’ By C. Carstensen, T. Gudi, and M. Jensen included error estimate with discontinuous Galerkin(DG) FEM [6]. The book ‘‘Discontinuous Galerkin methods for convection-dominated problems.’’ in Higher-order Methods has been existed vast information related to discontinuous Galerkin(DG) FEM by B. Cockburn [7]. The paper ‘‘Discontinuous Galerkin Methods.’’ Theory, computation and applications by B. Cockburn, G. E.

Karniadakis, and C. W. Shu (eds.) represent some key element on discontinuous Galerkin(DG) FEM [8].The thesis paper ‘‘Discontinuous Galerkin Methods on Shape-Regular and Anisotropic Meshes.’’ By E.H. Georgoulis included shape-regular meshes on discontinuous Galerkin(DG) FEM [9]. The paper ‘‘What is the Difference Between FEM, FDM, and FVM?’’ by Sjodin, Bjorn has been helped me to give the clear concept between FEM, FDM, and FVM [10].The paper ‘The Direct Discontinuous Galerkin (DDG) Method Hailiang Liu, Jue Yan for Diffusion with Interface Corrections’ is published by Hailiang Liu, Jue Yan. In this paper is focused on The Direct Discontinuous Galerkin (DDG) Method [11]. The book Discontinuous Galerkinmethods, Theory, computation and applications by B. Cockburn,G. E. Karniadakis and C. W. Shu (eds.) is represented theoretical, computational discussion and application [12]. The paper ‘Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations’ is a helpful paper for me which is published by U. M. Ascher and L. R. Petzold [13]. The research paper ‘The finite element method with Lagrangian multipliers’ by I. Babuˇska provides some information on the finite element method [14]. The research paper ‘A discontinuous hp finite element method for diffusion problems: 1-D analysis, Computers & Mathematics with Applications’ provided the finite element method for diffusion problems [15]. ‘The Mathematical Theory of Finite Element Methods’ is an important paper which is published by S. Brenner and L. Scottfor the theory of finite element method [16]. The paper ‘The local discontinuous Galerkin method for the Oseen equations, Mathematics of Computation’ by B. Cockburn, G. Kanschat, and D. Schötzau represented local discontinuous Galerkin method [17]. Finally, the book ‘Parabolic Equations. Contributions to the Theory of Partial Differential Equations’ by P. Lax and N. Milgram provides parabolic equations by discontinuous Galerkin(DG) FEM [18].

Formulation of the problem

Let Ω be a polygonal domain in $\mathbb{R}^d$, $d = 2$ or 3. The side of the boundary $\partial\Omega$ of the domain is Γ. Let $\boldsymbol{n}$ be the unit normal vector to the boundary exterior to Ω. For f given in L^2 (Ω), g given in $H^{\frac{1}{2}}(\Gamma)$.

We consider the parabolic problem,

$$\nabla . u + cu = f \quad in\, \Omega \quad \ldots\ldots (1)$$

Boundary condition: $u = g \; on \, \Gamma \; \ldots\ldots\ldots\ldots (2)$

Now, let us consider a weight function v. Multiplying (1) by v and integrating on Ω, we obtain,

$$(\nabla . u + cu)v = fv$$

$$\Rightarrow (\nabla . u)v + cuv = fv$$

$$\Rightarrow \int_\Omega ((\nabla . u)v + cuv)\, dx = \int_\Omega fv dx$$

$$\Rightarrow \int_\Omega (\nabla . u)v\, dx + \int_\Omega cuv\, dx = \int_\Omega fv\, dx$$

$$\Rightarrow \sum \int_\Omega (\nabla . u)v\, dx + \sum \int_\Omega cuv\, dx = \sum \int_\Omega fv\, dx \ldots\ldots (3)$$

For the proposed DG method, the following DG norm is introduced

$$\|u\|_{DG}^2 = Ch\, \|u\|_{H^s(\Omega)}^2 + Ch_1\, \|u\|_{H^s(\Omega)}^2$$

Using the divergence theorem on every element integral (as v is now elementwise discontinuous), using the anti-clockwise orientation, we have,

$$-\sum \int_\Omega ((\nabla . u). \nabla v)\, dx + \sum \int_{\partial\Omega} ((\nabla . u). \boldsymbol{n}) v\, ds + \sum \int_\Omega cuv\, dx = \int_\Omega fv\, dx$$

where, $\boldsymbol{n}$ = The outward normal to each element edge.

Then, introduce a bilinear form $B(u, v)$ as

$$B(u, v) = -\int_\Omega ((\nabla . u). \nabla v)\, dx + \int_{\partial\Omega} ((\nabla . u). \boldsymbol{n}) v\, ds + \int_\Omega cuv\, dx$$

Therefore, the DG finite element method is defined as

$$B(u, v) = \int_\Omega fv\, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Stability analysis

The following theorem is introduced for the stability analysis of this method.

Theorem: Assume that there exist positive constants A and B such that the solution u satisfies the following bounds.

$$B(u,u) \leq A\, \|u\|^2_{L^2(\Omega)} + B\, \|\nabla . u\|^2_{L^2(\Omega)}$$

Proof: Let us define the bilinear form of the problem:

$$B(u,v) = -\int_\Omega \big((\nabla . u). \nabla v\big) dx + \int_{\partial\Omega} \big((\nabla . u). \boldsymbol{n}\big) v\, ds + \int_\Omega cuv\, dx \dots\dots (5)$$

Putting $v = u$, we obtain,

$$B(u,u) = -\int_\Omega \big((\nabla . u). \nabla u\big) dx + \int_{\partial\Omega} \big((\nabla . u). \boldsymbol{n}\big) u\, ds + \int_\Omega cu^2\, dx \dots\dots (6)$$

Using Cauchy's inequality on $((\nabla . u) \cdot n)u$, $\|((\nabla . u) \cdot n)u\| \leq \|(\nabla . u) \cdot n\| \cdot \|u\|$

We have,

$$B(u,u) \leq -\int_\Omega \big((\nabla . u). \nabla v\big) dx + \int_{\partial\Omega} \big((\nabla . u). \boldsymbol{n}\big) v\, ds + \int_\Omega cuv\, dx \dots\dots (7)$$

Using trace inequality on 2^{nd} term of (7),

$$\forall u \in \mathbb{P}_k(\Omega), \forall e \subset \partial\Omega,$$

$$\|(\nabla . u) \cdot \boldsymbol{n}\|_{L^2(e)} \leq \tilde{C}_t |e|^{\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \|\nabla . u\|_{L^2(\Omega)}$$

and

$$\forall u \in \mathbb{P}_k(\Omega), \forall e \subset \partial\Omega,\ \|u\|_{L^2(e)} \leq C_t h_\Omega^{-\frac{1}{2}} \|u\|_{L^2(\Omega)}$$

We know that, $\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2\right)^{\frac{1}{2}}$

$\therefore \int_\Omega ((\nabla . u). \nabla u)\, dx = \|\nabla . u\|^2_{L^2(\Omega)}$ And, $\int_\Omega cu^2 dx = c\, \|u\|^2_{L^2(\Omega)}$

From the equation (7), we obtain,

$$B(u,u) = -\|\nabla . u\|^2_{L^2(\Omega)} + \left(\tilde{C}_t |e|^{\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \|\nabla . u\|_{L^2(\Omega)}\right) \cdot \left(C_t h_\Omega^{-\frac{1}{2}} \|u\|_{L^2(\Omega)}\right) + c\|u\|^2_{L^2(\Omega)}$$

$$\leq -\|\nabla . u\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\left(\tilde{C}_t |e|^{\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \|\nabla . u\|_{L^2(\Omega)}\right)^2 + \frac{1}{2\epsilon}\left(C_t h_\Omega^{-\frac{1}{2}} \|u\|_{L^2(\Omega)}\right)^2 + c\, \|u\|^2_{L^2(\Omega)}$$

So we have,

$$B(u,u) \leq -\|\nabla . u\|^2_{L^2(\Omega)} \{1 - \frac{\epsilon}{2}\left(\tilde{C}_t |e|^{\frac{1}{2}} |\Omega|^{-\frac{1}{2}}\right)^2\} + \|u\|^2_{L^2(\Omega)} \{\frac{1}{2\epsilon}\left(C_t h_\Omega^{-\frac{1}{2}}\right)^2 + c\}$$

If we consider, $A = \frac{1}{2\epsilon}\left(C_t h_\Omega^{-\frac{1}{2}}\right)^2 + c$ and $B = -(1 - \frac{\epsilon}{2}\left(\tilde{C}_t |e|^{\frac{1}{2}} |\Omega|^{-\frac{1}{2}}\right)^2)$

Then from the above equation, we can find,

$=> B(u,u) \leq A\, \|u\|^2_{L^2(\Omega)} + B\|\nabla . u\|^2_{L^2(\Omega)}$ ….……….(8)

Hence, this completes the proof.

1.1. Consistency of the solution

We have,

$$\Rightarrow -\int_\Omega \big((\nabla . u). \nabla v\big) dx + \int_{\partial\Omega} \big((\nabla . u). \boldsymbol{n}\big) v\, ds + \int_\Omega cuv\, dx = \int_\Omega fv\, dx$$

$$\Rightarrow -\int_\Omega \big((\nabla . u) \cdot \nabla v - cuv\big) dx + \int_{\partial\Omega} ((\nabla . u) \cdot \boldsymbol{n})\, v\, ds = \int_\Omega fv\, dx$$

$$\Rightarrow -\int_\Omega \big((\nabla . u) \cdot \nabla v - cuv\big) dx + \int_{\partial\Omega} \big((\nabla . u) \cdot \boldsymbol{n}_E\big) v\, ds = \int_\Omega fv\, dx \dots\dots (9)$$

We define, $\boldsymbol{n}_E$ is the outward normal to E. We sum over all elements, switch to the normal vectors $\boldsymbol{n}_e$,

$$\sum_{E\in\mathcal{E}_h}\int_{\partial\Omega}((\nabla.u)\cdot\boldsymbol{n}_E)\,v\,ds-\sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot\boldsymbol{n}_e)\,v\,ds=\sum_{e\in\Gamma_h}\int_e[(\nabla.u)\cdot\boldsymbol{n}_e)\,v]\,ds$$

$$\Rightarrow\sum_{E\in\mathcal{E}_h}\int_{\partial\Omega}((\nabla.u)\cdot\boldsymbol{n}_E)\,v\,ds=\sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot\boldsymbol{n}_e)\,v\,ds+\sum_{e\in\Gamma_h}\int_e[(\nabla.u)\cdot\boldsymbol{n}_e)\,v]\,ds$$

By regularity of the solution u, we have,

$$(\nabla.u)\cdot\boldsymbol{n}_e=\{(\nabla.u)\cdot\boldsymbol{n}_e\}$$

Substituting all of these values in the equation (9), we get,

$$-\sum_{E\in\varepsilon_h}\int_\Omega\big((\nabla.u)\cdot\nabla v-cuv\big)dx+\sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot\boldsymbol{n}_e)\,v\,ds+\sum_{e\in\Gamma_h}\int_e\{(\nabla.u)\cdot\boldsymbol{n}_e)\}[v]\,ds=\int_\Omega fv\,dx$$

We subtract, $\epsilon\sum_{e\in\Gamma}\int_e(\nabla v\cdot\boldsymbol{n}_e)\,u\,ds$ to both sides and use the Dirichlet boundary condition $u=g$ on Γ, we get,

$$-\sum_{E\in\varepsilon_h}\int_\Omega\big((\nabla.u)\cdot\nabla v-cuv\big)dx+\sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot\boldsymbol{n}_e)\,v\,ds$$
$$+\sum_{e\in\Gamma_h}\int_e\{(\nabla.u)\cdot\boldsymbol{n}_e)\}[v]\,ds-\epsilon\sum_{e\in\Gamma}\int_e(\nabla v\cdot\boldsymbol{n}_e)\,u\,ds=\int_\Omega fv\,dx-\epsilon\sum_{e\in\Gamma}\int_e(\nabla v\cdot\boldsymbol{n}_e)\,gds$$

Finally, we can say that the jumps $[u]=[(\nabla.u)\cdot\boldsymbol{n}_e]$ are zero, that is on the interior edges (or faces).

Then we clearly have,

$$-\sum_{E\in\varepsilon_h}\int_\Omega\big((\nabla.u)\cdot\nabla v-cuv\big)dx=\int_\Omega fv\,dx$$

$$\Rightarrow-\sum_{E\in\varepsilon_h}\int_\Omega\big((\nabla.u)\cdot\nabla v\big)dx+\int_\Omega cuv=\int_\Omega fv\,dx$$

Which immediately yields in the distributional sense, for all $E\in\mathcal{E}_h$,

$$\nabla.u+cu=f$$

This proves that the given parabolic problem is consistent.

Error analysis

The following theorem is proposed for the error estimate of the problem governed by the equation (1).

Theorem: Let u_h be the DG finite element solution and u be the exact solution of (1) arising from (4).Then there exists a constant C such that

$$\|u-u_h\|^2_{L^2(\Omega)}\leq C\mathrm{h}\,\|u\|^2_{DG}$$

Proof: Bilinear form of the problem be,

$$B(u,v)=-\int_\Omega\big((\nabla.u).\nabla v\big)dx+\int_{\partial\Omega}\big((\nabla.u).\boldsymbol{n}\big)v\,ds+\int_\Omega cuv\,dx\,\ldots\ldots(10)$$

Let, the exact solution of the problem be u and the approximation solution be u_h .

Putting $u=u_h$ in (10), we get,

$$B(u_h,v)=-\int_\Omega\big((\nabla.u_h).\nabla v\big)dx+\int_{\partial\Omega}\big((\nabla.u_h).\boldsymbol{n}\big)v\,ds+\int_\Omega cu_hv\,dx\,\ldots\ldots(11)$$

Now,

$$B(u,v)-B(u_h,v)=-\int_\Omega\big((\nabla.u-\nabla.u_h)\cdot\nabla v\big)dx+\int_{\partial\Omega}\Big(\big((\nabla.u)\cdot\boldsymbol{n}\big)-\big((\nabla.u_h)\cdot\boldsymbol{n}\big)\Big)v\,ds+\int_\Omega c(u-u_h)v\,dx$$

Let us define, $=u-u_h$. Then we have,

$B(u, u - u_h) - B(u_h, v)$

$$= -\int_\Omega \left((\nabla. u - \nabla. u_h) \cdot \nabla(u - u_h)\right)dx + \int_{\partial\Omega} \left(\left((\nabla. u) \cdot \boldsymbol{n}\right) - \left((\nabla. u_h) \cdot \boldsymbol{n}\right)\right)(u - u_h)ds$$
$$+ \int_\Omega c(u - u_h)(u - u_h)dx$$

$\Rightarrow B(u, u - u_h) - B(u_h, v)$

$$= -\int_\Omega \left((\nabla. u - \nabla. u_h) \cdot \nabla(u - u_h)\right)dx + \int_{\partial\Omega} \left(\left((\nabla. u) \cdot \boldsymbol{n}\right) - \left((\nabla. u_h) \cdot \boldsymbol{n}\right)\right)(u - u_h)ds$$
$$+ \int_\Omega c(u - u_h)^2 dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The first term of right-hand side of the equation (12) can be written as,

$$\int_\Omega \left((\nabla. u - \nabla. u_h) \cdot \nabla(u - u_h)\right) = \int_\Omega \left((\nabla. (u - u_h) \cdot \nabla(u - u_h)\right)$$
$$= \int_\Omega \nabla. (u - u_h)^2$$

We know that, $\int_\Omega \nabla. (u - u_h)^2 = \|\nabla. (u - u_h)\|^2_{L^2(\Omega)}$

From the 2^{nd} term of right-hand side of the equation (12), we get,

$$\int_{\partial\Omega} \left(\left((\nabla. u) \cdot \boldsymbol{n}\right) - \left((\nabla. u_h) \cdot \boldsymbol{n}\right)\right)(u - u_h)$$

Using Cauchy-Schwarz's inequality, we obtain,

$$|(((\nabla. u) \cdot \boldsymbol{n}) - ((\nabla. u_h) \cdot \boldsymbol{n}))(u - u_h)| \leq \|((\nabla. u) \cdot \boldsymbol{n}) - ((\nabla. u_h) \cdot \boldsymbol{n})\|_{L^2(\Omega)} \cdot \|u - u_h\|_{L^2(\Omega)}$$

Using Trace inequality, we get,

$$\|((\nabla. u) \cdot \boldsymbol{n}) - ((\nabla. u_h) \cdot \boldsymbol{n})\|_{L^2(\Omega)} \leq C_t h_\Omega^{-\frac{1}{2}} \|\nabla. (u - u_h)\|_{L^2(\Omega)}$$
$$\leq C_t h_\Omega^{-\frac{1}{2}} . C h_\Omega^{min(k+1,s)-1} |u|_{H^s(\Omega)} = C_t . C h_\Omega^{-\frac{1}{2}+min(k+1,s)-1} |u|_{H^s(\Omega)}$$

Let, $C_t \cdot C = \tilde{C}$. Then we have,

$$\|((\nabla. u) \cdot \boldsymbol{n}) - ((\nabla. u_h) \cdot \boldsymbol{n})\|_{L^2(\Omega)} \leq \tilde{C} h_\Omega^{min(k+1,s)-\frac{3}{2}} |u|_{H^s(\Omega)}$$

Now, substituting these values in (12), we get,

$$B(u - u_h, u - u_h) \leq -\|\nabla. (u - u_h)\|^2_{L^2(\Omega)} + \left\{\tilde{C} h_\Omega^{\min(k+1,s)-\frac{3}{2}} |u|_{H^s(\Omega)} . \|u - u_h\|_{L^2(\Omega)}\right\} + C\|u - u_h\|^2_{L^2(\Omega)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Now,

$$\|\nabla. (u - u_h)\|^2_{L^2(\Omega)} \leq \left\{C h^{\min(k+1,s)-1} \|u\|_{H^s(\Omega)}\right\}^2$$
$$= C^2 h^{\min(2k+2,2s)-2} \|u\|^2_{H^s(\Omega)}$$
$$= C\, h^{\min(2k+2,2s)-2} \|u\|^2_{H^s(\Omega)}$$

[By considering the constant C^2 as C]

And

$$\|u - u_h\|_{L^2(\Omega)} \leq C h_\Omega^{min(k+1,s)-2} \|u\|_{H^s(\Omega)}$$

Again,

$$\|u - u_h\|^2_{L^2(\Omega)} \leq \{C h_\Omega^{min(k+1,s)-2} \|u\|_{L^2(\Omega)}\}^2$$
$$= C^2 h_\Omega^{min(2k+2,2s)-4} \|u\|^2_{L^2(\Omega)}$$
$$= C\, h_\Omega^{min(2k+2,2s)-4} \|u\|^2_{L^2(\Omega)}$$

[By considering the constant C^2 as C]

The last term of right-hand side of the equation (12) can be written as,

$$\int_{\Omega} c(u-u_h)^2 dx = c\|u - u_h\|^2_{L^2(\Omega)}$$

Substituting all of these values in (12), we get,

$$B(u - u_h, u - u_h) \leq -Ch_{\Omega}^{\min(2k+2,2s)-2}\|u\|^2_{H^s(\Omega)} + \left\{\tilde{C}h_{\Omega}^{\min(k+1,s)-\frac{3}{2}}\|u\|_{H^s(\Omega)}.Ch_{\Omega}^{\min(k+1,s)-2}\|u\|_{H^s(\Omega)}\right\} + cC\,h_{\Omega}^{min(2k+2,2s)-4}\|u\|^2_{L^2(\Omega)}$$

$$\leq \left\{-Ch_{\Omega}^{\min(2k+2,2s)-2} + cC\,h_{\Omega}^{\min(2k+2,2s)-4}\right\}\|u\|^2_{L^2(\Omega)} + \left\{C\tilde{C}h_{\Omega}^{\min(k+1,s)-\frac{3}{2}}.h_{\Omega}^{\min(k+1,s)-2}\right\}\|u\|^2_{H^s(\Omega)}$$

$$\leq Ch\,\|u\|^2_{H^s(\Omega)} + Ch_1\|u\|^2_{H^s(\Omega)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The constant terms is considered as,

$$-C + cC = C \text{ and } C\tilde{C} = C.$$

So, we can write,

$$\|u - u_h\|^2_{L^2(\Omega)} \leq Ch\,\|u\|^2_{H^s(\Omega)} + Ch_1\|u\|^2_{H^s(\Omega)}$$

If we apply our proposed DG norm, then

$$\|u - u_h\|^2_{L^2(\Omega)} \leq Ch\,\|u\|^2_{DG}$$

This completes the proof.

CONCLUSION

In this paper, we have investigated the error of the numerical solution by applying the Discontinuous Galerkin finite element method for the first order parabolic differential equation. We considered discontinuous Galerkin finite element approximations of a model scalar linear parabolic equation. It is a different and straightforward approach to seek error analysis from all other finite element schemes which are given in the literature. The technique used in this paper can also be extended to obtain the $L^2(\Omega)$ error estimate of the time dependent and higher order problems with the optimal order of convergence.

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