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Minimum Surface Area of a Cake with Bigger Portion in Cylindrical Shape and Remaining Portion in Shape of a Spherical Cap

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Abstract **Review Article**

A cake is in the shape of a cylinder extended by spherical cap. Volume of the spherical cap is calculated in terms of its depth. With given volume of the stuff in the cake its minimum surface area is determined. Maximum volume of a cylinder surmounted by a cone and maximum volume of cylinder surmounted an inverted hemisphere are separately also determined.

Keywords: spherical cap, cake, Cylindrical Shape.

INTRODUCTION

Many textbooks on Differential Calculus [1] deals with optimisation problems, viz, evaluation of minimum surface area and of maximum volume subject to some constraints. SN Maitra [2 to 6] published papers related to Lagrange's Multiplier. To begin with we solve a textbook problem of optimisation. A pyramid with vertex at (0, 0, C) is inscribed in an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{x^2}{c^2}=1 \qquad (1)$$

and the sides parallel to the elliptic section. Show that the maximum volume of the pyramid is $\frac{64}{81}c^3$.

Solution to the Problem

Let x, y, z be the coordinates of one corner of the pyramid base. Then (c-z) becomes its height. Then volume of the pyramid inscribed in ellipsoid [1] is

$$V=\frac{4}{3}xy(c-z) \qquad (2)$$

In order to find the maximum volume of the pyramid inscribed in the ellipsoid, drawing the relevant figure we choose function F and Lagrange's Multiplierλ such that using (1) and (2),

$$F=\frac{4}{3}xy(c-z)+\lambda(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}) \qquad (3)$$

$$\frac{\delta F}{\delta x}=\frac{4}{3}y(c-z)-\lambda\frac{2x}{a^2}=0 \qquad (4)$$

$$\frac{\delta F}{\delta y}=\frac{4}{3}x(c-z)-\lambda\frac{2y}{b^2}=0 \qquad (5)$$

$$\frac{\delta F}{\delta z}=\frac{4}{3}xy-\lambda\frac{2z}{c^2}=0 \qquad (6)$$

Multiplying (4), (5), (6) respectively by x, y, z and adding and then using (1) one gets

$$\frac{2}{3}xy(c-z)+\frac{2}{3}xy(c-z)-\frac{2}{3}xyz=\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{x^2}{c^2})$$

Or, $\lambda=\frac{4}{3}xyc-2xyz$ (7)

Substituting (7) into (4), one gets

$\frac{2}{3}y(c-z) = \frac{x}{a^2}(\frac{4}{3}xyc - 2xyz)$

Or, $\frac{x^2}{a^2} = \frac{c-z}{2c-3z}$ (8)

$\frac{y^2}{b^2} = \frac{c-z}{2c-3z}$ (9)

Adding (8) and (9) and using (1) we get

$\frac{2(c-z)}{2c-3z} + \frac{z^2}{c^2} = 1$

Or, $\frac{2(c-z)}{2c-3z} = 1 - \frac{z^2}{c^2}$

Or, $\frac{2(c-z)}{2c-3z} = \frac{c^2-z^2}{c^2}$

$\frac{2}{2c-3z} = \frac{(c+z)}{c^2}$

Or, $-3zc + 2c^2 - 3z^2 + 2zc = 2c^2$

Or, $3z^2 + zc = 0$

Or, $z = \frac{-c}{3}$ (10)

Which (negative sign) indicates that z-coordinate is below the centre of the sphere.

Because (10), equations (8) and (9) give

$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{4}{9}$ (11)

By virtue of (10) and (11), equation (1) gives the maximum volume of the inscribed pyramid as

$V_{max} = \frac{64}{81}abc$ (12)

As far as introduction is concerned, let us solve another textbook problem [1] herein: Find the minimum distance from point (3,4,15) to the cone $x^2 + y^2 = 4z^2$.

Solution the problem:

Let us consider a point (x, y, z) on the cone of given equation which is a constraint

$x^2 + y^2 = 4z^2$ (13)

The distance S of the point (3,4,15) to this point on the cone is given by

$S^2 = (x-3)^2 + (y-4)^2 + (z-15)^2$ (14)

Using (13) and (14) and Lagrange's Multiplier λ we form function F such that

F= $(x-3)^2 + (y-4)^2 + (z-15)^2 + \lambda(x^2 + y^2 - 4z^2)$

$\frac{\delta F}{\delta x} = 2(x-3) + 2\lambda x = 0$ (15)

$\frac{\delta F}{\delta y} = 2(y-4) + 2\lambda y = 0$ (16)

$\frac{\delta F}{\delta z} = 2(z-15) - 8\lambda z = 0$ (17)

The above four equations lead to

$\frac{x-3}{x} = \frac{y-4}{y} = \frac{z-15}{-4z} = -\lambda$ (18)

Or, $\frac{3}{x} = \frac{4}{y}$ Or, $y = \frac{4}{3}x$. From second equation of (18),

-4yz+16z=yz-15y

Or,16z=5yz-15y= 5y(z-3)

Y= $\frac{16z}{5(z-3)}$ and x=$\frac{12z}{5(z-3)}$ (19)

Substituting (19) into (13), one gets

$(\frac{16z}{5(z-3)})^2 + (\frac{12z}{5(z-3)})^2 = 4z^2$

Or, $(z-3)^2 = 4$

0r z=5 or 1 (20)

which due to (19) gives

Y= 8 and x=6 (21)

0r,

5Y=- 8 and 5x=-6 (22)

Value (20) suggests that the minimum distance will occur when z=5 and the maximum distance when z=1.

Hence putting the values of x, y and z from (21) and (20) in (14) is obtained the required minimum distance:
S^2= $(6-3)^2+(8-4)^2+(5-15)^2=125$
Or, S_{min} =$5\sqrt{5}$ (23)

Furthermore, some problems of maximization are done in textbooks of Calculus and in Google search such as maximization of volume /surface area of right circular cylinder and right circular cone inscribed in a sphere. In this feature an attempt has been made to solve some maximization problems, that have not yet appeared in textbooks or been published elsewhere, though, not that cumbersome.

MAXIMIZATION OF VOLUME OF A PRALLELOPIPED SURMOUNTED BY A PYRAMID INSCRIBED IN A SPHERE

Let 2x, 2y,2z be dimensions of the base and height of the parallelopiped surmounted by a pyramid of height h inscribed in a sphere of given radius R where x, y, z are the coordinates of one corner of the

parallelopiped with reference to the centre of the sphere as the origin. Then volume of this combination is given by
V=8xyz+$\frac{4xy}{3}$(R-z) (24)

subject to the constraint equation of the sphere
$x^2+y^2+z^2=R^2$ (25)

With the help of Lagrange's Multiplierλ we introduce Function F so that
F=8xyz+$\frac{4xy}{3}$ (R-z) +$\lambda(R^2-x^2-y^2-z^2)$ (26)
$\frac{\delta F}{\delta x}=\frac{20zy}{3}+\frac{4yR}{3}$ -2λx =0 (27)
$\frac{\delta F}{\delta y}=\frac{20zx}{3}+\frac{4xR}{3}$ -2λy =0 (28)
$\frac{\delta F}{\delta z}=\frac{20xy}{3}-2\lambda$z =0 (29)

Equating (28) and (27) is obtained
x=y (30)
Using (30) in (29) one gets
$\frac{10x^2}{3z}=\lambda$ (31)

Employing (31) and (30) in (27) or (28),
10z^2 +2zR=10x^2 (32)
Using (32) and (30) in (2) is obtained
$10x^2+10y^2+10z^2=10R^2$
0r, $30z^2+4zR=10R^2$
Or, $15z^2+2zR-5R^2=0$
z=$\frac{(\sqrt{76}-1)R}{15}=\frac{(7.72)R}{15}$ (33)

The optimum value of the height is
z_{opt}=.515R (34)

Because of (30), (34) and (25), we get optimum values of the dimensions of the base:
$2x^2=2y^2$ =R^2 -$z_{opt}{}^2$ = R^2 $-(.515)^2$ R^2 = $.7348R^2$
Ox=y=.607R
Or, x_{opt}=y_{opt}=$\frac{3R}{5}$, z_{opt}=$\frac{R}{2}$ (approximately) (35)
without loss of generality and subtelty.

*Applying (35) in (24) is ac*quired the maximum volume of the above combination of the parallelopiped and pyramid inscribed in the sphere:

$V_{max} = \frac{42}{25}R^3$ (36)

MAXIMUM VOLUME OF A CYLINDER SURMOUNTED BY A RIGHT CIRCULAR CONE INSCRIBED IN A SPHERE

Let a cylinder be surmounted by a cone of height h be inscribed in a sphere of radius R. In that case by geometry, the centre of the cylinder (mid point) coincides with the centre of the sphere. Let the common radius of the cylinder and cone be r. Then by geometry the height of the cylinder is 2(R-h).

$r^2 = 2Rh - h^2$ (37)

Volume V of the combination of the cylinder and cone due to (37) as depicted above is given by

V=$2\pi(2Rh-h^2)(R-h)+\frac{\pi}{3}(2Rh-h^2)h$

Or, V=$\frac{\pi}{3}(2Rh-h^2)(6R-5h)$ (38)

For maximum or minimum of V, we have

$\frac{\delta V}{\delta h}=\frac{\pi}{3}\{(6R-5h)(2R-2h)-5(2Rh-h^2)\}$

$\frac{\delta V}{\delta h}=\frac{\pi}{3}\{-5(2Rh-h^2)+2(6R-5h)(R-h)\}$

$=15h^2-32Rh+12R^2=0$

With $\frac{d^2V}{dh^2}<0$

Or, $\frac{h}{R}=\frac{32\pm\sqrt{32^2-720}}{30}=\frac{16\pm\sqrt{16^2-180}}{15}=\frac{16\pm\sqrt{76}}{15}=.485$

h=$\frac{R}{2}$ (approximately) (39)

which gives the value of h for maximum volume of the combination of the cylinder and the cone which on account of (38) manifests

$V_{max} = \frac{7\pi}{8}R^3$ (40)

MINIMUM SURFACE AREA OF A SPHERICAL CAPE

Let us consider a cake/bread in shape of cylinder surmounted by a spherical cape of height h cut off from a sphere of radius parallel to its diametric plane. At the outset we shall find out with the help of Calculus volume and surface area of the above cap. Let us consider a circular disc at a distance x of thickness dx from the centre of the sphere. Then radius r of the spherical cap by geometry is calculated as

$r^2 = 2Rh - h^2$ (41)

Volume of the cap is

$v_1=\int_0^h \pi(2Rx-x^2)dx$=R$h^2$- $\frac{h^3}{3}$ (41.1)

which due to elimination of R by use of (41.1) becomes

$v_1=\frac{hr^2}{2}+\frac{h^3}{6}$ (42)

In order to compute surface area of the spherical cap, let the line joining the centre of the sphere to the element dx ie let radius R make an angle dθ with the axis of the disk while $angle\ \theta$ varies from 0 to $\alpha\ to\ cover\ the\ cap\ whose$ surface area can be obtained as

$S_1=\int_0^\alpha 2\pi R sin\theta.R\,d\theta=\int_0^\alpha 2\pi R^2 sin\theta.d\theta= 2\pi R^2(1-cos\alpha)$ (43)

where $cos\alpha=\frac{R-h}{R}$ (44)

so that (43) reduces to

$S_1 = 2\pi Rh$ (By (41))

Or, $S_1=\pi(r^2+h^2)$ (45)

Using (42) and (45) the total volume V and surface area S of the cake are given by

V= $\pi(r^2h+\frac{hr^2}{2}+\frac{h^3}{6})$

Or, V== $\pi(\frac{3hr^2}{2}+\frac{h^3}{6})$ *(46)*

$S=2\pi rh+\pi(2r^2+h^2)$ (47)

Involving Lagrange's Multiplier λ and equation (42), function F can be written as

F=$\pi(2rh+2r^2+h^2)+\lambda\{V-\pi\left(\frac{3hr^2}{2}+\frac{h^3}{6}\right)\}$ (48)

$\frac{\delta F}{\delta r} = \pi(2h + 4r) - 3\pi\lambda rh = 0$

Or, $\lambda = \frac{2h+4r}{3rh}$ (49)

$\frac{\delta F}{\delta h} = \pi(2r + 2h) - \lambda(\frac{3r^2}{2} + \frac{h^2}{2}) = 0$

$\lambda = \frac{4(h+r)}{3r^2+h^2}$ (50)

Equating (49) and (50) we get

$\frac{2h+4r}{3rh} = \frac{4(h+r)}{3r^2+h^2}$

Or, $12rh^2 + 12r^2h = 6r^2h + 12r^3 + 2h^3 + 4rh^2$

Or, $6r^3 + h^3 - 4rh^2 - 3r^2h = 0$

Or, $6p^3 - 3p^2 - 4p + 1 = 0$ (51)

Where $p = \frac{r}{h}$

Or, $6p^2(p-1) + 3p(p-1) - (p-1) = 0$

Or, $(p-1)\{6p^2 + 3p - 1\} = 0$

Or, $p=1$, $p = \frac{-3\pm\sqrt{9+24}}{12} = \frac{1}{4}$(approx)

Or, $h=r$, $h = \frac{r}{4}$ (52)

In view of (45), (52) suggests that the minimum surface area occurs when

$h = \frac{r}{4}$ (53)

and is given by

$S_{min} = \frac{41}{16}\pi r^2$ (54)

the maximum surface area when

$h=r$ (55)

$S_{max} = 5\pi r^2$ (56)

MAXIMUM VOLUME OF A CYLINDER SURMOUNTED BY AN INVERTED HEMISPHERE INSCRIBED IN A SPHERE

If r is the common radius of the cylinder and the hemisphere inscribed in a sphere of radius R, then height of the cylinder is, by geometry,

$h = \sqrt{R^2 - r^2} + R - r$ (57)

Hence volume of the inscribed cylinder surmounted by the hemisphere is

Or, $V = \pi r^2(\sqrt{R^2 - r^2} + R - r) + \frac{2\pi r^3}{3}$ (58)

Or, $V = \pi r^2(\sqrt{R^2 - r^2} + R) - \frac{\pi r^3}{3}$

For maximum/minimum of V, we can write

$\frac{\delta V}{\delta r} = 2\pi r\left(\sqrt{R^2 - r^2} + R\right) - \pi r^2 - \frac{1}{\sqrt{R^2-r^2}}\pi r^3 = 0$

Or, $2(R^2 - r^2) - r^2 + (2R - r)\sqrt{R^2 - r^2} = 0$

Or, Or, $(2R^2 - 3r^2) + (2R - r)\sqrt{R^2 - r^2}) = 0$

Squaring both sides, one gets

$(2R^2 - 3r^2)^2 = \{(2R - r)\sqrt{R^2 - r^2})\}^2$

Or, $9r^4 - 12R^2r^2 + 4R^4 == (4R^2 - 4Rr + r^2)(R^2 - r^2)$

Or, $9r^4 - 12R^2r^2 + 4R^4 == 4R^4 - 4R^3r + R^2r^2 - 4R^2r^2 + 4Rr^3 - r^4$

$10r^3 - 9R^2r + 4R^3 - 4Rr^2 = 0$ (59)

$10q^3 - 4q^2 - 9q + 4 = 0$ (60)

Where $q = \frac{r}{R} < 1$. Adopting a process of approximation, we can solve this cubical equation assuming without loss of generality and sufficient accuracy, for which let

$q = 1 - \mu$, $\mu \ll 1$ (61)

such that (60) reduces to the form

$10(1-\mu)^3 - 4(1-\mu)^2 - 9(1-\mu) + 4 = 0$ (62)

Or, expandinμg Binomially and neglecting square and other higher powers of μ is obtained

$10(1-3\mu)-4(1-2\mu)-9(1-\mu)+4=0$

0r, $\mu=\frac{1}{13}$

In consequence of which (61) gives

r=$\frac{12}{13}R$ (63)

which on substitution in (58) leads to the maximum volume

V=$\pi r^2(\sqrt{R^2-r^2}$ +R)$-\frac{\pi r^3}{3}$

V=$\pi(\frac{12R}{13})^2(\sqrt{R^2-(\frac{12R}{13})^2}$ +R)$-\frac{\pi(\frac{12R}{13})^3}{3}$

$=\pi\frac{144}{169}(\frac{18}{13}-\frac{12}{39})R^3$

$=\pi\frac{144.}{169}\frac{42}{39})R^3$

$V_{max}=\pi\frac{2016}{2177}R^3$ (64)

MIXIMUM SURFACE AREA OF A CYLINDER SURMOUNTED BY AN INVERTED HEMISPHERE INSCRIBED IN A SPHERE

In the light of equation (57), surface area S of the cylinder surmounted by an inverted hemisphere is given by

S=$2\pi r(\sqrt{R^2-r^2}$ +R -r) + $2\pi r^2+\pi r^2$=$2\pi r(\sqrt{R^2-r^2}$ +R) +πr^2 (65)

For maxima or minima of S,

$\frac{dS}{dr}$ =$2\pi\{(\sqrt{R^2-r^2}$ +R)$-\frac{r^2}{\sqrt{R^2-r^2}}+r\}$

$=2\pi\ (R^2-2r^2+(R+r)\sqrt{R^2-r^2})$ =0 (66)

$\frac{d^2S}{dr^2}=2\pi\ (-4r+\sqrt{R^2-r^2}-\frac{r(r+R)}{\sqrt{R^2-r^2}})$

$=2\pi\ (-4r+\sqrt{R^2-r^2}-\frac{r(r+R)}{\sqrt{R^2-r^2}}$

$=\frac{2\pi}{\sqrt{R^2-r^2}}\{-4r\sqrt{R^2-r^2}+(R^2-r^2)-r(r+R)\}$

$=\frac{2\pi}{\sqrt{R^2-r^2}}\{-4r\sqrt{R^2-r^2}+(R^2-2r^2)-rR)\}<0$ (67)

From (66) is obtained

$R^2-2r^2=-(R+r)\sqrt{R^2-r^2}$

Squaring both sides and simplifying one gets

$R^4-4r^2R^2+4r^4=(R^2+2Rr+r^2)(R^2-r^2)$

Or, $5r^4+2r^3R-4r^2R^2-2rR^3=0$

$5r^3+2r^2R-4rR^2-2R^3=0$ (68)

Denoting $\mu=\frac{r}{R}<1$, leading to $\mu=1-\epsilon,\epsilon>0$ (69)

$5\mu^3+2\mu^2-4\mu-2=0$

Or, $5(1-\epsilon)^3+2(1-\epsilon)^2-4(1-\epsilon)-2=0$ (70)

Expanding binomially neglecting square and other higher powers of ϵ,we get

$5(1-3\epsilon)+2(1-2\epsilon)-4(1-\epsilon)-2=0$

Or, $\epsilon=\frac{1}{15}$ (71)

Or, by virtue of (69), we can write

$\mu=\frac{14}{15}$ Or, r= $\frac{14}{15}R$ for which $\frac{d^2S}{dr^2}<0$ (72)

in consequence of which, by use of (65), is found the maximum surface area of the combination:

$S_{max}=\frac{28}{225}\pi(\sqrt{29}+1)R^2+\pi(\frac{14}{15}R)^2$

$=\frac{28}{225}\pi\{(\sqrt{29}+99)R^2$

$=\frac{196}{225}\{(\sqrt{29}+99)R^2$

$S_{max}=13\pi R^2\ (approx)$ (73)

MAXIMUM SURFACE AREA OF A PYRAMID INSCRIBED IN A SPHERE

Let (x, y, z) be the coordinates of one corner of the base of the pyramid with respect to the centre of a sphere of radius R as the origin, resulting in height (R-z) of the pyramid inscribed in the sphere of radius R. Then surface area S of the pyramid is given by

$$S=4xy+2x\sqrt{y^2+(R-z)^2}+2y\sqrt{x^2+(R-z)^2} \quad (74)$$

For maximum or minimum of S, choosing function F and Lagrange's Multiplierλ,

$$F=4xy+2x\sqrt{y^2+(R-z)^2}+2y\sqrt{x^2+(R-z)^2}+\lambda(R^2-x^2-y^2-z^2) \quad (75)$$

$$\frac{\delta F}{\delta x}=4y+2\sqrt{y^2+(R-z)^2}+\frac{2xy}{\sqrt{x^2+(R-z)^2}}-2x\lambda=0 \quad (76)$$

$$\frac{\delta F}{\delta y}=4x+2\sqrt{x^2+(R-z)^2}+\frac{2xy}{\sqrt{y^2+(R-z)^2}}-2y\lambda=0 \quad (77)$$

Combining and rationalizing (75) and (76) or by accounting for symmetry is obtained

$$x=y \quad (78)$$

$$\frac{\delta F}{\delta z}=-\frac{2x(R-z)}{\sqrt{y^2+(R-z)^2}}-\frac{2y(R-Z)}{\sqrt{x^2+(R-z)^2}}-2z\lambda=0$$

which because of (78) becomes

$$-\frac{2x(R-Z)}{Z\sqrt{x^2+(R-z)^2}}=\lambda \quad (79)$$

Using (78) and (79) in (77), we get

$$4x+2\sqrt{x^2+(R-z)^2}+\frac{2x^2}{\sqrt{x^2+(R-z)^2}}+2\frac{2x^2(R-Z)}{z\sqrt{x^2+(R-z)^2}}=0$$

Or, $2x\sqrt{x^2+(R-z)^2}+x^2+(R-z)^2+x^2+\frac{2x^2(R-Z)}{z}=0$

Or, $2xz\sqrt{x^2+(R-z)^2}+2x^2z+z(R-z)^2+2x^2(R-z)=0$

Or, $2xz\sqrt{x^2+(R-z)^2}+z(R-z)^2+2x^2R=0$

$Or, -2xz\sqrt{x^2+(R-z)^2}=z(R-z)^2+2x^2R$

Or,$4x^2z^2\{x^2+(R-z)^2\}=z^2(R-z)^4+4x^4R^2+4z(R-z)^2x^2R$

$0r,\ 4x^4(R^2-z^2)+z^2(R-z)^4+4z(R-z)^2x^2(R-z)=0$

$Or,\ ,\ 4x^4(R+z)+z^2(R-z)^3+4z(R-z)^2x^2=0$

$Or,\ x^2=\frac{-4z(R-z)^2\pm\sqrt{\{4z(R-z)^2\}^2-16z^2(R+z)(R-z)^3}}{8(R+z)}$

$$=\frac{-z(R-z)^2\pm z(R-z)\sqrt{(R-z)^2-(R+z)(R-z)}}{2(R+z)}=y^2 \quad (80)$$

$$=\frac{z(R-z)\{-(R-z)\pm\sqrt{2z^2-2Rz}\}}{2(R+z)}$$

Recalling the constraint equation and in consequence of (78)

$$x^2+y^2+z^2=R^2 \quad (81)$$

$$z^2=R^2-\frac{z(R-z)\{-(R-z)\pm\sqrt{2z^2-2Rz}\}}{(R+z)}$$

Or, $\frac{-z(R-z)\pm z\sqrt{2z^2-2Rz}}{(R+z)}=(R+z)$

Or, $\pm z\sqrt{2z^2-2Rz}=3Rz+R^2$

Or, $(2z^2-2Rz)z^2=R^4+6R^3z+9R^2z^2$

Or,$2z^4-2Rz^3-9R^2z^2-6R^3z-R^4=0$

Or, $2(R+z)z^3-4Rz^2(R+z)-5zR^2(R+z)-R^3(R+z)=0$

$$2z^3-4Rz^2-5zR^2-R^3=0 \quad (82)$$

Putting z=-R, L.H.S of (82) gives $-2R^3$, $putting\ z=\frac{-R}{2}$,it gives $\frac{1}{4}$, with z= $\frac{-R}{3}$ it gives $\frac{4}{27}$, with z= -$\frac{R}{4}$, it gives $\frac{-1}{32}$, (83) and with z=-$\frac{R}{5}$, it gives -$\frac{22R}{125}$ and as such we can assume

$$z=\frac{-R}{4}+\mu \quad (84)$$

where μ is a small <<R (82), after expanding binomially and neglecting the squares and other higher powers of the small terms is obtained

$$2\left(\frac{-R}{4}+\mu\right)^3-4R\left(\frac{-R}{4}+\mu\right)^2-5\left(\frac{-R}{4}+\mu\right)R^2-R^3=0$$

Or, $2(\frac{-R}{4})^3\left(1-\frac{12\mu}{R}\right)-4R(\frac{R}{4})^2\left(1-\frac{8\mu}{R}\right)+\frac{5}{4}R^3\left(1-\frac{4\mu}{R}\right)-R^3=0$

Or, $2\left(1-\frac{12\mu}{R}\right)+16\left(1-\frac{8\mu}{R}\right)-80\left(1-\frac{4\mu}{R}\right)+64=0$

Or, $\left(1-\frac{12\mu}{R}\right)+8\left(1-\frac{8\mu}{R}\right)-40\left(1-\frac{4\mu}{R}\right)+32=0$

Or, $\frac{84\mu}{R}=1$

Or, $\mu=\frac{R}{84}$ (85)

Substituting this value in (84) is obtained

$z=\frac{-R}{4}+\mu=\frac{-5R}{21}$ (86)

Hence the optimum height of the pyramid is given by

$h=R-z=\frac{26R}{21}$ (87)

By use of (87) and (86) in (80), is determined the optimum dimensions of the base of the pyramid ie

$x^2(\text{opt})=y^2(\text{opt})=\frac{z(R-z)\{-(R-z)\pm\sqrt{2z^2-2Rz}\}}{2(R+z)}$

$=\frac{-5R}{21}\cdot\frac{26R}{21}\cdot\frac{\{-\frac{26R}{21}\pm\sqrt{2(\frac{5R}{21})^2+2R\frac{5R}{21}}\}}{2(R-\frac{5R}{21})}$

$=-\frac{130R^2}{441}\frac{\{-\frac{26R}{21}\pm\sqrt{2(\frac{5R}{21})^2+2R\frac{5R}{21}}\}}{\frac{32R}{21})}$

$=\frac{-130R^3}{441}\frac{\{-\frac{26}{21}+\frac{\sqrt{260}}{21}\}}{\frac{32R}{21}}$

$=\frac{-130R^3}{441}\frac{\{-\frac{26}{21}+\frac{16}{21}\}}{\frac{32R}{21}}$ (approximately)$=\frac{1300R^2}{441}\frac{1}{32}$

$=\frac{325R^2}{441}\frac{1}{8}$

$=\frac{325R^2}{3528}$ (88)

$x^2(\text{opt})=y^2(\text{opt})=\frac{2R^2}{23}$

Or, x(opt)=y(opt)= $\frac{2R}{7}$ (approximately) (89)

Another value

$x^2(\text{opt})=\frac{65R^2}{168}=\frac{5R^2}{13}$

Or, x(opt)= y(opt)= $\frac{5R}{8}$ (approximately) (90)

without sacrifice of sufficient accuracy and generality.

Substituting (89) and (90) in (74) consecutively are obtained First value of S is with simplicity and without sacrifice of the sufficient accuracy

Local$S_{max}=4x^2+4x\sqrt{y^2+(R-z)^2}$

$=(\frac{196}{529}+\frac{28}{23}\sqrt{(\frac{7}{23})^2+(\frac{26}{21})^2})\,R^2$ (91)

Second value of S is with simplicity and without sacrifice of the sufficient accuracy:

Actual $S_{max}=4x^2+4x\sqrt{x^2+(R-z)^2}$

$=(\frac{25}{16}+\frac{5}{2}\sqrt{(\frac{5}{8})^2+(\frac{26}{21})^2})\,R^2$ (92)

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