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Circling a Regular Pentagon with Straightedge and Compass in Euclidean Geometry

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Abstract **Original Research Article**

"Circling The Regular Pentagon" is possibly used to describe a new challenge problem that comes from the exact solution for another new challenge problem "Circling the SQUARE", published in January 2024 in the International Journal of Mathematics Trends and Technology (IJMTT). "Circling The SQUARE" means constructing a circle that has the exact area of the given square". Therefore, the "Circling the Regular Pentagon" problem challenge has not existed in the mathematics field until it is solved and published nowadays. This paper, then, provides an exact solution to construct a circle that has the same area as a given regular pentagon. The solution does not use the number π and suits the exact constraint of Euclidean Geometry by straightedge & compass.

Keywords: Circling Pentagon, Regular Pentagon Circled, Rounding a Pentagon, Circulating Regular Pentagon, Circle Mature 0f Pentagon, Make Regular Pentagon Circled, Find Circle Area Same as Regular Pentagon, Make Regular Pentagon Rounded.

1. INTRODUCTION

Has the "*Circling the Regular Pentagon*" problem in Mathematics ever existed? No, the current "Circling The Regular Pentagon" problem in mathematics refers to the challenge of constructing a circle that has the same area as a given regular pentagon, using only a straightedge & a compass. This problem isn't a classic geometric construction task that uses the relationship properties of a regular pentagon and a circle of the same area. It is answered that the "Circling the Regular Pentagon" problem with straightedge & compass in the Mathematics field had not existed before this article paper published. However from now on this paper creates and solves the problem, exactly & accurately. "Circling The Regular Pentagon" is spontaneously a new research title that, at the same time arose from the exact solution to the recent "Circling The Square" problem published in January 2024 [2]. The way of posing this Mathematics problem is based on "it is certainly that people possibly construct a circle with an area exactly equal to the area of a given square" [2]. Because a square and a regular pentagon are both regular polygons, the proposed problem is not far from reality.

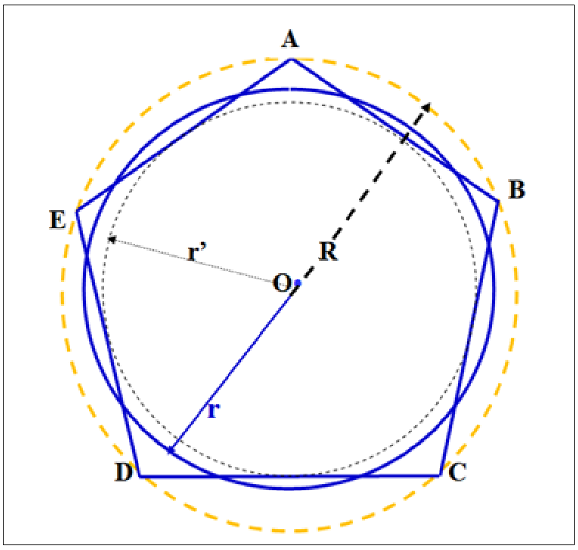

Figure 1: ***General Ideas & Feeling.***

And then, the research study found a new shape formed by a circle and an outside angle having its bisector go through the circle centre. The shape can be named "Conical-Arc" and this shape area is the same as the partial cut of the regular pentagon, exactly. There are all 5 partial cuts, mentioned circle segments. These circle segments are due to the hypothetical circle cutting 5 segments of the given pentagon, according to the Geometry Analysis method of Euclidean geometry. The hypothetical circle is concentric to the given regular pentagon and also intersects 5 sides of the pentagon. It is emphasized that this study research does not use the irrational number π in its construction by straightedge & compass.

Finally, it is also proved that this study research found the radius of the hypothetical circle for constructing the circle with a straightedge & compass and this circle is the exact solution to the "Circling the Regular Pentagon" problem.

2. PROPOSITION

2.1 Theorem 01: "*Regular Decagon Construction*"

Given a pentagon inscribed in a circle (O, r), r $\subset \mathbb{R}$, then the inscribed decagon of the circle is constructed accurately, by straightedge & compass.

PROOF

Given a pentagon ABCDE, side a, to be inscribed in a circle (O, r). With a straightedge & a compass, one can identify the midpoints I, J, K, L & M of 5 sides of this inscribed pentagon (Figure 02 below). Then 5 radius of the circle *(marked by 5 red arrows in Figure 02 below)* cut its circumference at 5 points P, Q, R, S & T (Figure 02 below). These points P, Q, R, S & T are also the midpoints of the 5 arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$ & $\overset{\frown}{EA}$ of the circle (O, r). These midpoints P, Q, R, S & T show that the 10 arc chords AP, PB, BQ, QC, CR, RD, DS, SE, ET & TA are equal, i.e.

AP = PB = BQ = QC = CR = RD = DS = SE = ET = TA

Therefore, APBQCRDSET is the regular decagon (inscribed in the given circle), that is constructed (Figure 02 below).

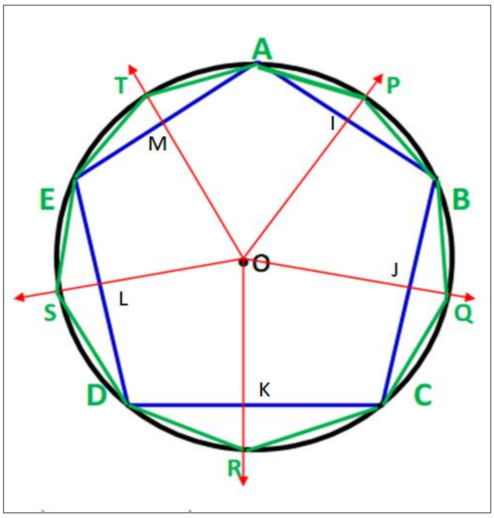

Figure 2: ***Constructive Method for regular decagon using a straightedge & a compass***

2.2 Definition 1: "Conical-Arc" shape

Given a circle (O, r) and an angle $\widehat{BAC}$ with its vertex outside the circle such that the bisector of the angle passes through the centre O of the circle, then the special shape formed by the 2 sides of the angle and arc $\overset{\frown}{DE}$ can be called a Conical-Arc (in Figure 03 below, the red shape ADE is a Conical-Arc). If $\widehat{BAC}$ is a right angle then the shape ADE is called a Right-Conical-Arc.

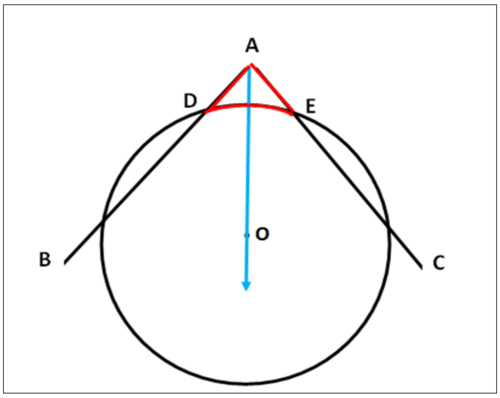

Figure 3: ***Conical-Arc shape (red colour)***

2.3 Theorem 03: *Analysis Theorem/Inverse Problem*

If there exists a circle (O, r) of which area πr^2 is equal to area A of a given regular pentagon ABCDE and this circle is concentric to the centre O of the pentagon, then

a. Circle (O, r) locates in between the circumscribed circle (O, R) & the inscribed circle (O, r') of the given regular pentagon ABCDE;
b. Circle (O, r) and the given regular pentagon ABCDE form 5 Conical-Arc shapes and 5 circle segments;
c. The 5 Conical-Arcs, which are formed by the circumference of the circle (O, r) and the pentagon ABCDE at 5 vertices A, B, C, D, E, are equal. And also, the 5 circle segments of the circle (O, r) are equal.

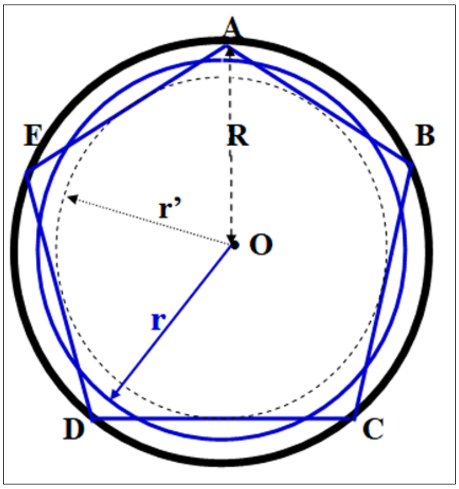

Figure 4: ***Locations of the concentric shapes: the given regular pentagon, the circles (O, R), (O, r), and (O, r')***

PROOF

a. Assume a concentric circle (O, r) with the same area A as the given regular pentagon ABCDE (blue colour in the above Figure 04), then A must be less than the area of the circumscribed concentric circle (O, R) of the given pentagon. And also, A is larger than the area of the concentric inscribed circle (O, r') of the given pentagon (Figure 04, above). These facts show that the circle (O, r) is located in between the (O, R) and the (O, r').

b. By Section a. above, the concentric circle (O, r) has to intersect with 5 sides of the given pentagon at 10 points to form 5 circle segments (Figure 04, above). These 5 circle segments of the circle (O, r) are attached to the 5 sides of the given regular pentagon ABCDE (Figure 04, above).

c. Consider 5 Conical-Arcs at 5 vertices A, B, C, D & E formed by the concentric circle (O, r) and the given regular pentagon. This concentric property shows the areas of these 5 Conical-Arcs are equal *(marked by the red arrows in Figure 05 below)*. The concentric property of the given pentagon and the circle (O, r) also shows that the areas of the 5 circle segments (blue circle segment in Figure 05 below) are equal.

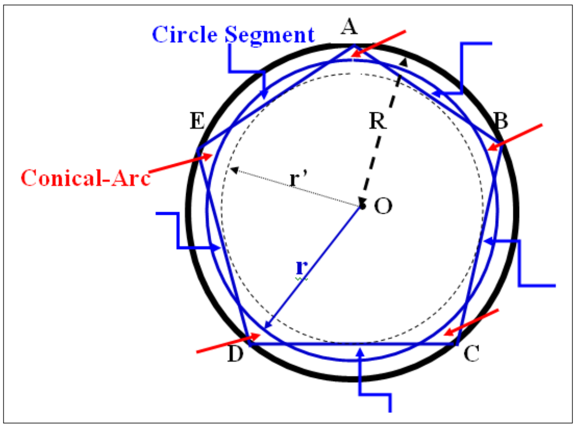

Figure 5: ***Five equal Conical-Arcs (marked by red arrows) and 5 equal circle segments (blue colour arrows)***

3.3 Theorem 04: *Core Theorem*

If there exists a circle (O, r) of which area πr^2 is equal to area A of a given regular pentagon ABCDE and this circle is concentric to the centre O of the pentagon, then

a. Areas of 5 Conical-Arcs and 5 circle segments, formed by the circle and the pentagon, are equal;

b. There exists a regular decagon that inscribes in the given regular pentagon ABCDE;

c. The circle's radius r is identified by the distance length from the centre O of the given regular pentagon ABCDE to any vertex of the regular decagon abcdefghij.

PROOF

Given a regular pentagon ABCDE. Assume there exists a circle (O, r) with area πr^2 equal to area A and concentric with the given pentagon. By Theorem 03 above, five equal Conical-Arcs and five equal circle segments are formed by the circle (O, r) and the given pentagon ABCDE (Figure 06 below).

a. By Theorem 03 above, five areas of the five circle segments, formed by the circle (O, r) and the pentagon at 5 sides AB, BC, CD, DE & EA of ABCDE, are equal. Also, five areas of the five Conical-Arcs at vertices A, B, C, D & E are equal.

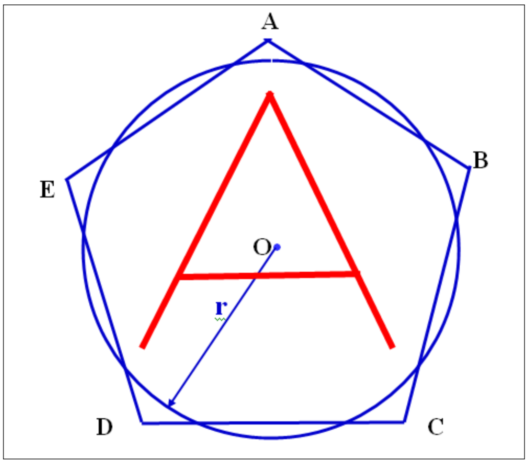

Figure 6: ***The intersecting area named with the*** **red capital letter A as "AreaA"**

Then consider the intersecting area of the given pentagon ABCDE and the circle (O, r), which is marked by the red capital letter A and named "AreaA" (Figure 06, above). And then,

"AreaA"+5 areas of the 5 circle segments = Area πr^2 of the Circle (O, r) (1)
"AreaA" + 5 areas of the 5 Conical-Arcs = Area of the given pentagon (2)

By the assumption of this Theorem 03 - (area πr^2 is assumed to be equal to the area A of the given regular pentagon ABCDE) – one gets expression in (1) is equal to expression in (2) above, to result:
"AreaA" + 5 areas of the 5 circle segments = **"AreaA"** + 5 areas of the 5 Conical-Arcs (3)

By Section c. of Theorem 03 above, (3) shows that "Areas of 5 Conical-Arcs and 5 circle segments, formed by the circle and the pentagon, are equal", as required.

a. Consider the Conical-Arc Ehi in Figure 07 above, then lengthen the arc chord ih of the circle (O, r) that cut the circumscribed circle (O, R) at D' & E' (yellow colour). Then connect E' to O to make the symmetric axe E'O (red colour in Figure 07 above). And then, draw a line segment E'A', which is symmetric to D'E' via the axe EO (yellow colour in Figure 07 below). Because D'E' and E'A' are symmetric and the arc chords hi & ja are also symmetric then E'A' has to overlap the arc chord ja (Figure 07, below).

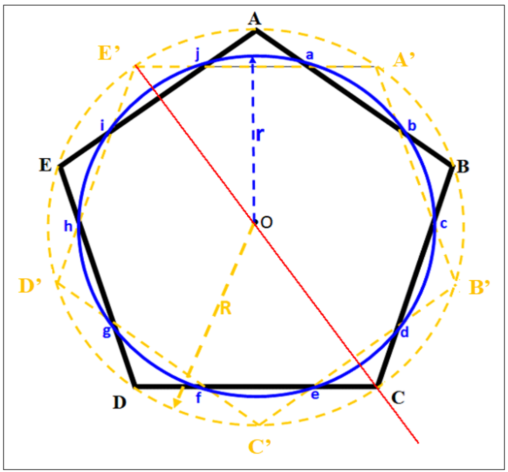

Figure 7: ***Two regular pentagon ABCDE & A'B'C'D'E' of the same area and inscribing in the circle (O, R) and decagon* abcdefghij**

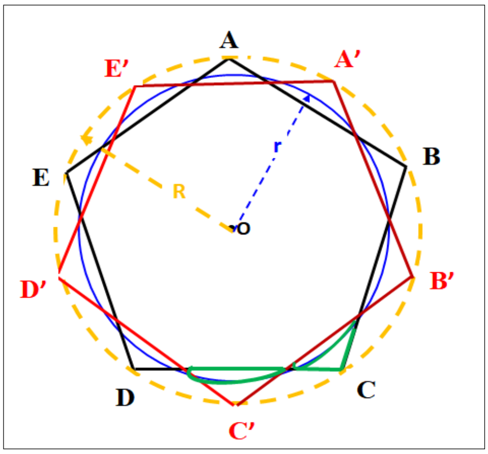

Figure 8: ***Five Conical-Arcs at A, B, C, D & E and five circle segments (black & blue colour) formed by the given pentagon ABCDE and circle (O, r).***

Similar constructions to the above description imply line segments A'B', B'C' & C'D' are all the arc chords of the circle (O, R) as same as the arc chords D'E' & E'A'. Then these 5 arc chords form the regular pentagon A'B'C'D'E' (dashed yellow pentagon in Figure 07 above) which is inscribed in the circle (O, R) as same as the given pentagon ABCDE (black colour in Figure 07 above). Therefore, the areas A& a of these pentagons ABCDE & A'B'C'D'E' are equal or the same.

Area A = Area a (4)

Then, by the assumption in this section b., both pentagons ABCDE & A'B'C'D'E' share the same assumed circle (O, r), which has the area $\pi r^2 = A = a$. In other words, the area of the assumed circle (O, r) is also equal to the area a of the pentagon A'B'C'D'E'. These pentagons ABCDE & A'B'C'D'E' and the assumed circle (O, r) share the decagon abcdefghij (Figures 07 & 08 above). This decagon abcdefghij is also inscribed in the circle (O, r), the given regular pentagon ABCDE and the constructed pentagon A'B'C'D'E'. In other words, all the 3 shapes, the circle (O, r), the given regular pentagon ABCDE, and the constructed pentagon A'B'C'D'E' share the same decagon abcdefghij as an inscribed shape, commonly. This show that 5 circle segments AB, BC, CD, DE & EA attached to the sides of pentagon ABCDE and 5 circle segments A'B', B'C', C'D', D'E' & E'A' attached to the sides of pentagon A'B'C'D'E' are equal. Then this equality of these 10 circle segments shows the 10 arc chords ab, bc, cd, de, ef, fg, gh, hi, ij & ja are equal (Figures 07 & 08 above). Therefore the decagon abcdefghij is a regular decagon, as required.

a. The above decagon abcdefghij is a commonly inscribed shape in the pentagon ABCDE and the circle (O, r), where all its 10 vertices a, b, c, d, e, f, g, h, i, & j are located in the 5 sides of the given pentagon (Figures 07 & 08 above and Figure 08a below). This decagon, therefore, is constructible with a straightedge & compass in the given pentagon, concentrically. Then, the distance from the centre O to 1 of the 10 vertices a, b, c, d, e, f, g, h, i, & j, is the exact length r, as required.

Note that the proof of Theorem 04 (Core Theorem) also shows that the regular decagon abcdefghij *in Figures 07 & 08 above & Figure 08a below, has 5 sides overlapping the given regular pentagon ABCDE.*

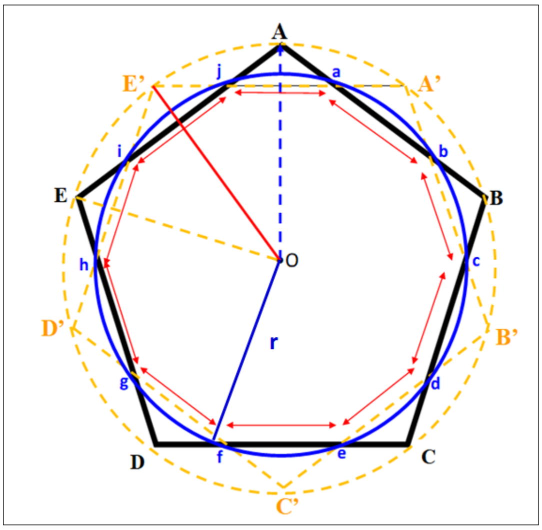

Figure 8a: ***Two regular pentagons ABCDE & A'B'C'D'E' form a regular decagon* abcdefghij**

(marked by 2-ended red arrows) inscribing in the circle (O, r).

3.4 Construction of the Resulting Circle:

Given a regular pentagon ABCDE with area A, then by the sections 3.1, 3.2 & 3.3 above, a straightedge and a compass are used to construct the circle that has the same area A as the pentagon, as follows:

Method 1

- Use the compass to take length R from the centre O of the pentagon to any of its 5 vertices A, B, C, D & E, then draw a circle (O, R), marked orange colour in Figure 09 below. This circle (O, R) is the circumscribed circle of the pentagon (Figure 09 below).
- Connect AO and extend AO to meet circle (O, R) at C', then from C' construct the red pentagon A'B'C'D'E' inscribed in (O, R) with compass & straightedge (Figure 09 below). Five sides of this red pentagon intersect with 5 sides of the given pentagon ABCDE at 10 points. Those 10 points identify a decagon that is inscribed in both pentagons (red & black in Figure 09 below). Let one of these 10 points be X.
- Connect OX (green colour in Figure 09) to get length OX = r, which is the radius of the resulting circle (O, r) in the following Figure 09.

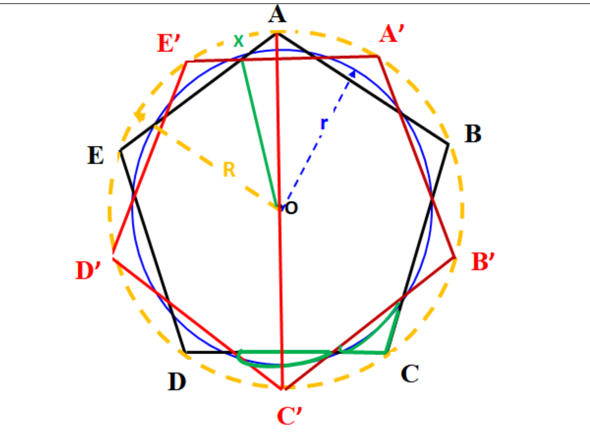

Figure 09: *Length OX = the required radius r to construct the resulting circle (O, r) with straightedge & compass.*

Method 2

For a given regular pentagon ABCDE (red colour in Figure 10 below), then use a straightedge & a compass to draw an inscribed decagon abcdefghij of the given pentagon (Figure 10 below). This red decagon identifies two circles which are its inscribed circle (O, r') and its circumscribed circle (O, r). The circumscribed circle (O, r) is the resulting circle, as required (Figure 10 below).

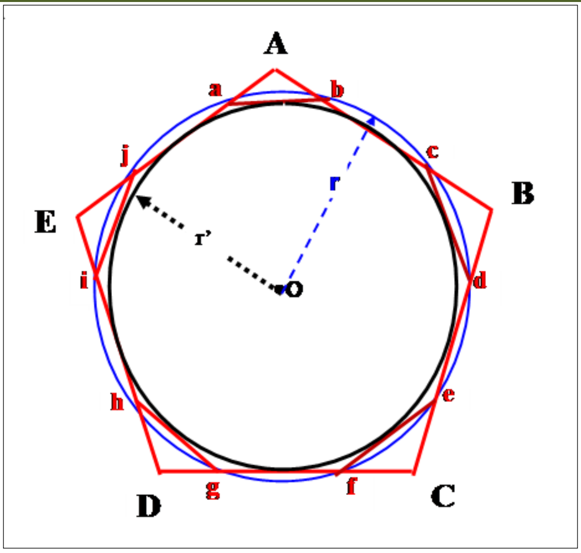

Figure 10: ***The inscribed decagon* abcdefghij *of the given pentagon* ABCDE. *The resulting circle (O, r) is marked blue colour.***

4. DISCUSSION & CONCLUSION

The Analysis Method *or* the inverse problem ***or*** the last-up backward research method is a good solution for complex problems involving a lot of logic. Complex and logical here means that the data of the problem can constantly change and be closely related to each other.

It is obvious that the problem "Circling the Regular Pentagon" did not ever exist in the Mathematics field till 2024 when I solved the newly challenged problem "Circling the Square" [2]. This "Circling the Square" problem came from my successful study research to construct exactly a circle with the same area as a given square without any difficulty from the irrational Piπ, only using a straightedge & compass. It also shows the impossibility of solving exactly the 3 ancient Greek challenges (Trisecting an Angle, Doubling the Cube, and Squaring the Circle) with straightedge & compass no longer exists [3-5]. "Circling The Regular Pentagon" is a topic related to geometry, specifically focusing on the geometric properties and constructions involving regular pentagons inscribed in circles or vice versa. While there may not be an extensive body of research specifically titled "Circling the Regular Pentagon," there are likely NO mathematical studies, papers, and articles that explore various aspects of this topic within the broader field of geometry and mathematical constructions. Research in this area might cover topics such as:

1. Geometric properties of regular pentagons and circles.
2. Constructions involving regular pentagons inscribed in circles or circumscribed around circles.
3. Relationships between the side length of a regular pentagon and the radius of the circumscribing or inscribing circle.

4. Applications of regular pentagons and circles in architecture, art, or other fields.

Can mathematicians use a compass and a straightedge to construct a circle having the same area as a given regular pentagon exactly/accurately? This question is the same as for constructing the mentioned circle or finding an accurate solution for the new challenge Mathematics problem "CIRCLING THE REGULAR PENTAGON". Surprisingly, only I, myself, have been still working on this question because the challenge problem had just arisen contemporarily when I ended my original research article *"Circling The Square With Straightedge and Compass in Euclidean Geometry"*, published by IJMTT in January 2024 [2].

The "Analysis" Method is applied correctly to Geometry to complete this research study to gain an exact/accurate solution to this new challenge "CIRCLING THE REGULAR PENTAGON" problem in Mathematics.

The results of my independent research show that the correct answer (O, r), constructed by compass and straightedge, has the area πr^2 = *area A of the given regular pentagon*, therefore if the given regular pentagon to circle is a unit shape, volume A = 1, then in terms of geometry, π can be constructive/expressed by a circle with area $\pi r^2 = 1$. This circle comes from the *RULER of the "CIRCLING THE REGULAR PENTAGON" problem* yielded by the given unit pentagon. Subsequently, the exact geometric length of $\pi = \frac{1}{r^2}$ was determined. In practice, if the *International Bureau of Weights and Measures (BIPM)*, the *International System of Units*, or any accurate laser measurement is used to measure the arithmetic value r of the answer circle (O, r), we can use this r to measure as accurately as possible to obtain the arithmetic value of $\pi = \frac{1}{r^2}$.

The above arithmetic value π could be the nearest arithmetical value of the irrational number π□ ever seen.

My construction method is quite different from approximation and is based on using a straightedge and compass within the Euclidean Geometry. Moreover, this method shows that the value $r = \frac{\sqrt{\pi}}{\pi}$ can be expressed accurately, and the value $\pi = \frac{1}{r^2}$ or $\pi = \frac{a^2}{r^2}$, $a \subset \mathbb{R}$ can also be expressed accurately in terms of Geometry. This Geometrical expression of the irrational number π could be an interesting field for mathematicians in this 21st century. In other words, algebraic geometry can express exactly any irrational number kπ, $k \subset \mathbb{R}$.

In addition, this research result can be used for further research in the topics "CIRCLING THE EQUILATERAL TRIANGLE", "CIRCLING THE REGULAR HEXAGON", "CIRCLING THE REGULAR HEPTAGON", "CIRCLING THE REGULAR OCTAGON" etc …, using only a straightedge and compass in Euclidean Geometry.

Therefore, the question "Does there exist any other research on the topic title "Circling the Regular Pentagon?" is answered that "No, there is not, except in this article paper".

5. Conflicts of Interest: The author declares that there is no conflict of interest regarding the publication of this paper.

6. Funding Statement: No funding from any financial bodies for this research.

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