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Generating Future and Present Values of Annuity Using Interest Rate Theory

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Abstract **Review Article**

The present value of an annuity is the current value of the future payments from an annuity, given a specified rate of discount. The Future value of an annuity is the estimated value of current asset at specific time in the future based on an assumed growth rate. The future and present values of annuity were mathematically generated on the basis of interest rate theory. This was anchored on the fact that every compound interest problem involves the annual rate and the rate per compounding period. The derivations of both present and future values of annuity were established with real life applications. It was also shown that as the frequency of compounding periods increases, the compound amount behavior tends to exponential growth rate. It was deduced that effective rate of interest is a function of nominal rates and compounding periods.

Keywords: Annuity Future Value, Annuity Present Value, Continuous Compounding, Compounding Periods, Periodic Payments.

1. INTRODUCTION

A common component to all financial transactions is the investment of money to interest. In order to analyze the financial transaction, a clear understanding of the concept of interest is required (Marcel, 2013; Kellison, 1991; Cairns *et al.*, 2006; Ledlie *et al.*, 2007).

Interest can be viewed from different perspectives. It is the cost of borrowing money for some period of time. However, the amount of interest depends among other things, on the amount of money borrowed and repayment time (Frank, 1993). Suppose $A(t)$ denotes the amount of value of an investment at time t years. Then the interest earned during the time t years and $t+s$ years is $A_{(t+s)}$ - A_t and the annual interest rate r

$$= \frac{A(t+1)-A(t)}{A(t)} \tag{1}$$

1.1 Definitions of Terms

Present Value Interest Factor (PVIF): This estimates the current worth of a sum of money that is to be received at some future date and it is denoted by

$$PVIF = \frac{a}{(1+r)^n} \tag{2}$$

where a = future sum to be received, r = discounted interest rate and n = the number of years of transaction

Present Value Interest Factor of Annuity (PVIFA). This is estimated current worth of a sum of money used to compute the present value of a series of annuities and it is denoted by

$$PVIFA = \frac{\left[1-(1+r)^{-n}\right]}{r} \tag{3}$$

1.2 Significance of the Study

The study will enable investors to make sound business judgments or decisions anchored on their anticipated needs. It will also help investors to determine whether or not it is beneficial to receive a lump sum payments or annuities.

1.3 Generating Compound Amount from the Interest Variables

If the simple interest is denoted by $I = prt$, where I is the amount of interest earned after t years, p is the amount invested and r is the annual interest rate, then future value of the account after t years denoted by

$$\begin{aligned} F &= p + I \\ &= p + prt \\ &= p(1+rt) \end{aligned} \tag{4}$$

If the interest is compounded annually, then the amounts at the end of t years are specified

$$\begin{aligned} A_1 &= p(1+r) \\ A_2 &= A_1 + A_1 r \\ &= A_1(1+r) \\ &= p(1+r)^2 \\ A_3 &= A_2 + A_2 r \\ &= A_2(1+r) \\ &= p(1+r)^2(1+r) \\ &= p(1+r)^2 \\ &. \\ &. \\ &. \\ A_t &= p(1+r)^t \end{aligned} \tag{5}$$

Where $t = 0,1,2,...$ Thus, the building block for the annual compound interest formula is the simple interest formula (Awogbemi, 2012; Lin and Cox, 2005a; Lin and Cox, 2005b).

Suppose the interest is compounded n times in a year. At the end of every period of time $\frac{1}{n}$ years, the period's interest is added to the principal to earn interest in future periods. Thus, we have divided the year into n intervals, each with duration of $\frac{1}{n}$ years. The simple interest formula is applied to p over each period (sub interval). Thus, for one period, the interest earned is:

$$\begin{aligned} I &= prt \\ &= p\frac{r}{n} \end{aligned} \tag{6}$$

The future value after one period $\left(t = \frac{1}{n}\text{ years}\right)$ denoted by

$$\begin{aligned} A_1 &= p\left(1 + r\frac{1}{n}\right) \\ &= p\left(1+\frac{r}{n}\right) \end{aligned}$$

After two periods: (7)

$$\begin{aligned} A_2 &= \left[p\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right) \\ &= p\left(1+\frac{r}{n}\right)^2 \end{aligned}$$

After three periods:

$$\begin{aligned} A_3 &= \left[p\left(1+\frac{r}{n}\right)^2\right]\left(1+\frac{r}{n}\right) \\ &= p\left(1+\frac{r}{n}\right)^3 \\ &\vdots \\ A_n &= p\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)\cdots\left(1+\frac{r}{n}\right) \\ &= p\left(1+\frac{r}{n}\right)^n \end{aligned} \tag{8}$$

After n periods, p is compounded n times.

Thus, the amount due in t years if there are n compounding periods per year is computed as:

$$A_t = p\left(1+\frac{r}{n}\right)^{nt}, \tag{9}$$

Where $\frac{r}{n}$ is the rate per period, r is the annual interest rate, n the number of compounding periods in a year, t is the number of years and nt is the total number of compounding periods in t years.

1.4 Continuous Compounding of Interest

An interest is continuously compounded if the number of compounding periods increases infinitely. The implication of this is that as the frequency of compounding increases, $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$ tends to e^r with the compound amount $A=\lim_{n\to\infty} p\left(1+\frac{r}{n}\right)^{nt}$

Thus, the amount due for continuously compounded interest denoted by $A = pe^{rt}$ (Buchanan, 2010; Willet, 2004). The future value for continuously compounded interest assumes that a limit exists.

Proof:
Let a limit be L so that

$$L=\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$$

$$\ln(L)=\ln\left[L=\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}\right]$$

$$\ln(L)=\lim_{n\to\infty}\ln\left[P\left(1+\frac{r}{n}\right)^{nt}\right] \quad (10)$$

$$\ln(L)=\lim_{n\to\infty}\left[\ln(P)+\ln\left(1+\frac{r}{n}\right)^{nt}\right]$$

$$\ln(L)=\lim_{n\to\infty}\left[\ln(P)+nt\ \ln\left(1+\frac{r}{n}\right)\right]$$

But as n gets larger, $\frac{r}{n}$ gets really small. Therefore, log approximation $\ln(1+h)\to h$ is used to get

$$\ln(L)=\lim_{n\to\infty}\left(\ln(P)+nt.\frac{r}{n}\right)$$

$$\ln(L)=\lim_{n\to\infty}(\ln P+rt)$$

$$L=Pe^{rt} \quad (11)$$

Lemma 1

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n=e \quad (12)$$

Proof:
Suppose of f and g are continuous and differentiable functions on an open interval containing $x=a$, and that $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$

Let a new variable $k=\frac{1}{n}\Rightarrow n=\frac{1}{k}$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{k\to\infty}(1+k)^{\frac{1}{k}} \quad (13)$$

Introducing a dependent variable $y=(1+k)^{\frac{1}{k}}$ and taking the log of both sides gives

$$\lim_{k\to 0}\log y=\lim_{k\to 0}\frac{\log(1+k)}{k}$$

$$\log y=\log\left[(1+k)^{\frac{1}{k}}\right]$$
$$=\frac{1}{k}\log(1+k)$$
$$=\frac{\log(1+k)}{k} \quad (14)$$

(an indeterminate form of the type $\frac{0}{0}$)

By L'Hôpital's Rule, we have

$$\lim_{k\to 0}\log y=\lim_{k\to 0}\frac{\log(1+k)}{k}$$
$$=\lim_{k\to 0}\frac{\frac{1}{(1+k)}}{1} \quad (15)$$
$$=1$$

We have shown that $\log y\to 1$ as $k\to 0$ .The continuity of the exponential function implies that $e^{\log y}\to e^1$ as $k\to 0$. This implies that $y\to e$ as $k\to 0$ (Zenou, 2006; McCutcheon and Scott, 1986)

Hence, we have proved that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{k\to 0}(1+k)^{\frac{1}{k}}=e \quad (16)$$

2. ANNUITIES

An annuity is a sequence of payments made at regular intervals of time. The time period in which these payments are made is referred to the term of the annuity (Teresa, 2010; Lando, 2004).

An annuity in which payments are made at the end of each payment is called ordinary annuity, where as an annuity in which payments are made at the beginning of each period is called an annuity due (Nasiru and

Awogbemi, 2023). An annuity in which the payment coincides with the interest compounding period is called simple annuity, where as an annuity in which the payment period differs from the interest compounding period is called a complex annuity. The annuities considered in this study have terms given by fixed time intervals, periodic payments equal in size, payments made at the end of the period, and payment periods coincide with the interest compounding periods.

2.1 Future Value of Annuity

Future value of annuity is the sum of compound amount of payments made each accumulating to the end of the term. (Awogbemi and Ojo, 2012)

The future value of an annuity of n payments paid at the end of each investment period into an account that earns interest at the rate of i per period is

$$F_v = m\left[\frac{(1+i)^N - 1}{i}\right] \quad i = \frac{r}{n},\ N = nt \tag{17}$$

Let the periodic payments of an annuity be denoted by m , the total number of compounding periods in t years by $N = nt$ and the number of compounding periods in a year by n . Then the future value to which the deposit would have grown at the time of N^{th} deposit is

$$\begin{aligned} F_v &= m + m(1+i) + m(1+i)^2 + \cdots + m(1+i)^{N-1} \\ &= m\frac{r^N - 1}{r-1}, \quad r = (1+i) > 1 \\ &= m\frac{(1+i)^N - 1}{1+i-1} \\ &= m\frac{(1+i)^N - 1}{i} \end{aligned} \tag{18}$$

2.2 Present Value of Annuity

The present value of an annuity is the amount of money at hand that is equivalent to a series of equal payments in the future. The interest is in depositing lump sum that will have the same value as the annuity at the end of some time period (Awogbemi and Kekere, 2023)

Setting the future value based on compound interest to future value of annuity, we have

$$P_v(1+i)^N = m\left[\frac{(1+i)^N - 1}{i}\right]$$

$$\begin{aligned} P_v &= \frac{m}{i}\left[\frac{(1+i)^N - 1}{(1+i)^N}\right] \\ &= \frac{m}{i}\left[1 - \frac{1}{(1+i)^N}\right] \\ &= m\left[\frac{1-(1+i)^{-N}}{i}\right] \end{aligned} \tag{19}$$

If a loan is amortized, then the present value of annuity is used to generate the repayment of periodic equal installments m by algebraically solving for it in P_v as

$$m = \frac{(P_v)i}{1-(1+i)^{-N}} \tag{20}$$

2.3 Loan Repayment

Suppose a loan of amount P is to be repaid discretely in n times per year over t years. The unpaid portion of the loan is charged interest at an annual rate r compounded n times per year. The discrete payment m could be obtained. The present value of all the payments should equal the amount borrowed.

If the payment is made at the end of the first compounding period, then the present value of all the payments denoted by P is

$$\begin{aligned} P &= m\left(1+\frac{r}{n}\right)^{-1} + m\left(1+\frac{r}{n}\right)^{-2} + \cdots + m\left(1+\frac{r}{n}\right)^{-nt} \\ &= m\left(1+\frac{r}{n}\right)^{-1}\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{1-\left(1+\frac{r}{n}\right)^{-1}}\right] \\ &= m\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}} \\ &= m\frac{n}{r}\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right] \end{aligned}$$

$$\Rightarrow m = P\frac{r}{n}\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]^{-1} \tag{21}$$

2.4 Mortgage Loan

Suppose a mortgage loan is secured in the amount of L, with an n equal monthly payments of amount m, where the annual interest rate is r compounded monthly. Setting the sum of the present values of all the payments to the amount loaned implies that

$$L = \sum_{i=1}^{n} \frac{m}{\left(1+\frac{r}{12}\right)^{i}}, \quad i \text{ is rate per period}$$

$$= m\sum_{i=1}^{n}\left(1+\frac{r}{12}\right)^{-i}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\sum_{i=0}^{n-1}\left(1+\frac{r}{12}\right)^{-i}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\left[\frac{1-\left(1-\frac{r}{12}\right)^{-n}}{1-\left(1+\frac{r}{12}\right)^{-1}}\right]$$

$$= m\frac{\left[1-\left(1+\frac{r}{12}\right)^{-n}\right]}{\left(1+\frac{r}{12}\right)-1} \qquad (22)$$

$$= \frac{12m}{r}\left[1-\left(1+\frac{r}{12}\right)^{-n}\right]$$

Expressing m as a function of L, r and $n,$ we have

$$m = L\frac{r}{12}\left[1-\left(1+\frac{r}{12}\right)^{-n}\right]^{-1}$$

If L_k denotes the outstanding balance immediately after the k^{th} payments, then L_k is the sum of the present values of the remaining payments

$$L_k = \sum_{i=1}^{n-k} \frac{m}{\left(1+\frac{r}{12}\right)^{i}}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\sum_{i=0}^{n-k-1}\left(1+\frac{r}{12}\right)^{-i}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\frac{1-\left(1+\frac{r}{12}\right)^{-n+k}}{1-\left(1+\frac{r}{12}\right)^{-1}}$$

$$= \frac{m\left[1-\left(1+\frac{r}{12}\right)^{-n+k}\right]}{\left(1+\frac{r}{12}\right)-1} \qquad (23)$$

$$= \frac{12m}{r}\left[1-\left(1+\frac{r}{12}\right)^{-n+k}\right]$$

The amount of interest in the k th payment denoted by

$$P_k = L_{k-1}\left(\frac{r}{12}\right)$$

$$= m\left[1-\left(1+\frac{r}{12}\right)^{-n+k-1}\right] \qquad (24)$$

The repaid amount in the k^{th} repayment denoted by

$$R_k = m - P_k$$

$$= m\left(1+\frac{r}{12}\right)^{-n+k-1} \qquad (25)$$

3. APPLICATIONS OF ANNUITIES

To provide for future education cost, a family at Nigerian Universities Commission considers various methods of savings. Assuming savings will continue for a period of 10 years at an interest rate of 7.5% per annum, the value of the fund can be computed after 10 years given a deposit of ₦2000.00 annually. If the future value of the fund is ₦40000.00, the annual deposit can also be computed.

Substituting the periodic deposit m; the annual interest rate i and time t in (19), we have

$$F_{10} = m\frac{(1+i)^N - 1}{i} = 2000\frac{(1+0.075)^{10} - 1}{0.075} = ₦28293.40$$

This implies that the fund will be worth ₦28293.40 at the end of 10 years.

Substituting F_{10}, i and t into (19), we

$$4000 = m\frac{(1+0.075)^{10} - 1}{0.075}$$

m = ₦2827.52

This implies that ending up with ₦40000, the annual periodic deposit should be ₦2827.52

Two business partners X and Y decide to set aside a fund for their manager who would retire in 10 years. The first partner (X) wants to make ₦50 monthly periodic deposit into the account while the second partners (Y) wants to make a lump sum deposit today, but both want to deposit equal amount. The amount that would be in the account in 10 years at interest rate of 9% compounded monthly can be computed. The amount that the second partner (Y) should deposit in order to equal first partner's contribution can also be computed.

In this case, we are interested in depositing lump sum today that will have the same value as an annuity at the end of some time period.

Substituting m, r, i and t into (20), we have

$$= m\left[\frac{1-(1+i)^{-N}}{i}\right] = 50\left[\frac{1-(1+0.0075)^{-120}}{i}\right] = ₦9675.71$$

To compute second partner's contribution, substitute for r and nt in (9) to have ₦3,947.08.

This shows that for the two partners to contribute equal amount, the first partner (X) would deposit ₦50 monthly for 10 years and the second partner(Y) would make a lump sum contribution of ₦3947.

Remark: It should be noted that the lump sum of the second partner can be computed without first computing the value of the first partner's annuity. This can be implemented by setting (9) to (19) to have P =₦3947

4. CONCLUSION

Interest rate theory has been employed as a veritable tool in this research work to generate the future and present values of annuity, the results of which are applied in repayment of loans and mortgage loans. The effective rate of interest, which is useful in comparing alternative investment opportunities, depends on the nominal rate and the conversion periods.

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