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Observations on the Hyperbola $Y^2 = 72X^2+1$

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Abstract: The binary quadratic equation $y^2 = 72x^2 + 1$ is considered and a few interesting properties among the solutions are presented.Employing the integral solutions of the equation under consideration, a special pythagorean triangle is obtained.
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INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is a non-square positive integer, has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. In [5] infinitely many pythagorean triangle in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 1$. In [6], a special pythogorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [7], different patterns of infinitely many pythagorean triangle are obtained by employing the non-trivial solutions of $y^2 = 12x^2 + 1$. In this context one may also refer [8-16]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 72x^2 + 1$ representing a hyperbola .A few interesting properties among the solutions are presented.Employing the integral solutions of the equation under the consideration, a special pythagorean triangle is obtained.

Notations

$t_{m,n}$ – Polygonal number of rank n with size m.
P_n^m – Pyramidal number of rank n with size m.
CP_n^m - Centered pyramidal number of rank n with size m.
$CP_{m,n}$ - Centered polygonal number of rank n with size m.
GNO_n – Gnomonic number of rank n.
S_n – star number of rank n.

METHOD OF ANALYSIS

The binary quadratic equation representing hyperbola under consideration is

$$y^2 = 72x^2 + 1 \qquad (1)$$

whose general solution (x_n, y_n) is given by $x_n = \frac{g}{2\sqrt{72}}, y_n = \frac{f}{2}$,

where

$$f = (17 + 2\sqrt{72})^{n+1} + (17 - 2\sqrt{72})^{n+1}$$
$$g = (17 + 2\sqrt{72})^{n+1} - (17 - 2\sqrt{72})^{n+1} \quad , n = 0,1,2,3,\ldots\ldots$$

The recurrence relations satisfied by x and y are given by

$$y_{n+2} - 34y_{n+1} + y_n = 0 \, , y_0 = 17, y_1 = 577$$
$$x_{n+2} - 34x_{n+1} + x_n = 0 \, , x_0 = 2, x_1 = 68$$

Some numerical examples of x and y satisfying (1) are given in the following table

n	x_n	y_n
0	2	17
1	68	577
2	2310	19601
3	78472	665857
4	2665738	22619537
5	3550000614930	30122754096401

From the above table, we observe some interesting properties :

1. x_n is always even.
2. y_n is always odd.
3. $y_{2n} \equiv (mod\ 17)$.
4. $x_{2n+1} \equiv$ (mod 68).

A few interesting properties between the solutions and special numbers are given below:

1. $34y_{2n+2} - 288x_{2n+2} + 2$ is a perfect square.
2. $6(34y_{2n+2} - 288x_{2n+2} + 2)$ is a nasty number.
3. $34y_{3n+3} - 288x_{3n+3} + 3(34y_{n+1} - 288x_{n+1})$ is a cubic integer.
4. $S_f = 6[34(y_{2n+2} - y_{n+1}) - 288(x_{2n+2} - x_{n+1})]$.
5. $GNO_f = 68y_{n+1} - 576x_{n+1} - 1$.
6. $6P_f^m = 34[(m-2)x_{3n+3} + 3y_{2n+2} + (2m-1)y_{n+1}]$
   $- \quad 288[(m-2)x_{3n+3} + 3\,x_{2n+2} + (2m-1)x_{n+1}] + 6$.
7. $6CP_f^m = \quad 34[m(y_{3n+3} + 2y_{n+1}) + y_{n+1}] - 288[m(x_{3n+3} + 2x_{n+1})]$
   i. $+x_{n+1}$.
8. $2t_{m,f} = 34[(m-2)y_{2n+2} - y_{n+1}(m-4)]$
   $- \quad -288[(m-2)x_{2n+2} - x_{n+1}(m-4)] + 2m - 4$.
9. $2Cp_{m,f} = m\begin{bmatrix} (m-2)(34y_{2n+2} - 288x_{2n+2}) \\ -(m-4)(34y_{n+1} - 288\,x_{n+1}) + 2(m-2) \end{bmatrix} + 2$.
10. Let y = $34y_{n+1} - 288x_{n+1}$ $and\ x = 17x_{n+1} - 2y_{n+1}$ .Then the pair (x,y) satisfies the hyperbola $y^2 = 288x^2 + 4$.
11. $y_{n+1} = 17y_n + 144x_n$.
12. $y_{n+2} = 577y_n + 4896x_n$.
13. $x_{n+1} = 17x_n + 2y_n$.
14. $x_{n+2} = 577x_n + 68_n$.
15. $y_{3n+2} + 3y_n = 2y_n[y_{2n+1} + 1]$.
16. $(y_{3n+2} + 3y_n)^2 = 16y_n^6$.

REMARKABLE OBSERVATIONS

1. Let α be any non-zero positive integer such that $\alpha_n = \frac{y_n - 1}{2}$,
   n = 0,1,2,….. ,it is seen that $6t_{3,\alpha_n}$ is a nasty number.

2. Let p,q be the generators of the pythagorean triangle S (α ,β ,γ) with α =2pq , β = $p^2 - q^2$,γ = $p^2 + q^2$, $p > q > 0$ .Let $q_s = x_s$,
   $p_s = x_s + y_s$ .Then S satisfies the following relations.

   (a) 36β =35γ +α +1.
   (b) (γ –α) = $36\left(\alpha - \frac{4A}{P}\right) + 1$ where A and P represent the area and perimeter of the pythagorean triangle.
   (c) $x_n = p - q, p > q > 0$ .Let N be a positive integer defined by $N = \frac{y_n - 1}{2}$. Then 36(γ – α) is four times a triangular number.

n	x_n	p	q	y_n	$N=\frac{y_n-1}{2}$	α	γ	$36(\gamma-\alpha)=4t_{3,N}$
0	2	6 3	4 1	17	8	48 6	52 10	$144=4t_{3,8}$
1	68	70 74	2 6	577	288	280 888	4904 5512	$66464=4t_{3,288}$

3. Employing the solutions of (1), the following relations among the special polygonal and pyramidal numbers are observed,

(i) $t_{3,\frac{y_s-1}{2}} = 9\left(\frac{p^5_{x_s}}{t_{3,x_s}}\right)^2$

(ii) $t_{3,\frac{y_s-1}{2}} = 9\left(\frac{3p^3_{x_s}}{t_{3,x_s+1}}\right)^2$

(iii) $t_{3,\frac{y_s-1}{2}} = 9\left(\frac{6p^4_{x_s-1}}{t_{3,2x_s-2}}\right)^2$

(iv) $\left(\frac{p^5_{y_s}}{t_{3,y_s}}\right)^2 = 72\left(\frac{3p^3_{x_s}}{t_{3,x_s+1}}\right)^2+1$

(v) $\left(\frac{6p^4_{y_s-1}}{t_{3,2y_s-2}}\right)^2 = 72\left(\frac{p^5_{x_s}}{t_{3,x_s}}\right)^2+1$

(vi) $\left(\frac{p^5_{y_s}}{t_{3,y_s}}\right)^2 = 72\left(\frac{3p^3_{x_s-2}}{t_{3,x_s-2}}\right)^2+1$

(vii) $t_{3,\frac{y_s-1}{2}} = 9\left(\frac{3p^3_{x_s-2}}{t_{3,x_s-2}}\right)^2$

CONCLUSION

To conclude, one may search for other choices of hyperbola for patterns of solutions and their corresponding properties.

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