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Lattice Points on the Homogeneous Cone $8(x^2+y^2)-15xy=56z^2$

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Abstract: The ternary quadratic equation $8(x^2+y^2)-15xy=56z^2$ representing a cone is analysed for its non-zero distinct integer points on it. Employing the integer solutions, a few interesting relations between the solutions and special polygonal numbers are presented. Also, knowing an integer solution, formulas for generating sequence of solutions are given.
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INTRODUCTION

The ternary quadratic Diophantine equation offers an unlimited field for research because of their variety [1-2]. In particular, one may refer [3-21] for finding integer points on some specific three dimensional surfaces. This communication concerns with yet another ternary quadratic equation $8(x^2+y^2)-15xy=56z^2$ representing cone for determining its infinitely many integral solutions. Employing the integral solutions on the given cone, a few interesting relations among the special polygonal and pyramidal numbers are given.

Notations

$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$ = Polygonal number of rank n with sides m.

$P_n^m=\left(\frac{n(n+1)}{6}\right)[(m-2)n+(5-m)]$ = Pyramidal number of rank n with sides m.

METHOD OF ANALYSIS

Consider the equation

$$8(x^2+y^2)-15xy=56z^2 \tag{1}$$

The substitution of linear transformations

$$x=u+v \quad \text{and} \quad y=u-v \quad (u\neq v\neq 0) \tag{2}$$

In (1) leads to

$$u^2+31v^2=56z^2 \tag{3}$$

The above equation is solved through different methods and using (2), different patterns of integer solutions to (1) are obtained.

PATTERN 1

Write 56 as

$$56=(5+i\sqrt{3})(5-i\sqrt{3}) \tag{4}$$

Assume $z=\ a^2+31b^2$ where a,b> 0 (5)

Using (4) and (5) in (3),and applying the method of factorization, define

$$\left(u+i\sqrt{31}v\right)=(5+i\sqrt{31})\left(a+i\sqrt{31}b\right)^2 \tag{6}$$

Equating the real and imaginary parts, we have

$$u=u(a,b)=5a^2-155b^2-62ab$$
$$v=v(a,b)=a^2-31b^2+10ab$$

Substituting the above u and v in equation (2), the values of x and y are given by

$$\left.\begin{aligned} x = x(a,b) = 6a^2 - 186b^2 - 52ab \\ y = y(a,b) = 4a^2 - 124b^2 - 72ab \end{aligned}\right\} \quad (7)$$

Thus (5) and (7) represent non-zero distinct integral solution of (1) in two parameters

Properties

1. $x(a+1,a) - T_{(362,a)} + 52pr_a \equiv 6(mod167)$
2. $y(a,1) - T_{(10,a)} \equiv 57(mod67)$
3. $y(a+1,a^2) - T_{(10,a)} + 124T_{(4,a^2)} + 144p_a^5 \equiv 49(mod11)$

PATTERN 2

Instead of (4), we write 56 as

$$56 = \frac{(37+i\sqrt{31})(37-i\sqrt{31})}{25} \quad (8)$$

Following the procedure presented above in pattern 1, the corresponding values of x and y are obtained as

$$\left.\begin{aligned} x = x(a,b) = \frac{1}{5}[38a^2 - 1178b^2 + 12ab] \\ y = y(a,b) = \frac{1}{5}[36a^2 - 1116b^2 - 136ab] \end{aligned}\right\} \quad (9)$$

Replacing 'a' by 5A and 'b' by 5B in (5) and (9), the integer values of x, y, z satisfying (1) are given by

$$x = x(A,B) = 190A^2 - 5890B^2 + 60AB$$
$$y = y(A,B) = 180A^2 - 5580B^2 - 680AB$$
$$z = z(A,B) = 25A^2 + 775B^2$$

Properties

1. $x(a,1) - S_a \equiv -3(mod46)$
2. $y(2^n,1) = 6J_{2n} + 2j_{2n} - 108J_n + 36j_n - 124$
3. $x(a,1) - z(a,1) - 10t_{3,a-1} \equiv 18(mod47)$

PATTERN 3

$Introducing\ the\ linear\ transformations$

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$$u = 5\alpha \quad \text{and} \quad \left.\begin{aligned} z = X + 31T \\ v = X + 56T \end{aligned}\right\} \quad (10)$$

In (3), we get $X^2 = \alpha^2 - 1736T^2$ (11)

which is satisfied by

$$\left.\begin{aligned} T = T(r,s) = 2rs \\ X = X(r,s) = 1736r^2 + s^2 \\ \alpha = \alpha(r,s) = 1736r^2 - s^2 \end{aligned}\right\} \quad (12)$$

Substituting (12) in (10) and using (2), the corresponding integer solutions to (1) are given by

$$x = x(r,s) = 10416r^2 - 4s^2 + 112rs$$
$$y = y(r,s) = 6944r^2 - 6s^2 - 112rs$$
$$z = z(r,s) = 1736r^2 + s^2 + 62rs$$

Properties

1. $x(2^n,1) = 15624J_{2n} + 5208j_{2n} + 168J_n + 56j_n - 4$
2. $\frac{1}{2}[x(a,1) - y(a,1)] - 3472t_{3,a-1} \equiv -1847(mod1848)$
3. $\frac{1}{5}[z(1,a) - x(1,a)] - 10t_{3,a-1} \equiv -1725(mod11)$

Note

In addition to (10), one may also consider the linear transformations $= X - 31T$, $v = X - 56T$. Following the method presented above a different set of solutions is obtained.

PATTERN 4

Consider (3) as

$$u^2 - 25z^2 = 31(z^2 - v^2) \quad (13)$$

Write (13) in the form of ratio as

$$\frac{u+5z}{z+v} = \frac{31(z-v)}{u-5z} = \frac{A}{B}, \ B \neq 0$$

which is equivalent to the following two equations

$$Bu - Av + (5B - A)z = 0$$
$$-Au - 31Bv + (31B + 5A)z = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} u &= -5A^2 + 155B^2 - 62AB \\ v &= A^2 - 31B^2 - 10AB \end{aligned}\right\} \tag{14}$$

$$z = -A^2 - 31B^2 \tag{15}$$

Substituting the values of u and v from (14) in (2), the non-zero distinct integral values of x,y are given by

$$\left.\begin{aligned} x &= x(A,B) = -4A^2 + 124B^2 - 72AB \\ y &= y(A,B) = -6A^2 + 186B^2 - 52AB \end{aligned}\right\} \tag{16}$$

Thus (16) and (15) represent the non-zero distinct integer solution of equation (1) in two parameters.

Properties

1. $x(a+1,a^2) - T_{(18,a)} + 124T_{(4,a^2)} - 144p_a^5 \equiv 4(mod11)$
2. $z(a,a+1) - T_{(66,a)} \equiv 31(mod93)$
3. $x(a,1) - T_{(18,a)} \equiv -26(mod75)$

Note

(13) can also be expressed in the form of ratio in $three$ different ways as follows.

(i) $\frac{u+5z}{31(z+v)} = \frac{z-v}{u-5z} = \frac{A}{B}, B \neq 0$

(ii) $\frac{u+5z}{z-v} = \frac{31(z+v)}{u-5z} = \frac{A}{B}, B \neq 0$

(iii) $\frac{u+5z}{31(z-v)} = \frac{z+v}{u-5z} = \frac{A}{B}, B \neq 0$

Repeating the analysis as above, we get different patterns of solutions to (1).

Remarkable observation:

If (x_0, y_0, z_0) is any given integer solutions of (1), then each of the following triples satisfies (1)

Triple 1: $(5^{2n}u_0 + 5^{2n}v_0, 5^{2n}u_0 - 5^{2n}v_0, 5^{2n}z_0)$

Triple 2: $(5^{2n-1}u_0 + 5^{2n-2}M_1v_0, 5^{2n-1}u_0 - 5^{2n-2}M_1v_0, 5^{-2n-2}M_1z_0), M_1 = \begin{bmatrix} 625 & -840 \\ 465 & -625 \end{bmatrix}$.

Triple 3: $(5^{2n}u_0 + 5^{2n}v_0, 5^{2n}u_0 - 5^{2n}v_0, 5^{2n}z_0)$

Triple 4: $(5^{2n-2}M_2u_0 + 5^{2n-1}v_0, 5^{2n-2}M_2u_0 - 5^{2n-1}v_0, 5^{2n-2}M_2z_0), M_2 = \begin{bmatrix} 75 & -560 \\ 10 & -75 \end{bmatrix}$

In the above Triples 1 – 4, (u_0, v_0, z_0) is the initial solutions of (3).

CONCLUSION

In this paper, we have presented different patterns of integer solutions to the ternary quadratic equation $8(x^2 +$ *y2−15xy=56z2* representing the cone. As this Diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of equations along with suitable properties.

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