Scholars Journal of Physics, Mathematics and Statistics
Sch. J. Phys. Math. Stat. 2014; 1(2):45-47
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

On the Integral Points of the Bi-Quadratic Equation with Four Unknowns $(x-y)(x^2+y^2)+(x^2-xy+y^2)z=12zw^3$

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Abstract: The Bi-quadratic Equation with four unknowns given by $(x-y)(x^2+y^2)+(x^2-xy+y^2)z=12zw^3$ is analyzed for its patterns of non –zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.
Keywords: Bi-Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers

INTRODUCTION

There is a great interest for Mathematicians since ambiguity in homogeneous and non- homogeneous Bi-quadratic Diophantine Equations [1-5]. In the context, so many references [6-19] available for varieties of problems on the diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another biquadratic equation in four unknowns represented by $(x-y)(x^2+y^2)+(x^2-xy+y^2)z=12zw^3$. A few interesting relations between the solutions and special polygonal numbers are presented.

NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

CP_n^m : Centered Pyramidal number of rank n with size m.

S_n : Star number of rank n

J_n :Jacobsthal number of rank n

j_n : Jacobsthal-Lucas number of rank n

ky_n : keynea number of rank n.

METHOD OF ANALYSIS

The non-homogeneous biquadratic Diophantine equation to be solved for its distinct non-zero integral solutions is

$$(x-y)(x^2+y^2)+(x^2-xy+y^2)z=12zw^3 \tag{1}$$

We present below different patterns of solution of (1)

Pattern 1

Introduction of the linear transformations

$$x=u+v,\ \ y=u-v,\ \ z=4v \tag{2}$$

in (1) leads to

$$u^2+2v^2=6w^3 \tag{3}$$

Let

$$w=a^2+2b^2 \tag{4}$$

Using (4) in (3) and applying the method of factorization, define

$$u + i\sqrt{2}v = (2 + i\sqrt{2})(a + i\sqrt{2}b)^3$$

Equating real and imaginary parts, we get

$$\left.\begin{array}{r} u = 2a^3 - 12ab^2 - 6a^2b + 4b^3 \\ v = a^3 - 6ab^2 + 6a^2b + 4b^3 \end{array}\right\} \quad (5)$$

Using (5) in (2), we have the integer solutions of (1) to be given by

$$x(a,b) = 3a^3 - 18ab^2$$
$$y(a,b) = a^3 - 6ab^2 - 12a^2b + 8b^3$$
$$z(a,b) = 4(a^3 - 6ab^2 + 6a^2b + 4b^3)$$
$$w(a,b) = a^2 + 2b^2$$

Properties

$(i)\ x(n+1,n) + y(n+1,n) - z(n+1,n) = -12(2CP_n^3 + CP_{4,n} + 4t_{4,n} - 1)$

$(ii)\ z(n+1,n) - y(n+1,n) = 3(CP_n^6 + CP_{30,n} + 6t_{4,n})$

$(iii)\ x(n,n-1) + w(n,n-1) = -CP_n^5 - CP_4^n - CP_n^3 + 7CP_n^6 + S_n + t_{22,n} + t_{10,n} + t_{6,n} + 17t_{4,n} + 1$

$(iv)\ w(2^n, 2^{n+1}) = 3J_{2n} + 2j_{2n+2} - 1$

Pattern 2

(3) can be written as

$$u^2 + 2v^2 = 6w^3 * 1 \quad (6)$$

Write 1 as $\quad 1 = \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9} \quad (7)$

Substituting (4) and (7) in (6) and employing the factorization method, define

$$u + i\sqrt{2}v = (2 + i\sqrt{2})(a + i\sqrt{2}b)^3 \frac{(1 + i2\sqrt{2})}{3}$$

Equating real and imaginary parts, we have

$$\left.\begin{array}{r} u = \frac{1}{3}[-2a^3 + 12ab^2 - 30a^2b + 20b^3] \\ v = \frac{1}{3}[5a^3 - 30ab^2 - 6a^2b + 4b^3] \end{array}\right\} \quad (8)$$

Replacing a by 3a and b by 3b in (8) and employing (2), the corresponding non-zero integer solutions to (1) are obtained as

$$x(a,b) = 27a^3 - 162ab^2 - 324a^2b + 216b^3$$
$$y(a,b) = -63a^3 + 378ab^2 - 216a^2b + 144b^3$$
$$z(a,b) = 4[45a^3 - 270ab^2 - 54a^2b + 36b^3$$
$$w(a,b) = 9a^2 + 18b^2$$

Properties

$(i)\ x(n,1) + y(n,1) = 36[CP_n^6 - t_{16,n} - 8t_{4,n} + 10]$

$(ii)\ w(2^n,1) = 9[Ky_n - j_{n+1} + 2]$ if n is even

$(iii)\ w(2^n,1) = 9[Ky_n - j_{n+1} + 4]$ if n is odd

$(iv) x(n,1) + w(n,1) = 9[2CP_n^9 - CP_{28,n} - CP_{6,n} - 18t_{4,n} + 28]$

Note: It is worth to observe that in addition to (7), we write 1 in two different ways as

$$1 = \frac{(7 + i4\sqrt{2})(7 - i4\sqrt{2})}{81}$$

and

$$1 = \frac{(1 + i12\sqrt{2})(1 - i12\sqrt{2})}{17^2}$$

Repeating the process as in pattern.2, the corresponding two sets of non-zero distinct integer solutions to (1) are found to be as follows;

Set 1

$$x(a,b) = 63a^3 - 378ab^2 - 216a^2b + 144b^3$$
$$y(a,b) = -27a^3 + 162ab^2 - 324a^2b + 216b^3$$
$$z(a,b) = 4[45a^3 - 270ab^2 + 54a^2b - 36b^3$$
$$w(a,b) = 9a^2 + 18b^2$$

Set 2

$$x(a,b) = 867a^3 - 5202ab^2 - 62424a^2b + 41616b^3$$
$$y(a,b) = -13583a^3 + 81498ab^2 - 24276a^2b + 16184b^3$$
$$z(a,b) = 4[7225a^3 - 43350ab^2 + 19074a^2b - 12716b^3$$
$$w(a,b) = 289a^2 + 578b^2$$

CONCLUSION

In this paper, a bi-quadratic equation with four unknowns is studied for its non-zero integer solutions by employing the linear transformations x=u+v, y=u-v, z=4v. Instead of z=4v one may also consider z=4kv and attempt for getting non-zero integer solutions to the considered equation. As bi-quadratic diophantine equations are in rich in variety, one may consider other forms of bi-quadratic equations with four unknowns to determine their integer solutions and obtain their relations with special numbers, namely polygonal numbers, Pyramidal numbers, Jacobsthal numbers and so on.

ACKNOWLEDGEMENT

The finical support from the UGC, New Delhi (F.MRP-5123/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged.

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