Scholars Journal of Physics, Mathematics and Statistics **ISSN 2393-8056 (Print)**
Sch. J. Phys. Math. Stat. 2014; 1(2):74-80 **ISSN 2393-8064 (Online)**

Arriving and Convert Times in Hyper-networks

Linli Zhu[1*], Yun Gao[2], Wei Gao[3]
[1]School of Computer Engineering, Jiangsu University of Technology, Changzhou 213001, China
[2]Department of Editorial, Yunnan Normal University, Kunming 650092, China
[3]School of Information, Yunnan Normal University, Kunming 650500, China

***Corresponding Author:**
Linli Zhu
Email: zhulinli@163.com

Abstract: In hyper-networks, arriving time and the convert distance are used to measure the structure of hyper-graphs. For two vertices *u* and *v*, the arriving time H_{uv} is defined as the expected time for it takes a random walk to travel from *u* to *v*. The convert distance is a symmetrized version denoted as $C_{uv} = H_{uv} + H_{vu}$. In this article, we consider the characters of arriving times and convert distances when the number *n* of vertices in the hyper-networks tends to ∞. We discuss random geometric hyper-graphs, such as ε-hyper-graphs, *k*-NN hyper-graphs and Gaussian similarity hyper-graphs, and the hyper-graphs with a given expected degree distribution or other special hyper-graphs structures. Several results on convergence are determined, and these illustrate the promising application prospects for hyper-networks algorithm.
Keywords: arriving time, convert time, hyper-networks, random hyper-graph, spectral gap.

INTRODUCTION

Let *H*=(*V*, *E*) be a fixed an undirected, weighted hyper-graph with *n* vertices, which express a hyper-networks. The convert distance between two vertices *u* and *v* is denoted as the expected time it takes the natural random walk from vertex *u* to vertex *v* and then back to vertex *u*. It is equivalent to the resistance distance, which interprets the hyper-graph as an electrical hyper-network and denotes the distance between vertices *u* and *v* as the effective resistance between these vertices.

In our paper, we learn the convergence for convert distance when the order of the hyper-networks increases. We focus on the special cases such that the random geometric hyper-networks can be expressed as *k*-nearest neighbor hyper-graphs, ε-hyper-graphs, and Gaussian similarity hyper-graphs. For two vertices *u* and *v*, the arriving time H_{uv} is defined as the expected time for it takes a random walk to travel from *u* to *v*. The convert distance is a symmetrized version denoted as $C_{uv} = H_{uv} + H_{vu}$. Let vol(*H*)= $\sum_{v\in V(H)} d(v)$ be the volume of the hyper-graph *H*. The main result in this paper to show the fact that in hyper-networks setting, as the number *n* of vertices tends to ∞, there exist a scaling term *c* such that the arriving times and convert distances in random geometric hyper-graphs meet

$$c\cdot\left|\frac{H_{uv}}{\mathrm{vol}(H)}-\frac{1}{d_v}\right|\to 0,\qquad c\cdot\left|\frac{C_{uv}}{\mathrm{vol}(H)}-(\frac{1}{d_u}+\frac{1}{d_v})\right|\to 0,$$

and simultaneously cd_u and cd_v converge to positive constants. It reveals that the rescaled convert distance approximated by the sum of the inverse rescaled degrees.

The organization of this paper is as follows: the terminologies and notations for this setting are given in Section 2; and in Section 3, we present the main results in our paper.

HINTSSETTING AND DEFINETIONS

Let $V=\{v_1,v_2,\ldots,v_m\}$ be a limited set, *E* is family of subset of *V*, i.e., $E\subseteq 2^V$. Then *H*=(*V*,*E*) is a hyper-graph on *V*. the element of *V* is called a vertex, the elements of *E* is called a hyper-edge. Let $|V|$ be the order of *H*, $|E|$ be the scale of *H*.

$|e|$ is basic number of hyper-edge *e*. $r(H)=\max_j |e_j|$ is rank of hyper-edge *e*, and $s(H)=\min_j |e_j|$ lower rank of hyper-edge *e*. If $|e|=k$ for each hyper-edge *e* of *E* (that is $r(H)=s(H)=k$), then *H* is a *k*-uniform hyper-graph. If $k=2$, then *H* is just a normal graph.

Hyper-graph as a expansion concept of graph, it applied in many fields of computer science. Several results can refer to [1-7]. A hyper-graph *H* is called a simple hyper-graph or a sperner hyper-graph, if any two hyper-edges are not contained with each other. Let $H'=(V, E')$ is a hyper-graph on *V*, if $E' \subset E$, then H' is a part-hyper-graph of *H*. For $S \subseteq V$, $H[S]=\{e\in E: e\subseteq S\}$ is called a sub-hyper-graph of *H* induced by *S*.

Hyper-graph *H* can be represented by graph by using the set of vertices to represent the elements of *V*. If $|e_j|=2$, using a continuous curve which attach to the elements of e_j to representing e_j; If $|e_j|=1$, using a loop which contain e_j to represent e_j; If $|e_j|\geq 3$, using a simple close curve which contains all the elements of e_j to represent e_j.

In this paper, we assume *H* is a weighted hyper-graph, each edge given a wight $w(e)$. The degree of vertex v_j in hyper-graph *H* is denoted as

$$d_j(H)=\sum_{e\in E} w(e)h(v,e),$$

where

$$h(v,e)=\begin{cases}1, & if \quad v\in e\\ 0, & if \quad v\notin e\end{cases}.$$

Let $\delta(e)=\sum_{v\in V} h(v,e)$. Then, the normalized laplacian $L(H)\in \mathbb{R}^{m\times m}$ on hyper-graph *H* is defined by :

$$L_{ij}(H)=\begin{cases}-\sum_{\{i,j\}\subseteq e} w(e)\dfrac{1}{\delta(e)} & i\neq j\\ d_j(H) & \text{otherwise}\end{cases}.$$

Let $d_{\min}$ and $d_{\max}$ be the minimal and maximal degrees, respectively. Let *D* be a diagonal matrix with diagonal entries d_i. The unnormalized hyper-graph Laplacian is denoted as $L=D-W$, and the normalized one as $L_{\text{sym}}=D^{-1/2}LD^{-1/2}$. Consider the natural random walk on hyper-graph *H*. Its transition matrix is expressed as $P=D^{-1}W$. Then, λ is an eigenvalue of L_{sym} if and only if $1-\lambda$ is an eigenvalue of *P*. Set $1=\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n>-1$ the eigenvalues of *P*. The spectral gap of *P* is defined as $1-\max\{\lambda_2,|\lambda_n|\}$.

Let *U* be the projection on the eigenspace corresponding to eigenvalue 0, then the Moore-Penrose inverse of symmetric, non-invertible matrix ***A*** is defined as $A^{\dagger}=(A+U)^{-1}-U$. Reset e_i as the *i*-th unit vector in $\mathbb{R}^n$, then convert times can be expressed by virtue of the Moore-Penrose inverse $L^{\dagger}$ of the unnormalized hyper-graph Laplacian:

$$C_{ij}=\text{vol}(H)\left\langle e_i-e_j, L^{\dagger}(e_i-e_j)\right\rangle,$$

Our first result present the closed form expression for arriving and convert times.

Lemma 1. Let *H* be a connected, undirected hyper-graph with order *n*. For $i\neq j$, we infer

$$H_{ij}=\text{vol}(H)\left\langle \frac{1}{\sqrt{d_j}}e_j, L_{\text{sym}}^{\dagger}\left(\frac{1}{\sqrt{d_j}}e_j-\frac{1}{\sqrt{d_i}}e_i\right)\right\rangle,$$

$$C_{ij}=\text{vol}(H)\left\langle \frac{1}{\sqrt{d_i}}e_i-\frac{1}{\sqrt{d_j}}e_j, L_{\text{sym}}^{\dagger}\left(\frac{1}{\sqrt{d_j}}e_j-\frac{1}{\sqrt{d_i}}e_i\right)\right\rangle.$$

Proof of Lemma 1. For the arriving time presentation, set $u_1,\ldots,u_n$ as an orthonormal set of eigenvectors of the matrix

$D^{-1/2}WD^{-1/2}$ corresponding to the eigenvalues $\mu_1,\ldots,\mu_n$. Let u_{ij} be the j-th entry of u_i. Then, we deduce

$$H_{ij}=\mathrm{vol}(H)\sum_{k=2}^{n}\frac{1}{1-\mu_k}\left(\frac{u_{kj}^2}{d_j}-\frac{u_{ki}u_{kj}}{\sqrt{d_i d_j}}\right).$$

In terms of the spectral representation of L_{sym}, we yield

$$H_{ij}=\mathrm{vol}(H)\left\langle \frac{1}{\sqrt{d_j}}e_j,\sum_{k=2}^{n}\frac{1}{1-\mu_k}\left\langle u_k,\frac{1}{\sqrt{d_j}}e_j-\frac{1}{\sqrt{d_i}}e_i\right\rangle u_k\right\rangle=\mathrm{vol}(H)\left\langle \frac{1}{\sqrt{d_j}}e_j,L_{\mathrm{sym}}^{\dagger}\left(\frac{1}{\sqrt{d_j}}e_j-\frac{1}{\sqrt{d_i}}e_i\right)\right\rangle.$$

The result for the convert time can be obtained in similar way. □

Then, we introduce some geometric hyper-graphs. Consider a given set of points $X_1,\ldots,X_n \in \mathbb{R}^d$ for a deterministic geometric hyper-graph. These points form the vertex set V in the hyper-graph. In the ε-hyper-graph, any two vertices in the same hyper-edge satisfies that their Euclidean distance is less than or equal to ε. In the undirected symmetric k-nearest neighbor hyper-graph, two vertices v_i to v_j in the same hyper-edge if X_i is among the k nearest neighbors of X_j. For the undirected mutual k-nearest neighbor hyper-graph, two vertices v_i to v_j in the same hyper-edge if X_i is among the k nearest neighbors of X_j. For a general similarity hyper-graph, we define the weight matrix W with entries $w_{ij}=k(X_i,X_j)$. In large applications, this weight function is the Gaussian similarity function $w_{ij}=\exp\left(-\frac{\|X_i-X_j\|^2}{\sigma^2}\right)$, where $\sigma>0$ is called bandwidth parameter.

Assume that the underlying set of vertices V has been drawn i.i.d. according to certain probability density p on $\mathbb{R}^d$. If the vertices are known, the hyper-edges in the hyper-graphs are constructed as described above. A connected subset $X\subset\mathbb{R}^d$ is called a valid region if it meets following characters:

- For any $x\in X$ we get that $0<p_{\min}\le p(x)\le p_{\max}<\infty$ for certain constants $p_{\min}$ and $p_{\max}$.
- X has bottleneck larger than certain positive value h: the collection $\{x\in X: \mathrm{dist}(x,\partial X)>h/2\}$ is connected.
- The boundary of X is regular in the following sense. Assume that there exist positive constants α and ε_0 satisfies that if $\varepsilon<\varepsilon_0$, then for all points $x\in\partial X$ meet $\mathrm{vol}(B_\varepsilon(x)\cap X)\ge\alpha\,\mathrm{vol}(B_\varepsilon(x))$ (here vol is the Lebesgue volume).

In what follows, we assume that $X := \mathrm{supp}(p)$ is a valid region, X does not contain any holes and does not become arbitrarily narrow: there exist a homeomorphism $h: X\to[0,1]^d$ and constant $0<L_{\min}<L_{\max}<\infty$ satisfies that for any $x,y\in X$, we get

$$L_{\min}\|x-y\|\le\|h(x)-h(y)\|\le L_{\max}\|x-y\|.$$

Let η_d be the volume of the unit ball in $\mathbb{R}^d$. Positive constants c_i are independent of n and the hyper-graph connectivity parameter (ε or k or h, respectively) but rely on the dimension, the geometry of X, and p.

MAIN RESULTS AND PROOFS

The following result drives the Absolute and relative bounds in any given hyper-graph for arriving and convert times.

Lemma 2. Let H be a finite, connected, undirected, possibly weighted hyper-graph that is not bipartite.

(1) For $i\ne j$

$$\left|\frac{1}{\mathrm{vol}(H)}H_{ij}-\frac{1}{d_j}\right|\le 2\left(\frac{1}{1-\lambda_2}+1\right)\frac{w_{\max}}{d_{\min}^2}.$$

(2) For $i\ne j$

$$\left|\frac{1}{\mathrm{vol}(H)}C_{ij}-\left(\frac{1}{d_i}+\frac{1}{d_j}\right)\right|\le\left(\frac{1}{1-\lambda_2}+1\right)\frac{w_{\max}}{d_{\min}^2}\left(\frac{1}{d_i}+\frac{1}{d_j}\right)\le 2\left(\frac{1}{1-\lambda_2}+1\right)\frac{w_{\max}}{d_{\min}^2}.$$

Proof of Lemma 2. Let $A=D^{-1/2}WD^{-1/2}$ and $u_i=\frac{e_i}{\sqrt{d_i}}$. Denote the projectionon the eigenspace of the j-the eigenvalue

λ_j of A by P_j. we obtain

$$\langle u_i, Au_j\rangle = \frac{w_{ij}}{d_i d_j} \le \frac{w_{\max}}{d_{\min}^2},$$

$$\|Au_i\|^2 = \sum_{k=1}^{n} \frac{w_{ik}^2}{d_i^2 d_k} \le \frac{w_{\max}}{d_{\min} d_i^2} \sum_k w_{ik} = \frac{w_{\max}}{d_{\min}} \frac{1}{d_i} \le \frac{w_{\max}}{d_{\min}^2}$$

$$\|A(u_i - u_j)\|^2 \le \frac{w_{\max}}{d_{\min}} \left(\frac{1}{d_i} + \frac{1}{d_j}\right) \le \frac{2w_{\max}}{d_{\min}^2}.$$

In terms of $P_1(u_i - u_j)$ =0, we yield following equation for the arriving time

$$\left|\frac{H_{ij}}{\mathrm{vol}(H)} - \frac{1}{d_j}\right| = \left|\langle u_j, M(u_j - u_i)\rangle\right| \le \frac{1}{1-\lambda_2}\|Au_j\| \cdot \|A(u_j - u_i)\| + \left|\langle u_j, A(u_j - u_i)\rangle\right|$$

$$\le \frac{1}{1-\lambda_2} \frac{w_{\max}}{d_{\min}} \left(\frac{1}{\sqrt{d_j}} \sqrt{\frac{1}{d_i} + \frac{1}{d_j}}\right) + \frac{w_{ij}}{d_i d_j} + \frac{w_{ij}}{d_j^2} \le \frac{w_{\max}}{d_{\min}^2} \left(\frac{1}{1-\lambda_2} + 1\right).$$

On the convert time, we deduce

$$\left|\frac{C_{ij}}{\mathrm{vol}(H)} - \left(\frac{1}{d_i} + \frac{1}{d_j}\right)\right| = \left|\langle u_i - u_j, M(u_i - u_j)\rangle\right| \le \frac{1}{1-\lambda_2}\|A(u_i - u_j)\|^2 + \left|\langle u_i - u_j, A(u_i - u_j)\rangle\right|$$

$$\le \frac{w_{\max}}{d_{\min}} \left(\frac{1}{1-\lambda_2} + 2\right)\left(\frac{1}{d_i} + \frac{1}{d_j}\right).$$

□

Now, we discuss some classes of random geometric hyper-graphs by view of Lemma 2. Theorem 1 obtains the spectral gap of the ε-hyper-graph and Theorem 2 gets the spectral gap of the *k*NN-hyper-graph.

Theorem 1. Assume that the general assumptions establish. Then there exist positive constants $c_1,\ldots, c_6$ such that with probability at least $1 - c_1 n\exp(-c_2 n\varepsilon^d) - \frac{c_3 n\exp(-c_4 n\varepsilon^d)}{\varepsilon^d}$,

$$1 - \lambda_2 \ge c_5 \varepsilon^2,\ 1 - |\lambda_n| \ge \frac{c_6 \varepsilon^{d+1}}{n}.$$

Furthermore, if $\frac{n\varepsilon^d}{\log n} \to \infty$, then this probability converges to 1.

Theorem 2. Assume that the general assumptions establish. Then for both the symmetric and the mutual *k*NN-hyper-graph there exist positive constants $c_1,\ldots, c_4$ such that with probability at least $1 - c_1 n\exp(-c_2 k)$,

$$1 - \lambda_2 \ge c_3 \left(\frac{k}{n}\right)^{\frac{2}{d}},\ 1 - |\lambda_n| \ge \frac{c_4 k^{\frac{2}{d}}}{n^{\frac{d+2}{d}}}.$$

Moreover, if $\frac{k}{\log n} \to \infty$, then the probability converges to 1.

Following corollaries describe the Arriving and convert times on ε-hyper-graphs and *k*NN-hyper-graphs, respectively.

Corollary 1. Assume that the general assumptions establish. Let H be an unweighted ε-hyper-graph constructed from the sequence $X_1,\ldots,X_n$ drawn i.i.d. from the density p. Then there exist positive constants $c_1,\ldots, c_5$ such that with probability at least $1 - c_1 n\exp(-c_2 n\varepsilon^d) - \frac{c_3 n\exp(-c_4 n\varepsilon^d)}{\varepsilon^d}$, for any $i \ne j$, we have

$$\left|\frac{n\varepsilon^d}{d_j}-\frac{n\varepsilon^d}{\mathrm{vol}(H)}H_{ij}\right|\leq\frac{c_5}{n\varepsilon^{d+2}}.$$

Suppose that the density p is continuous and $n\to\infty$, $\varepsilon\to 0$ and $n\varepsilon^{d+2}\to\infty$, then following fact almost surely

$$\frac{n\varepsilon^d}{\mathrm{vol}(H)}H_{ij}\to\frac{1}{\eta_d p(X_j)}.$$

By $C_{uv}=H_{uv}+H_{vu}$, we can get analogous results for the convert times.

Corollary 2. Assume that the general assumptions establish. Let H be an unweighted kNN-hyper-graph constructed from the sequence $X_1,\ldots,X_n$ drawn i.i.d. from the density p. Then for both the symmetric and mutual kNN-hyper-graph there exist positive constants c_1, c_2, c_3 such that with probability at least $1-c_1 n\exp(-c_2 k)$, for any $i\neq j$, we get

$$\left|\frac{k}{d_j}-\frac{k}{\mathrm{vol}(H)}H_{ij}\right|\leq\frac{c_3 n^{\frac{d}{2}}}{k^{1+\frac{d}{2}}}.$$

If the density p is continuous and $n\to\infty$, $\varepsilon\to 0$ and $k(\frac{k}{n})^{\frac{2}{d}}\to\infty$, then following fact almost surely

$$\frac{k}{\mathrm{vol}(H)}H_{ij}\to 1.$$

By $C_{uv}=H_{uv}+H_{vu}$, we can get analogous results for the convert times.

Now, we discuss the weighted hyper-graphs. The result stated as follows concern fully connected weighted hyper-graphs.

Theorem 3. Consider a given fully connected weighted hyper-graph with weight matrix $\boldsymbol{W}$. Let $w_{\min}$ and $w_{\max}$ be the upper and lower bound of entries of $\boldsymbol{W}$. Then, for any $i, j\in\{1,\ldots,n\}$ and $i\neq j$, we infer

$$\left|\frac{n}{\mathrm{vol}(H)}H_{ij}-\frac{n}{d_j}\right|\leq 4n\frac{w_{\max}}{w_{\min}}\frac{w_{\max}}{d_{\min}^2}\leq\frac{w_{\max}^2}{w_{\min}^3}\frac{4}{n}.$$

Proof of Theorem 3. By the hyper-graph learning tricks, we get the following facts:

- For any row-stochastic matrix P,

$$\lambda_2\leq\frac{1}{2}\max_{i,j}\sum_{k=1}^{n}\left|\frac{w_{ik}}{d_i}-\frac{w_{jk}}{d_j}\right|\leq 1-n\min_{i,j}\frac{w_{ij}}{d_i}\leq 1-\frac{w_{\min}}{w_{\max}}.$$

- Let H be a weighted hyper-graph with hyper-edge weights $0<w_{\min}\leq w_{ij}\leq w_{\max}$ and $\lambda_{2,weighted}$ be its second eigenvalue. Consider the corresponding unweighted hyper-graph where the weights of all hyper-edge are 1, and denote its second eigen value by $\lambda_{2,unweighted}$. Then, we yield

$$(1-\lambda_{2,unweighted})\frac{w_{\min}}{w_{\max}}\leq(1-\lambda_{2,unweighted})\leq(1-\lambda_{2,unweighted})\frac{w_{\max}}{w_{\min}}.$$

Then, the result follows directly from above facts and Lemma 2. □

The next theorem discusses the case of Gaussian similarity hyper-graphs with adapted bandwidth.

Theorem 4. Let $X\subset\mathbb{R}^d$ be a compact set and p be a continuous, strictly positive density on X. Let H be a fully connected, weighted similarity hyper-graph constructed from the points $X_1,\ldots,X_n$ drawn i.i.d. from density p. Its weight function is Gaussian similarity function: $k_\sigma(x,y)=\frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}}\exp(-\frac{\|x-y\|^2}{2\sigma^2})$. Suppose the density p is continuous and $n \to \infty$, $\sigma \to 0$ and $\dfrac{n\sigma^{d+2}}{\log n} \to \infty$, then following fact almost surely

$$\frac{n}{\mathrm{vol}(H)} C_{ij} \to \frac{1}{p(X_i)} + \frac{1}{p(X_j)}.$$

Proof of Theorem 4. Note that

$$\left| nR_{ij} - \frac{1}{p(X_i)} - \frac{1}{p(X_j)} \right| \le \left| nR_{ij} - \frac{n}{d_i} - \frac{n}{d_j} \right| + \left| \frac{n}{d_i} + \frac{n}{d_j} - \frac{1}{p(X_i)} - \frac{1}{p(X_j)} \right|.$$

By the given assumption, the second term on the right hand side converges to 0.

Let $Q_{ij} = \dfrac{1}{d_i - w_{ij}} + \dfrac{1}{d_j - w_{ij}}$, then we have $R_{ij} \ge \dfrac{Q_{ij}}{1 + w_{ij} Q_{ij}}$. By virtue of the given conditions, for any two given points X_i and X_j the Gaussian hyper-edge weight w_{ij} converges to 0 as σ tend to 0. Hence, we get

$$n(R_{ij} - \frac{1}{d_i} - \frac{1}{d_j}) \ge n(\frac{Q_{ij}}{1 + w_{ij} Q_{ij}} - \frac{1}{d_i} - \frac{1}{d_j}) \to 0 \text{ a.s.}$$

Let H^{ε} be the ε-truncated Gauss hyper-graph with hyper-edge weights

$$w_{ij}^{\varepsilon} = \begin{cases} w_{ij} & \text{if } \|X_i - X_j\| \le \varepsilon \\ 0 & \text{else} \end{cases}.$$

Let $d_i^{\varepsilon} = \sum_{j=1}^{n} w_{ij}^{\varepsilon}$. Let R^{ε} be the resistance of the ε-truncated Gauss hyper-graph. By $w_{ij}^{\varepsilon} \le w_{ij}$, we infer $R_{ij} \le R_{ij}^{\varepsilon}$. Thus, we get

$$nR_{ij} - \frac{n}{d_i} - \frac{n}{d_j} \le \left| nR_{ij} - (\frac{n}{d_i} + \frac{n}{d_j}) \right| \le \underbrace{\left| nR_{ij} - (\frac{n}{d_i^{\varepsilon}} + \frac{n}{d_j^{\varepsilon}}) \right|}_{I} + \underbrace{\left| (\frac{n}{d_i^{\varepsilon}} + \frac{n}{d_j^{\varepsilon}}) - (\frac{n}{d_i} + \frac{n}{d_j}) \right|}_{II}.$$

A probabilistic bound for term II can be determined by standard concentration arguments since the degrees in the truncated hyper-graph converge to the ones in the non-truncated hyper-graph.

For bound term I, we assume that that ε meets $\sigma^2 = w(\dfrac{\varepsilon^2}{\log(n\varepsilon^{d+2})})$. Let $\lambda^{\varepsilon,weighted}$ be the eigenvalues of the ε-truncated Gauss hyper-graph, and $w_{\min}^{\varepsilon}$, $w_{\max}^{\varepsilon}$ be its minimal and maximal hyper-edge weights, respectively. Let H'' be a hyper-graph which is the unweighted version of the ε-truncated Gauss hyper-graph H^{ε}. Therefore, H'' coincides with the standard ε-hyper-graph. Let $\lambda^{\varepsilon,weighted}$ be its eigenvalues. In view of Lemma 2 and Corollary 1, we deduce

$$\left| nR_{ij} - (\frac{n}{d_i^{\varepsilon}} + \frac{n}{d_j^{\varepsilon}}) \right| \le \frac{w_{\max}^{\varepsilon}}{d_{\min}^{\varepsilon}} (\frac{1}{1 - \lambda_2^{\varepsilon,\text{unweighted}}} + 2)(\frac{n}{d_i^{\varepsilon}} + \frac{n}{d_j^{\varepsilon}}) \le \frac{w_{\max}^{\varepsilon}}{d_{\min}^{\varepsilon}} (\frac{w_{\max}^{\varepsilon}}{d_{\min}^{\varepsilon}} \frac{1}{1 - \lambda_2^{\varepsilon,\text{unweighted}}} + 2)(\frac{n}{d_i^{\varepsilon}} + \frac{n}{d_j^{\varepsilon}}).$$

It is not hard to verify that the last factor in above inequality converges to a constant:

$$(\frac{n}{d_i^{\varepsilon}} + \frac{n}{d_j^{\varepsilon}}) \to \frac{1}{p(X_i)} + \frac{1}{p(X_j)}.$$

And, we use the following quantities to deal with the other factors of this inequality:

$$w_{\min}^{\varepsilon} \ge \frac{1}{\sigma^d} \exp(-\frac{\varepsilon^2}{2\sigma^2}),\ w_{\max}^{\varepsilon} \le \frac{1}{\sigma^d},\ d_{\min}^{\varepsilon} \ge n\varepsilon^d w_{\min}^{\varepsilon},\ 1 - \lambda_2^{\varepsilon,\text{unweighted}} \ge \varepsilon^2.$$

By combining these quantities with above inequality and using $\sigma^2 = w(\dfrac{\varepsilon^2}{\log(n\varepsilon^{d+2})})$, we lead to the desired conclusion.

REFERENCES

1. Gao W, Liang L, Zhang C; The research on fractional chromatic number of hypergraph, Journal of Yunnan Normal University (Natural Science Edition), 2009; 29(1): 33-36.
2. Wang YY, Jia ZY, Gao W; Generalization bound analysis for classification algorithm under semi-supervised hypergraph normalization laplacian dimension reduction, Journal of Nantong University (Natural Science Edition), 2013; 12(2): 57-61.
3. Zhu LL, Gao W; Analysis for classification algorithm on hypergraph, Open Cybernetics and Systemics Journal, In press.
4. Zhou D, Huang J, Scholkopf B; Beyond pair-wise classification and clustering using hypergraphs, Canada: University of Waterloo, 2005.
5. Zhou D, Huang J, Scholkopf B; Learning with hypergraphs, clustering, Classification, and Embedding, Proceedings of 20th Annual Conference on Neural Information Processing Systems, 2006, Vancouver / Whistler, Canada: IEEE, 2006; 1601-1608.
6. Liang S, Ji S, Ye J; Hypergraph spectral learning for multi-label classification, Proceeding of the 14th ACM SIGKDD international conference on Know ledge discovery and data mining . Las Vegas, Nevada, USA: ACM, 2008; 668- 676.
7. Chen G, Zhang J, Wang F; Efficient multi- label classification with hypergraph regularization, IEEE Conference on Computer Vision and Pattern Recognition. Miami, FL, USA: IEEE, 2009;9: 1658-1665.