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A numerical solution of multi-term fractional ordinary differential equations by Generalized Taylor matrix method

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Abstract: In this article, we present an efficient algorithm for solving a multi-term fractional ordinary differential equations (semidifferential equations) using the Generalized Taylor matrix method. This method is based on first taking the truncated Generalized Taylor expansions of the solution function in the multi-term fractional ordinary differential equation and then substituting their matrix forms into the equation. The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of algebraic equations thus greatly simplifying the problem. Numerical examples are used to illustrate the preciseness and effectiveness of the proposed method. Additionally, we successfully solve the modelling of physical phenomena such as Bagley-Torvik equation, relaxation-oscillation equation.
Keywords: Multi-term fractional differential equations; Bagley-Torvik equation; relaxation-oscillation equation; Generalized Taylor matrix method.

1. Introduction

Fractional calculus has become the focus of interest for many researchers in different disciplines of science and technology. In earlier work the main application of the fractional calculus has been as a technique for solving integral equations. The theory for the derivatives of fractional order was developed in the 19th century[1,2]. The fractional differential equations (FDEs) have received considerable interest in recent years. In recent studies are attracted to study fractional differential equations in, psychology physics, chemistry, engineering, finance, and other sciences such as dynamics model of love[3], nonlinear oscillation of earthquake can be modeled with fractional derivatives [4], the fluid-dynamic traffic model with fractional derivatives[5], relaxation–oscillation model[6], modeling of viscoelastic dampers[7], self-similar protein dynamics[8], bioengineering[9], viscoelastically damped structures[10] and others[11-20]. This mathematical phenomenon allows to describe a real object more accurately than the classical integer methods. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is nonlocal. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This makes studying fractional order systems an active area of research.
A fractional differential equation of the form

$$\left(D^{n/2} + A_{n-1}D^{(n-1)/2} + \Lambda + A_0 D^0\right)y(x) = f(x),\ n \in N \tag{1}$$

with the conditions

$$D^i y(0) = \lambda_i,\ i = 0,1,\mathrm{K},\ q = \lfloor n/2 \rfloor \tag{2}$$

is called a multi-term fractional ordinary differential equations of order n where $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$.
The analytic results on existence and uniqueness of solutions to fractional differential equations have been investigated by many authors[1,2]. The multi-term fractional ordinary differential equations appear the modelling of many physical phenomena. An important example is the Bagley-Torvik equation which is defined by[1,21-26]

$$\left(AD^2 + A_3 D^{3/2} + A_0 D^0\right)y(x) = f(x) \tag{3}$$

with the initial conditions

$$y(0)=0,\; y'(0)=0 \tag{4}$$

where $A=m$, $A_1=2A\sqrt{\mu\rho}$, $A_0=k$ and where μ is the viscosity, ρ is the fluid density.
This equation arises in the modelling of the motion of a rigid plate immersed in a Newtonian fluid. The motion of a rigid plate of mass m and area A connected by a mass less spring of stiffness k, immersed in a Newtonian fluid.

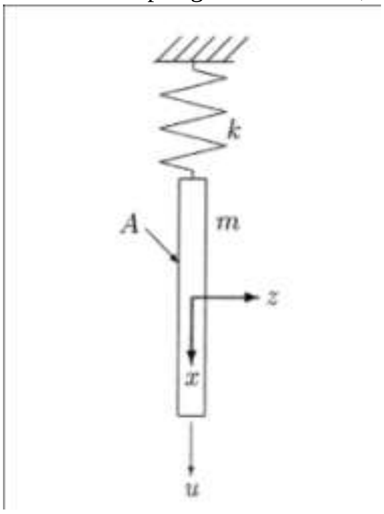

Fig. 1. Rigid plate of mass m immersed into a Newtonian fluid

A rigid plate of mass m immersed into an infinite Newtonian fluid as shown in the Fig.1. The plate is held at a fixed point by means of a spring of stiffness k. It is assumed that the motions of spring do not influence the motion of the fluid and that the area A of the plate is very large, such that the stress-velocity relationship is valid on both sides of the plate.
Another important example is fractional relaxation-oscillation model can be depicted as

$$D^{\alpha}y(x)+Ay(x)=f(x)$$

$$y(0)=a \text{ if } 0<\alpha\le 1$$

or

$$y(0)=a \text{ and } y'(0)=b \text{ if } 0<\alpha\le 2$$

where A is a positive constant. For $0<\alpha\le 2$ this equation is called the fractional relaxation–oscillation equation. When $0<\alpha\le 1$, the model describes the relaxation with the power law attenuation. When $1<\alpha\le 2$ 2, the model depicts the damped oscillation with viscoelastic intrinsic damping of oscillator. This model has been applied in electrical model of the heart, signal processing, modeling cardiac pacemakers, predator–prey system, spruce–budworm interactions etc.
In this study, we seek the approximate solution of Eq.(1) with the Generalized Taylor series as $y_N(x)\in C(a,b]$,

$$y_N(x)=\sum_{k=0}^{N}\frac{(x-c)^{k\alpha}}{\Gamma(k\alpha+1)}\left(D^{k\alpha}y_N(x)\right)_{x=c} \tag{5}$$

where $0<\alpha\le 1$. We use the Generalized Taylor matrix method (power of fractional number) instead of the standart Taylor matrix method (power of positive integer). Because, if exact solution of Eq.(1) can be written as a fractional Taylor series, then we don't obtain the fractional terms by approximate the standart Taylor matrix method. This method transform each part of equation into matrix form then, we get the linear algebraic equation. Solving this equation, we obtained the Generalized Taylor coefficients then so, we obtain the approximate solutions for various N. All computations are performed on the computer algebraic system Maple 13.

2. Basic Definitions

In this section, we first give some basic definitions and then present properties of fractional calculus[1-2].
Definition 2.1 The Riemann-Liouville fractional derivative of order α with respect to the variable x and with the starting point at $x=a$ is

$$
{}_aD_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha+m+1)} \dfrac{d^{m+1}}{dx^{m+1}} \displaystyle\int_a^x (x-\tau)^{m-\alpha} f(\tau)d\tau, & 0 \le m \le \alpha < m+1 \\ \dfrac{d^m f(x)}{dx^m}, & \alpha = m+1 \in N \end{cases}
$$

Definition 2.2 The fractional derivative of $f(x)$ by means of Caputo sense is defined as

$$
D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau
$$

for n-1< $\alpha \le n$, $n \in N$, $t > 0$, $f \in C_{-1}^n$.
For the Caputo derivative we further have:

$D^{\alpha} C = 0$, as C is a constant,

$$
D^{\alpha} x^n = \begin{cases} 0, & n \in N, n < \lceil \alpha \rceil \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \in N, n < \lfloor \alpha \rfloor \end{cases}
$$

Theorem 1. (Generalized Taylor Formula) Suppose that $D^{k\alpha} f(x) \in C(a,b]$ for $k = 0,1,\ldots,n+1$ where $0 < \alpha \le 1$, then we have[17]

$$
f(x) = \sum_{i=0}^{n} \frac{(x-c)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha} f(x)\right)(c) + \frac{\left(D^{n+1} f(x)\right)(\xi)}{\Gamma((n+1)\alpha+1)} (x-c)^{(n+1)\alpha}
$$

with $a \le \xi \le x$, $\forall x \in (a,b]$, where

$$
D^{n\alpha} = \underbrace{D^{\alpha} \cdot D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha}}_{n \ times}
$$

3. Fundemental relations

In this section, we consider the multi-term fractional ordinary differential equations (1). We use the Generalized Taylor matrix method to find the truncated Generalized Taylor series expansions of each term in expression at $x = c$ and their matrix representations for solving $nth-$ order multi-term fractional ordinary differential equations with variable coefficients. For this propose, we using the definition of Caputo sense. The definition of Caputo is suitable for the numerical calculation [2]. The aim is to find the Generalized Taylor coefficients, that is the matrix $\mathbf{A}$. We first consider the solution $y_N(x)$ of Eq.(1) defined by a truncated Generalized Taylor series (5). Then, we have the matrix form of the solution $y_N(x)$

$$
[y_N(x)] = \mathbf{X}\mathbf{M}_0\mathbf{A} \tag{6}
$$

where

$$
\mathbf{X}(x) = \begin{bmatrix} 1 & (x-c)^{\alpha} & (x-c)^{2\alpha} & \cdots & (x-c)^{N\alpha} \end{bmatrix}
$$

$$
\mathbf{M}_0 = \begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(N\alpha+1)} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} D_*^{0\alpha} y(c) \\ D_*^{1\alpha} y(c) \\ D_*^{2\alpha} y(c) \\ \vdots \\ D_*^{N\alpha} y(c) \end{bmatrix}
$$

where $D^0 y_N(x) = y_N(x)$.

Now, we consider the differential part of $D^{i\alpha}y_N(x)$ in Eq. (1) where $\alpha = 1/2$ and $i = n, n-1, \ldots, 0$.

For $i = 1$, we obtained the matrix representation of the function $D^{1\alpha}y_N(x)$

$$D^{1\alpha}y_N(x) = D^{1\alpha}\mathbf{X}(x)\mathbf{M}_0\mathbf{A} \tag{7}$$

and we compute the $D^{1\alpha}\mathbf{X}(x)$, then

$$\begin{aligned} D^{1\alpha}\mathbf{X}(x) &= \left[D^{\alpha}1 \quad D^{\alpha}(x-c)^{\alpha} \quad D^{\alpha}(x-c)^{2\alpha} \quad \cdots \quad D^{\alpha}(x-c)^{N\alpha} \right] \\ &= \left[0 \quad \frac{\Gamma(\alpha+1)}{\Gamma(1)} \quad \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x-c)^{\alpha} \quad \cdots \quad \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}(x-c)^{(N-1)\alpha} \right] \\ &= \mathbf{X}(x)\mathbf{M}_1 \end{aligned}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Then, the matrix representation of $D^{1\alpha}y_N(x)$ as

$$D^{1\alpha}y_N(x) = \mathbf{X}(x)\mathbf{M}_1\mathbf{M}_0\mathbf{A} \tag{8}$$

Similiarly, for $i = k$, we obtain

$$D^{k\alpha}y_N(x) = \mathbf{X}(x)\mathbf{M}_k\mathbf{M}_0\mathbf{A}, \quad 0 \le k \le n$$

where

$$\mathbf{M}_k = \begin{bmatrix} 0 & \cdots & \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-k)\alpha+1)} \\ 0 & \cdots & 0 & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Then, so the matrix representation of $D^{i\alpha}y_N(x)$ become

$$D^{i\alpha}y_N(x) = \mathbf{X}(x)\mathbf{M}_i\mathbf{M}_0\mathbf{A} \tag{9}$$

Moreover, let assume that the function $f(x)$ can be written as a truncated Generalized Taylor series

$$f(x) = \sum_{n=0}^{N} \frac{1}{\Gamma(n\alpha+1)} \left(D^{n\alpha} f(x) \right)_{x=c} (x-c)^{n\alpha} \tag{10}$$

Then, so we obtaine the matrix form of Eq.(10)

$$\left[f(x) \right] = \mathbf{X}(x)\mathbf{M}_0\mathbf{F} \tag{11}$$

where

$$F = \left[f(c) \quad \left(D^{\alpha} f(x)\right)_{x=c} \quad \left(D^{2\alpha} f(x)\right)_{x=c} \quad \cdots \quad \left(D^{N\alpha} f(x)\right)_{x=c} \right]^{T}.$$

Thus, we obtain the matrix-vector form of Eq.(1) as

$$\left(\mathbf{M}_n\mathbf{M}_0 + A_{n-1}\mathbf{M}_{n-1}\mathbf{M}_0 + \ldots + A_1\mathbf{M}_1\mathbf{M}_0 + A_0\mathbf{M}_0\right)\mathbf{A} = \mathbf{M}_0\mathbf{F} \tag{12}$$

On the other hand, the matrix representation of the conditions Eq.(2) become:

$$D^{i} y_N(0) = D^{(\frac{i}{\alpha})\alpha} y_N(0) = \mathbf{X}(0)\mathbf{M}_{i/\alpha}\mathbf{A} \tag{13}$$

Let us define $\mathbf{U}_i$ as

$$\mathbf{U}_i = \mathbf{X}(c)\mathbf{M}_{i/\alpha} = \left[u_{i0} \quad u_{i1} \quad u_{i2} \quad \cdots \quad u_{iN}\right] = \left[\lambda_i\right],\ i = 0,1,\ldots,q = \lfloor n-1 \rfloor \tag{14}$$

4. Method of solution

We can write Eq. (12) in the form

$$\mathbf{WA} = \mathbf{G} \tag{15}$$

where

$$\mathbf{W} = \left[w_{ij}\right] = \mathbf{M}_n\mathbf{M}_0 + A_{n-1}\mathbf{M}_{n-1}\mathbf{M}_0 + \ldots + A_1\mathbf{M}_1\mathbf{M}_0 + A_0\mathbf{M}_0,\ i,j = 0,1\ldots,N$$

and $\mathbf{G} = \mathbf{M}_0\mathbf{F}$.

Consequently, to find the unknown Generalized Taylor coefficients $D^{k\alpha} y_N(c)$, $k = 0,1,\ldots,N$ related with the approximate solution of the problem consisting of Eq. (1) and conditions (2), by replacing the m row matrices (14) by the last n rows of the matrix (15) and we have augmented matrix[28]

$$\left[\mathbf{W}^*;\mathbf{G}^*\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & (D^{0\alpha} f(c))/\Gamma(0\alpha+1) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & (D^{1\alpha} f(c))/\Gamma(1\alpha+1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{N-q0} & w_{N-q1} & \cdots & w_{N-qN} & ; & (D^{(N-q)\alpha} f(c))/\Gamma((N-q)\alpha+1) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ u_{q0} & u_{q1} & \cdots & u_{qN} & ; & \lambda_q \end{bmatrix}$$

or the corresponding matrix equation

$$\mathbf{W}^*\mathbf{A} = \mathbf{G}^* \tag{16}$$

So, we obtained to a system of $(N+1)\times(N+1)$ linear algebraic equations with $(N+1)$ unknown Generalized Taylor coefficients. If $rank\mathbf{W}^* = rank[\mathbf{W}^*;\mathbf{G}^*] = N+1$, the we can be write $\mathbf{A} = (\mathbf{W}^*)^{-1}\mathbf{G}^*$. Thus, the matrix $\mathbf{A}$ is uniquely determined. Also the Eq.(1) with conditions (2) has a unique solution. On the other hand, when $\left|\mathbf{W}^*\right| = 0$, if $rankW^* = rank[\mathbf{W}^*;\mathbf{G}^*] < N+1$, then we may find a particular solution. Otherwise if $rank\mathbf{W}^* \neq rank[\mathbf{W}^*;\mathbf{G}^*]$, then it is not a solution.

4.1 Accuracy of the solution and error bound

To investigate the convergence, we define the error function as:

$$e_N(x) = y(x) - y_N(x) \tag{17}$$

where $y(x)$ and $y_N(x)$ are the exact and the computed solution of the Eq.(1), respectively. Substituting $y_N(x)$ into Eq.(1) leads to

$$\left(D^{n/2} + A_{n-1}D^{(n-1)/2} + \Lambda + A_0 D^0\right) y_N(x) = f(x) + p_N(x),\ n \in N \tag{18}$$

where $p_N(x)$ is the perturbation term that can be obtained by substituting the computed solution $y_N(x)$ into Eq.(1) , i.e.

$$p_N(x) = \left(D^{n/2} + A_{n-1}D^{(n-1)/2} + \Lambda + A_0 D^0\right) y_N(x) - f(x),\ n \in N \tag{19}$$

Now, by subtracting (18) from (1) and using (17), the error function $e_N(x)$ satisfies:

$$-p_N(x) = \left(D^{n/2} + A_{n-1}D^{(n-1)/2} + \Lambda + A_0 D^0\right) e_N(x),\ n \in N \tag{20}$$

Theorem 4.1 f be continuous functions on their domains. Suppose that for some positive M , we have

$$\left|D^{(N+1)\alpha} y(x)\right| \le M,\ \forall x \in [0,b] \tag{21}$$

Then,

$$\lim_{N\to\infty} p_N = 0.$$

Proof:

Suppose that the solution $y(x)$ and computed solution $y_N(x)$ of Eq.(1) are approximated by their Taylor expansions about zero. Then we may write

$$e_N(x) = \sum_{k=N+1}^{\infty} \frac{x^{k\alpha}}{\Gamma(k+1)} \left(D^{k\alpha} y(x)\right)_{x=0} \tag{22}$$

which can be represented as

$$e_N(x) = \frac{x^{(N+1)\alpha}}{\Gamma(N+1)} \left(D^{(N+1)\alpha} y(\xi)\right),\ \xi \in (0,x) \tag{23}$$

for some $\xi \in (0,x)$ by generalized Taylor theorem.

Replacing $e_N(x)$ by (23) into (20) gives

$$-p_N(x) = \left(D^{n/2} + A_{n-1}D^{(n-1)/2} + \Lambda + A_0 D^0\right) \frac{x^{(N+1)\alpha}}{\Gamma(N+1)} \left(D^{(N+1)\alpha} y(\xi)\right) \tag{24}$$

Therefore, we have

$$\left|p_N(x)\right| \le \left(D^{(N+1)\alpha} y(\xi)\right) \frac{\Gamma((N+1)\alpha+1)}{\Gamma(N\alpha+1)} \left(\frac{b^{(N+1)\alpha - n/2}}{\Gamma((N+1)\alpha + 1 - \frac{n}{2})} + \right.$$

$$\left. A_{n-1} \frac{b^{(N+1)\alpha-(n-1)/2}}{\Gamma((N+1)\alpha + 1 - \frac{n-1}{2})} + \Lambda + A_0 \frac{b^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} \right)$$

We assume $A^* = \max\{A_{n-1},..., A_1, A_0\}$ and under assumption, we have

$$\left|p_N(x)\right| \le A^* M \frac{\Gamma((N+1)\alpha+1)}{\underbrace{\Gamma(N\alpha+1)\Gamma(N\alpha+1)...\Gamma(N\alpha+1)}_{n-times}\Gamma(N\alpha+1)} b^{(N+1)\alpha}$$

thus, the prof is complete.

Theorem 4.2 Under the assumptions of Theroem 4.1, we have $\lim\limits_{N\to\infty} e_N = 0$.

Proof:

Let assume

$$D^* = D^{n/2} + A_{n-1}D^{(n-1)/2} + \Lambda + A_0 I$$

Then, the Eq.(20) can be written as

$$D^* e_N(x) = -p_N(x)$$

Under the assumption, $\lim_{N\to\infty} p_N = 0$ and the Eq.(1) has unique solution [2]. Then, the operator D^* is invertible. Hence $\lim_{N\to\infty} e_N = 0$.

5. Examples

In order to illustrate the effectiveness of the method proposed in this paper, several numerical examples are carried out in this section. In the following computations, for convenience, absolute errors between N th-order approximate values $y_N(x)$ and the corresponding exact values $y(x)$ as $N_e = |y_N(x) - y(x)|$ are determined and all computations performe computer algebraic system with mathematical programing in Maple 13.

Example 1: Consider the following boundary Bagley-Torvik equation[26]:

$$\left(D^2 + D^{3/2} + D^0\right)y(x) = f(x),\ x \in [0,1] \tag{25}$$

with initial conditions

$$y(0) = 0,\ y'(0) = 0.$$

where $f(x) = x^2 + 4\sqrt{\frac{x}{4}} + 2$.

Now, we can apply our technique described in Section 4 in Eq.(25) for $N = 6,\ c = 0$ that is;

$$y_6(x) = \sum_{k=0}^{6} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} \left(D^{k\alpha} y(x)\right)(0).$$

Fundamental matrix relation of this equation is

$$\left(\mathbf{M}_4\mathbf{M}_0 + \mathbf{M}_3\mathbf{M}_0 + \mathbf{M}_0\right)\mathbf{A} = \mathbf{M}_0\mathbf{F}$$

where

$$\mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix} \mathbf{M}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{M}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{M}_0\mathbf{F} = \begin{bmatrix} 2 \\ 4/\sqrt{\pi} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Also, we have the matrix representation of conditions as,

$$y(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{A} = [0]$$
$$y'(0) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{A} = [0].$$

then, the augmented matrix becomes

$$[\mathbf{W}^*;\mathbf{F}^*] = \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & ; & 2 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 & ; & 4/\sqrt{\pi} \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & ; & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 4/3\sqrt{\pi} & ; & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & ; & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & ; & 0 \end{array}\right]$$

and solving this equation, we obtained the coefficients of the generalized Taylor series

$$\mathbf{A} = [0 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0].$$

Hence, for $N = 6$, the approximate solution of example 1 is given

$$y_6 = x^2$$

which is the exact solution of this problem. Since the exact solution is a polynomial of degree 2, this method gives the exact solution for $N \geq 4$. The condition numbers
of the matrices $\mathbf{W}^*$ for $n = 5, \ldots, 10$ are given in Fig. 2.

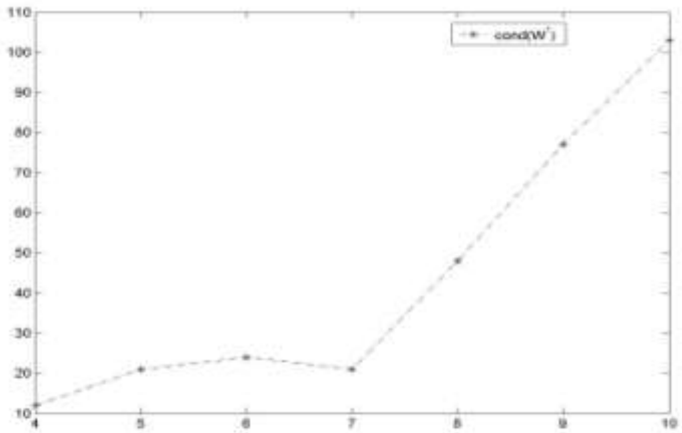

Fig. 2. The condition number of matrices $\mathbf{W}^*$ for $n = 4, 5, \ldots, 10$ in infinity norm.

Example 2: Let us consider the Bagley-Torvik equation[21,25]

$$A_2 D^2 y(x) + A_1 D^{3/2} y(x) + A_0 y(x) = f(x)$$

We consider the case $f(x) = A_0(x+1)$, $A_2 = 1$, $A_1 = 1$, $A_0 = 1$ with the initial conditions

$$y(0) = 1, \ y'(0) = 1.$$

In the following, using the present method, we gain the approximate solution $y_4(x) = x + 1$ for $N = 4$ which is the exact solution of this equation. Since the exact solution is a polynomial of degree 1, this method gives the exact solution for $N \geq 4$. The condition numbers of the matrices $\mathbf{W}^*$ for $n = 5, \ldots, 10$ are given in Fig. 3.

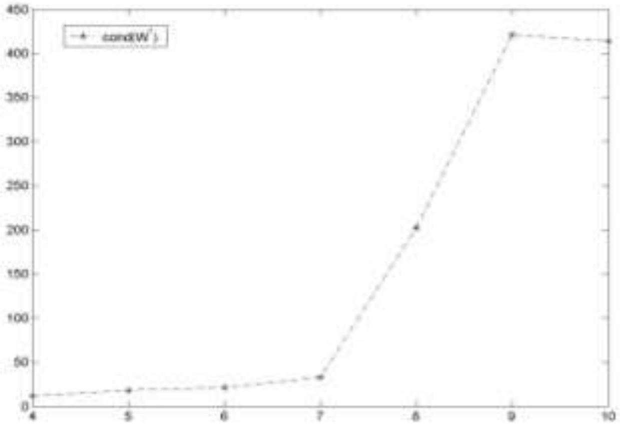

Fig. 3. The condition number of matrices $\mathbf{W}^*$ for $n = 4,5,\ldots,10$ in infinity norm.

Example 3: Let us consider the Bagley-Torvik equation [33-34]

$$D^\alpha y(x) + y(x) = 0,\ \ 0 < \alpha \le 2 \tag{19}$$

$$y(0) = 1,\ y'(0) = 0$$

If $0 < \alpha \le 1$, the first initial condition is needed, while all the initial conditions are neccessary when $1 < \alpha < 2$. The exact solution of this problem is $y(x) = E_\alpha(-x^\alpha)$. Here $E_\alpha(z)$ denotes the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

The numerical results by Taylor matrix method for and the exact solutions $E_\alpha(-x^\alpha)$ for $\alpha = 0.25, 0.5, 0.75$ and 1 are plotted in Fig.4, which shows that the numerical results are consistent with the exact ones and as α approaches to 1 the corresponding solutions of (19) approach to that of integer-order differential equation. For $1 < \alpha < 2$, Fig. 5 illustrate the numerical solutions by present method and exact ones $E_\alpha(-x^\alpha)$ for $\alpha = 1.25, 1.5$ and 2. Obviously the numerical results agree with the exact ones. For $\alpha = 2$, the Eq.(19) is the oscillation equation and the exact solution is $y(x) = \cos(x)$. We obtain the error function estimated for $N = 27$ and $\alpha = 0.5, 1.5$ as:

$$R_{27}(x) = 0.2E-14 + 0.2E-14x + 0.2E-14x^2 + 0.4E-14x^{\frac{3}{2}} + 0.1E-14x^3 + 0.6E-14x^{\frac{7}{2}}$$
$$+0.1E-16x^{\frac{11}{2}} + 0.2E-17x^{\frac{13}{2}} - 0.1E-18x^{\frac{15}{2}} + 0.1E-18x^8 + 0.5E-19x^{\frac{17}{2}} + 0.1E-20x^{\frac{19}{2}}$$
$$+0.1E-20x^{10} + 0.2E-21x^{\frac{21}{2}}\ 0.1E-22x^{\frac{23}{2}} - 0.433044E-10x^{\frac{27}{2}}$$

$$R_{27}(x) = 0.2E-14 + 0.4E-14x^{\frac{3}{2}} - 0.1E-18x^{\frac{15}{2}} + 0.1E-19x^9 + 0.21E-21x^{\frac{21}{2}}$$
$$+0.1E-22x^{12} - 0.433044E-10x^{\frac{27}{2}}$$

We define the maximum errors for $y_N(x)$ as,

$E_N = \|y(x) - y_N(x)\|_\infty = \max\{|y(x) - y_N(x)|, x \in [0,1]\}$. In Table 1 and Table 2, we give some numerical results such as comparison of maximum absolute errors, maximum error estimation values for $\alpha = 0.6, 1.5$ respectively. Moreover, we compare absolute errors with Operational Matrix Method[37] and Present Method ($N = 27$) in Table 3.

Table 1: Compare of some numerical values for $\alpha = 0.2$.

N	25	27	30
E_N	10^{-12}	10^{-13}	10^{-16}
$\lVert R_N \rVert_\infty$	10^{-12}	10^{-13}	10^{-15}

Table 2: Compare of some numerical values for $\alpha = 1.5$.

N	25	27	30
E_N	10^{-19}	10^{-24}	10^{-29}
$\lVert R_N \rVert_\infty$	10^{-19}	10^{-24}	10^{-29}

Table 3: Campare of some methods of Example 3

α	x	Oper. matrix method[37]	Present method ($N = 27$)
0.2	0.1	2.9×10^{-1}	5.6×10^{-7}
	0.3	4.5×10^{-1}	2.2×10^{-5}
	0.5	7.4×10^{-1}	3.7×10^{-3}
	0.7	3.7×10^{-1}	2.7×10^{-2}
	0.9	2.0×10^{-1}	9.5×10^{-2}
0.6	0.1	6.7×10^{-3}	1.0×10^{-14}
	0.3	2.0×10^{-5}	4.0×10^{-14}
	0.5	5.2×10^{-3}	3.0×10^{-14}
	0.7	4.4×10^{-3}	1.0×10^{-14}
	0.9	4.6×10^{-3}	8.0×10^{-14}

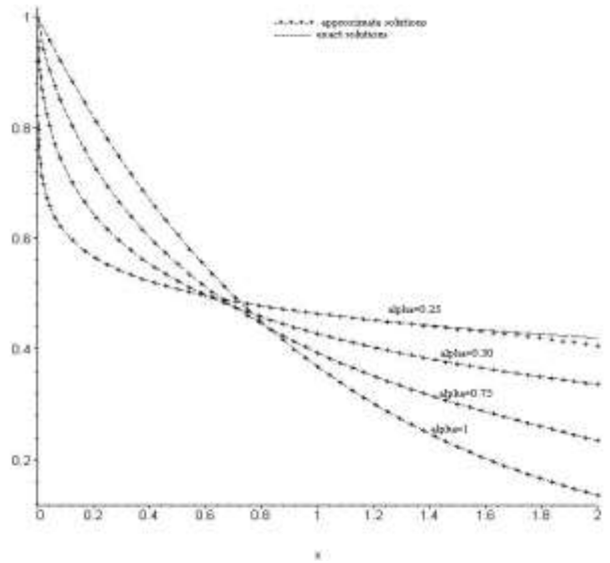

Fig. 4. Comparison of $y_N(x)$ for $N = 27$ and $\alpha = 0.25, 0.50, 0.75, 1$ with exact solution in Example 3.

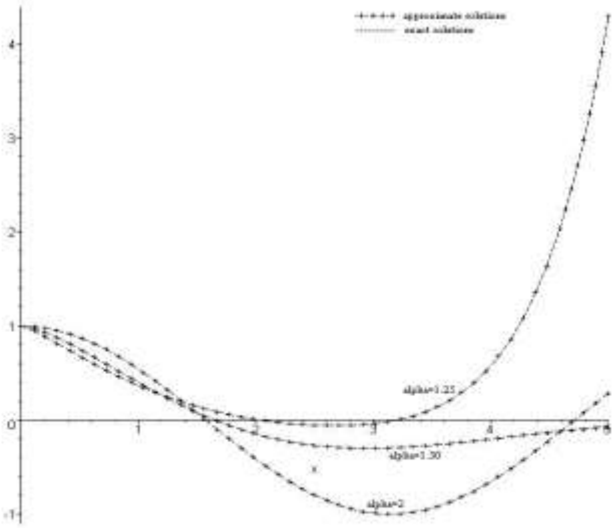

Fig. 5. Comparison of $y_N(x)$ for $N=27$ and $\alpha=1.25,1.50,2$ with exact solution in Example 3.

Example 4: Consider the following composite fractional relaxation-oscillation equation [15]

$$\left(D^1 - A_1 D^{1/2} - A_0 D^0\right) y(x) = 0\,,\, x>0,\; 0<\alpha\le 1$$

with the condition

$$y(0)=1.$$

Taking $A_1=-1$, $A_0=-1$ and $N=22$, $c=0$ we approximately solve this equation and

$$\begin{aligned} y_{22}(x) &= 1-1.128379x^{1/2}+0.752252x^{3/2}-0.5x^2+0.166666x^3-0.859717E-1x^{7/2} \\ &+0.191048E-1x^{9/2}-0.833333E-2x^5+0.138888E-2x^6-0.534400E-3x^{13/2} \\ &+0.712534E-4x^{15/2}-0.248015E-4x^8+0.275573E-5x^9-0.882395E-6x^{19/2} \\ &+0.840376E-7x^{21/2}-0.250521E-7x^{11} \end{aligned}$$

We give the numerical results with comparison Fractional Difference Method (FDM), Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) and Present Method (PM) in Table 4 and plotted the numerical results in Table 4 as Figure 6. It is clear that the approximations obtained using the decomposition method, the variational iteration method and fractional different method are in high agreement with those obtained using the present method. Moreover, we obtain the error function estimated for $N=22$ as:

$$\begin{aligned} R_{22}(x) &= +0.1E-9-0.1E-9x-0.3E-9x^{\frac{3}{2}}+0.1E-9x^2+0.1E-9x^{\frac{5}{2}}+0.1E-14x^4+0.1E-11x^5 \\ &+0.1E-11x^{\frac{11}{2}}-0.1E-12x^{\frac{13}{2}}-0.2E-13x^{\frac{15}{2}}-0.3E-15x^{\frac{19}{2}}-0.2E-15x^{10}-0.5E-16x^{\frac{21}{2}}-0.250E-7x^{11} \end{aligned}$$

and $\max\{E_{22}(x), x\in[0,1]\}=0.251402E-7$.

In Fig.7, we check accuracy of the approximate solution by obtained the Generalized Taylor matrix method and comparing of $R_{22}(x)$ and $R_{24}(x)$.

Table 4: The numerical results with comporison FDM, ADM, VIM and PM ($N = 22$)

x	FDM	ADM	VIM	PM
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.668686	0.662104	0.662104	0.662103
0.2	0.549125	0.543694	0.543694	0.543693
0.3	0.468508	0.463863	0.463863	0.463862
0.4	0.408125	0.404072	0.404072	0.404072
0.5	0.360495	0.356911	0.356911	0.356911
0.6	0.321710	0.318509	0.318509	0.318509
0.7	0.289424	0.286543	0.286543	0.286542
0.8	0.262106	0.259495	0.259495	0.259494
0.9	0.238694	0.236315	0.236315	0.236314
1.0	0.218421	0.216243	0.216243	0.216242

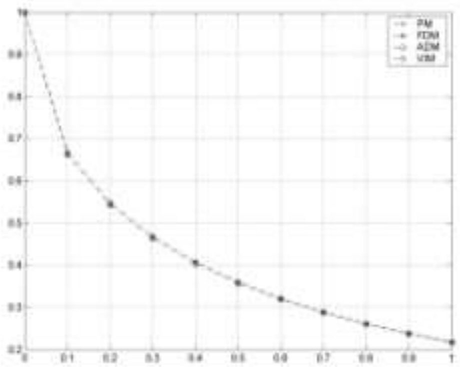

Fig. 6. Comparison of the some appro ximate methods for Example 4.

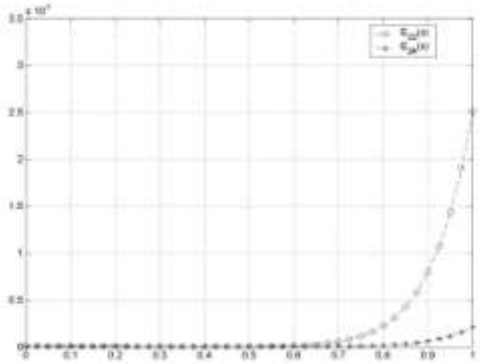

Fig. 7. Comparison of $R_{22}(x)$ and $R_{24}(x)$ for Example 4.

6. Conclusion

A scheme for approximate numerical solution of a class of fractional differential equations as multi-term fractional ordinary differential equations was presented. The semidifferential equation was expressed in terms the truncated Generalized Taylor series and the properties of this derivative and the truncated Generalized Taylor series were used to reduce multi-term fractional differential equation into linear algebraic equation. Mentioned method transforms the multi-term fractional differential equations into a algebraically system which is independent on collocation points. It is easy to write PC codes which are related to obtained system for necessary computation. So, we have some considerable advantage of the method is that the Generalized Taylor polynomial coefficients of the solution are found very easily, shorter computation times are so low such as 1 sn for Example 1, 1.3 sn for Example 3 (CPU Core2 Duo 2.13 Ghz, RAM 2Gb) and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. It shows simplicity and effectiveness of this method. An interesting feature of this method is to find the analytical solutions if the system has exact solutions that are polynomial functions. Examples show that the Generalized Taylor matrix method has been successfully applied to finding the approximate solutions of the multi-term fractional differential equation. The numerical solutions are compared with exact solution and some other methods.

References

1. Diethelm K; The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin Heidelberg, 2010.
2. PodlubnyI; Fractional differential equations. New York: Academic Pres, 1999.
3. Ahmad WM, El-Khazali R; Fractional-order dynamical models of love, Chaos, Solitons & Fractals, 2007; 33:1367-1375.
4. He JH; Nonlinear oscillation with fractional derivative and its applications, in: International Conference on Vibrating Engineering, Dalian, China, 1998.
5. He JH; Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol, 1999; 15(2):86–90.
6. El-Sayed AMA, Gaafar FM; Fractional order differential equations with memory and fractional-order relaxation–oscillation model, Pure Math. Appl, 2001; 12:296–310.
7. Lewandowski R, Chorazyczewski B; Identification of the parameters of the Kelvin–Voigt and the Maxwell fractional models, used to modeling of viscoelastic dampers, Comput. Struct, 2010; 88:1–17.

8. Glockle WG, Nonnenmacher TF; A fractional calculus approach to self-similar protein dynamics, Biophys. J, 1995; 68:46–53.
9. Magin RL; Fractional Calculus in Bioengineering, Begell House Publishers, 2006.
10. Anderson BD, Moore JB; Optimal Control-Linear Quadratic Methods, Prentice-Hall, New Jersey, 1990.
11. Bagley RL, Torvik BJ; Fractional calculus in the transient analysis of viscoelastically damped structures, AIAA J, 1985; 23(6) :918–925.
12. He JH; Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Eng, 1998; 167:57–68.
13. Bagley RL, Torvik PJ; On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech, 1984; 51:294–298.
14. Mainardi F; Fractional calculus: Some basic problems in continuum and statistical mechanics', in: A. carpinteri, F. Mainard, Fractals and Fractional calculus in Continuum Mechanics, Springer-Verlag, New York, 1997.
15. Momoni S, Odibat Z: Numerical comparison of methods for solving linear differential equations of fractional order, Chaos, Solitons and Fractals, 2007; 31:1248-1255.
16. Deng J, Ma L; Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Appl. Math. Lett, 2010; 23:676–680.
17. Bouafoura MK, Braiek NB; PIkDl controller design for integer and fractional plants using piecewise orthogonal functions, Commun. Nonlinear Sci. Numer. Simul, 2010; 15:1267–1278.
18. Jumarie G; Fractional Brownian motions via random walk in the complex plane and via fractional derivative. Comparison and further results on their Fokker–Planck equations. Chaos, Solitons & Fractals, 2004; 22:907–25.
19. Anderson BD, Moore JB; Optimal Control-Linear Quadratic Methods, Prentice-Hall, New Jersey, 1990.
20. Oustaloup A; From fractality ton on integer derivation through recursivity, a property common to these two consepts: a fundemental idea for a new process conta-rol strategy, Proceedings of the 12th IMACS World Congress, Paris, 1988; 3:203-208.
21. Diethelm K, Ford NJ; Numerical solution of the Bagley–Torvik equation, BIT, 2002; 42:490–507.
22. Saha Ray S, Bera RK; Analytical solution of the Bagley Torvik equation by Adomian decomposition method, Appl. Math. Comput., 2005;168:398–410.
23. Li Y; Zhao W; Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, Appl. Math. Comp., 2010; 216:2276–2285.
24. Saha Ray S; On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley-Torvik equation, Appl. Math. Appl, 2012; 218:5239-5248.
25. Keskin Y, Karaoğlu O, Servi S, Oturanç G; The approximate solution of high-order linear fractional differential equations with variable coefficients in terms of generalized Taylor polynomials, Math. Comp. Appl, 2011;16(3) : 617-629.
26. Doha EH, Bhrawy AH, Ezz-Eldien SS; A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, Comp. Math. Appl, 2011; 62:2364-2373.
27. Odibat Z, Shawagfeh NT; Generalized Taylor's Formula. Appl. Math. Comp, 2007; 186(1):286–293.
28. Gülsu M, Sezer M; A Taylor polynomial approach for solving differential difference equations, J. Comp. Appl. Math, 2006;186:349–364.
29. Gülsu M, Öztürk Y, Sezer M; A new collocation method for solution of mixed linear integro-differential-difference equations, Appl. Math. Comp, 2010; 216:2183-2198.
30. Gülsu M, Öztürk Y, Sezer M; On the solution of the Abel equation of the second kind by the shifted Chebyshev polynomials, Appl. Math. Comp, 2011; 217: 4827-4833.
31. Sezer M, Daşcıoğlu AA; A Taylor method for numerical solution of generalized pantograph equations with linear functional argument, J. Comp. Appl. Math, 2007; 200:217-255.
32. Sezer M, Gülsu M; A new polynomial approach for solving difference and Fredholm integro-difference equation with mixed argument, Appl. Math. Comp, 2005; 171(1):332–344.
33. Razzaghi M, Yousefi S; Legendre wavelets operational matrix of integration, Int. J. Syst. Sci, 2001; 32(4):495–502.
34. Babolian E, Fattahzadeh E; Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Comp, 2007; 188:417-426.
35. Blank L; Numerical treatment of differential equations of fractional order, Manchester Centre Comput. Math. Numer. Anal. Rep, 1996; 2876.
36. Li X; Numerical solution of fractional differential equations using cubic B-spline wavelet collocation method, Commun. Nonlinear Sci. Numer. Simulat, 2012;17:3934-3946.
37. Saadatmandi A, Dehghan M; A new operational matrix for solving fractional order differential equations, Comp. Math. Appl, 2010; 59 : 1326-1336.
38. Vanani SK, Aminataei A; Operational Tau approximation for a general class of fractional integro-differential equations, Comp. Appl. Math, 2011; 30(3):655-674.