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# The Proof and Application of Schwarz Integral Inequality

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**Abstract:** In mathematical analysis, Schwarz integral inequality is an important inequality. It is the basis for many inequalities. In this paper, six methods are given for proving Schwarz integral inequality. Furthermore, several examples are given in practical problems.
**Keywords:** integral; inequality; Schwarz integral inequality

## Preliminary Knowledge

**Schwarz integral inequality** If $f(x)$ and $g(x)$ are continuous on the closed interval $[a,b]$.Then

$$(\int_a^b f(x)g(x)dx)^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx.$$

**Lemma 1** (cauchy inequality) Let $a_i$ and $b_i$ ($i$=1,2,…,$n$) are real numbers .Then

$$(\sum_{i=1}^n a_i b_i)^2 \le \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2.$$

**Lemma 2**[1] Let $f(x)$ is continuous on the closed interval $[a,b]$ and suppose that it is nonnegative and isn't always zero . Then

$$\int_a^b f(x)dx > 0 .$$

## Several Methods about the Proof of Schwarz Integral Inequality

**Proof 1**[2] In this part, we give out a method to prove Schwarz integral inequality by using Cauchy inequality. We divide the interval $[a,b]$ into isometric subinterval by means of points

$$x_i = a + \frac{i}{n}(b-a), i = 0,1,2,\Lambda\ ,n.$$

If $f(x)$ and $g(x)$ are continuous on the closed interval $[a,b]$, then they are integrabel.
We have

$$\int_a^b f(x)g(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n f(x_i)g(x_i),$$

$$\int_a^b f^2(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n f^2(x_i),$$

$$\int_a^b g^2(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n g^2(x_i).$$

By lemma 1,we get

$$(\frac{b-a}{n}\sum_{i=1}^n f(x_i)g(x_i))^2 \le \frac{b-a}{n}\sum_{i=1}^n f^2(x_i)\cdot\frac{b-a}{n}\sum_{i=1}^n g^2(x_i).$$

Therefore, let $n \to \infty$,we obtain

$$(\int_a^b f(x)g(x)dx)^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx.$$

**Proof 2** In this part, we give out a method to prove Schwarz integral inequality by using the properties of definite integrals.

If $\int_a^b f^2(x)dx = 0,$ by lemma 2 we get $f(x) \equiv 0$ and then $\int_a^b f(x)g(x)dx = 0,$ so the formula is true.

In the same way, if $\int_a^b g^2(x)dx = 0$ , then the formula is true.

Suppose that $\int_a^b f^2(x)dx > 0,$ and $\int_a^b g^2(x)dx > 0.$

Let $s = (\int_a^b f^2(x)dx)^{\frac{1}{2}},\ t = (\int_a^b g^2(x)dx)^{\frac{1}{2}},$ then $st > 0.$ we have

$$\int_a^b [\frac{|f(x)|}{s} - \frac{|g(x)|}{t}]^2 dx = \int_a^b \frac{f^2(x)}{s^2} dx + \int_a^b \frac{g^2(x)}{t^2} dx - \frac{2}{st}\int_a^b |f(x)g(x)| dx$$

$$= 2 - \frac{2}{st}\int_a^b |f(x)g(x)| dx \geq 0.$$

Therefore, we obtain

$$\int_a^b |f(x)g(x)| dx \leq st = [\int_a^b f^2(x)dx]^{\frac{1}{2}} \cdot [\int_a^b g^2(x)dx]^{\frac{1}{2}}.$$

$$\left|\int_a^b f(x)g(x)dx\right| \leq \int_a^b |f(x)g(x)| dx \leq [\int_a^b f^2(x)dx]^{\frac{1}{2}} \cdot [\int_a^b g^2(x)dx]^{\frac{1}{2}}$$

That is,

$$(\int_a^b f(x)g(x)dx)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

**Proof 3**[3] In this part, we give out a method to prove Schwarz integral inequality by using double integrals.

Let $I = \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$ , $D = \{(x,y) | a \leq x \leq b, a \leq y \leq b.\}$. Then we have

$$I = \iint_D f^2(x)g^2(y)dxdy = \iint_D f^2(y)g^2(x)dxdy.$$

So we get

$$I = \frac{1}{2}\iint_D [f^2(x)g^2(y) + f^2(y)g^2(x)]dxdx \geq \frac{1}{2}\iint_D 2\sqrt{f^2(x)g^2(y) \cdot f^2(y)g^2(x)}dxdx$$

$$= \iint_D |f(x)g(y)f(y)g(x)| dxdy = \int_a^b |f(x)g(x)| dx \cdot \int_a^b |f(y)g(y)| dy = (\int_a^b |f(x)g(x)| dx)^2$$

Because of $\left|\int_a^b f(x)g(x)dx\right| \leq \int_a^b |f(x)g(x)| dx$ , we obtain

$$(\int_a^b f(x)g(x)dx)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

**Proof 4** In this part, we give out a method to prove Schwarz integral inequality by using the monotonicity of a function .

Let $F(x) = \int_a^x f^2(t)dt \cdot \int_a^x g^2(t)dt - (\int_a^x f(t)g(t)dt)^2$ .

If $f(x)$ and $g(x)$ are continuous on the closed interval $[a,b]$, then $F(x)$ is derivable on this closed interval. We have

$$F'(x) = f^2(x)\int_a^x g^2(t)dt + g^2(x)\int_a^x f^2(t)dt - 2f(x)g(x)\int_a^x f(t)g(t)dt$$

$$= \int_a^x [f(x)g(t) - f(t)g(x)]^2 dt \geq 0.$$

Therefore, $F(x)$ is a monotonic and nondecreasing function on the closed interval $[a,b]$.

That is $F(b) \geq F(a) = 0.$ So we obtain

$$(\int_a^b f(x)g(x)dx)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

**Proof 5** In this part, we give out a method to prove Schwarz integral inequality by using the mean value theorem of differentials.

Let $F(x)=\int_a^x f^2(t)dt\cdot\int_a^x g^2(t)dt-(\int_a^x f(t)g(t)dt)^2$, and $F(x)$ meet the conditions of the mean value theorem of differentials, then we have

$$F(b)-F(a)=F'(\xi)(b-a)=(b-a)\int_a^\xi [f(\xi)g(t)-f(t)g(\xi)]^2dt\geq 0.$$

That is $F(b)\geq F(a)=0.$ So we obtain

$$(\int_a^b f(x)g(x)dx)^2\leq\int_a^b f^2(x)dx\int_a^b g^2(x)dx.$$

**Proof 6** In this part, we give out a method to prove Schwarz integral inequality by using the discriminant of quadratic equation with one unknown. For any real number $t$, we have $[tf(x)+g(x)]^2\geq 0.$ Integrating the above formula with respect to $x$ once, we yield

$$t^2\int_a^b f^2(x)dx+2t\int_a^b f(x)g(x)dx+\int_a^b g^2(x)dx\geq 0.$$

The discriminant is

$$\Delta=(2\int_a^b f(x)g(x)dx)^2-4\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx\leq 0.$$

So we obtain

$$(\int_a^b f(x)g(x)dx)^2\leq\int_a^b f^2(x)dx\int_a^b g^2(x)dx.$$

**Several Examples about the Application of Schwarz Integral Inequality**

**Example 1** Let *f*(*x*) is continuous on the closed interval [*a*,*b*] and $f(a)=0,$ show that

$$\int_a^b f^2(x)dx\leq\frac{(b-a)^2}{2}\int_a^b [f'(x)]^2dx.$$

**Proof** Because of $f(a)=0,$ then we obtain $f(x)=\int_a^x f'(t)dt.$

By using Schwarz integral inequality, we have

$$f^2(x)=[\int_a^x f'(t)dt]^2\leq(\int_a^x [f'(x)]^2dx)(x-a)\leq(x-a)\int_a^b [f'(x)]^2dx$$

Integrating the above formula with respect to $x$ once, we yield

$$\int_a^b f^2(x)dx\leq\int_a^b [f'(x)]^2dx\cdot\int_a^b (x-a)dx=\frac{(b-a)^2}{2}\int_a^b [f'(x)]^2dx.$$

**Example 2** Let *f*(*x*) is continuous on the closed interval [*a*,*b*] and $f(x)\geq 0,\int_a^b f(x)dx=1.$

Show that $(\int_a^b f(x)\sin\lambda xdx)^2+(\int_a^b f(x)\cos\lambda xdx)^2\leq 1.$

**Proof** By using Schwarz integral inequality, we have

$$(\int_a^b f(x)\sin\lambda xdx)^2=(\int_a^b \sqrt{f(x)}\cdot\sqrt{f(x)}\sin\lambda xdx)^2$$
$$\leq\int_a^b f(x)dx\cdot\int_a^b f(x)\sin^2\lambda xdx=\int_a^b f(x)\sin^2\lambda xdx.$$

In the same way, we get

$$(\int_a^b f(x)\cos\lambda xdx)^2\leq\int_a^b f(x)\cos^2\lambda xdx.$$

Therefore,

$$(\int_a^b f(x)\sin\lambda xdx)^2+(\int_a^b f(x)\cos\lambda xdx)^2\leq\int_a^b f(x)\sin^2\lambda xdx+\int_a^b f(x)\cos^2\lambda xdx=1.$$

**Example 3** If $f(x)$ and $g(x)$ are continuous on the closed interval [*a*,*b*] and suppose that $f(x)$ isn't always zero and $g(x)>0$ .Let $T_n=\int_a^b |f(x)|^n g(x)dx,\ n=1,2,\Lambda$ .

Show that $\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\max_{a\leq x\leq b}|f(x)|.$

**Proof** By using Schwarz integral inequality, we have

$$T_n=\int_a^b |f(x)|^n g(x)dx=\int_a^b \sqrt{g(x)}|f(x)|^{\frac{n-1}{2}}\cdot\sqrt{g(x)}|f(x)|^{\frac{n+1}{2}}dx$$
$$\le (\int_a^b g(x)|f(x)|^{n-1}dx)^{\frac{1}{2}}\cdot(\int_a^b g(x)|f(x)|^{n+1}dx)^{\frac{1}{2}}=T_{n-1}^{\frac{1}{2}}\cdot T_{n+1}^{\frac{1}{2}}.$$

Because $f(x)$ isn't always zero and $g(x)>0$, then we obtain $|f(x)|^n g(x)\ge 0$, and $|f(x)|^n g(x)$ isn't always zero.

By lemma 2,we get $T_n>0$. And then, $T_n^2\le T_{n-1}T_{n+1}$, $\frac{T_{n+1}}{T_n}\ge\frac{T_n}{T_{n-1}}$, so the sequence $\{\frac{T_{n+1}}{T_n}\}$ is monotonic increasing.

If $f(x)$ is a continuous function on a closed interval $[a,b]$,then $|f(x)|$ has an absolute maximum on this interval. So there is a $x_0\in[a,b]$,such that $f(x_0)=\max_{a\le x\le b}|f(x)|=M>0.$

Then we obtain

$$0\le\frac{T_{n+1}}{T_n}=\frac{\int_a^b g(x)|f(x)|^{n+1}dx}{\int_a^b g(x)|f(x)|^n dx}\le\frac{M\int_a^b g(x)|f(x)|^n}{\int_a^b g(x)|f(x)|^n dx}=M.$$

From the above formula, we know the sequence $\{\frac{T_{n+1}}{T_n}\}$ has an upper bound . Therefore, its limit exists. That is,

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\lim_{n\to\infty}\sqrt[n]{T_n}=\lim_{n\to\infty}(\int_a^b g(x)|f(x)|^n dx)^{\frac{1}{n}}.$$

Next, we shall prove the following formula

$$\lim_{n\to\infty}(\int_a^b g(x)|f(x)|^n dx)^{\frac{1}{n}}=M$$

On the one hand, the function $|f(x)|$ is continuous at $x_0$ .For any given positive number $\varepsilon$ ,there is an interval $[\alpha,\beta]\subset[a,b]$,such that if $x\in[\alpha,\beta]$ ,then $|f(x)|>M-\varepsilon$. We have

$$(\int_a^b g(x)|f(x)|^n dx)^{\frac{1}{n}}\ge(\int_\alpha^\beta g(x)|f(x)|^n dx)^{\frac{1}{n}}\ge(M-\varepsilon)(\int_\alpha^\beta g(x)dx)^{\frac{1}{n}}\to(M-\varepsilon)(n\to\infty).$$

Because of the arbitrariness of $\varepsilon$ ,we get

$$(\int_a^b g(x)|f(x)|^n dx)^{\frac{1}{n}}\ge M.$$

On the other hand ,

$$(\int_a^b g(x)|f(x)|^n dx)^{\frac{1}{n}}\le(\int_a^b M^n g(x)dx)^{\frac{1}{n}}=M(\int_a^b g(x)dx)^{\frac{1}{n}}\to M(n\to\infty).$$

Therefore,

$$\lim_{n\to\infty}(\int_a^b g(x)|f(x)|^n dx)^{\frac{1}{n}}=M,$$

That is,

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\max_{a\le x\le b}|f(x)|.$$

**CONCLUSIONS**

Schwarz integral inequality is a kind of important inequality in mathematical analysis, which is broadly used in athematical analysis. The study of integral inequality can help us not only solve some integral inequality of equation, but also put the primary mathematics knowledge and higher mathematics knowledge together toimprove our ability of thinking and innovation. This paper
gives out a few methods to prove inequation and by which we can simply and quickly solve the problem. Meanwhile it introduces the method of applying Schwarz integral inequality to prove some questions by giving several examples.

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