Scholars Journal of Physics, Mathematics and Statistics
Sch. J. Phys. Math. Stat. 2015; 2(2A):129-135
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

Region of Variability of a Subclass of Starlike Univalent Functions

S. Sunil Varma, Thomas Rosy
Department of Mathematics, Madras Christian College, Tambaram, Chennai, Tamil Nadu-600 059, India

***Corresponding Author:**
Thomas Rosy
Email: thomas.rosy@gmail.com

Abstract: Let $\mathcal{R}$ denote the class of all analytic univalent functions $f(z)$ in the unit disk Δ with $f(0) = f'(0) = 1$ and $\frac{zf'(z)}{f(z)}$ is starlike. For any fixed z_0 in the unit disk and $\lambda \in \bar{\Delta}$, we determine the region of variability $V(z_0, \lambda)$ for $log\frac{f'(z_0)}{f(z_0)}$ when f ranges over the class

$$\mathcal{R}(\lambda) = \{f \in \mathcal{R} : f''(0) = 2\lambda + 1\}.$$

Keywords: Analytic functions, Schwarz lemma, Variability region..
Mathematics Subject Classification : Primary 30C45

INTRODUCTION

Denote by $\mathcal{H}(\Delta)$ the class of analytic functions in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The set $\mathcal{H}(\Delta)$ may be thought as a topological vector space endowed with the topology of uniform convergence over compact subsets of Δ. Let $\mathcal{A}$ denote the set of all functions in $\mathcal{H}(\Delta)$ such that $f(0) = f'(0) = 1$ and S be the class of all functions in $\mathcal{A}$ that are univalent. Several researchers have studied the region of variability problems of f at a specified point inside the unit disk for several subclasses of S. In [2], the problem of determining the region of values of $log\left[\frac{f(z_0)}{z_0}\right]$ for a fixed $z_0 \in \Delta$ as f ranges over the class S^* of starlike functions is given.Duren in his paper [3] discusses the region of variability of $f'(z_0)$ for $f \in S$ and $g(z_0)$ for $g \in S_0 = \{f \in \mathcal{A} : f(z) \neq 0 \text{ in } \Delta, f(0) = 1\}$. Bhowmik determined the region of variability for concave univalent functions [1]. In [5 , 6 , 7 , 8 , 9 , 10], S. Ponnusamy *et al.* had obtained the region of variabilities for several standard subclasses of S. H. Yanagihara had discussed the region of variability for functions with bounded derivatives, convex functions and families of convex functions in [11, 12, 13].

In this paper , we define a new subclass of univalent analytic functions f satisfying certain normalization condition and determine the region of variability of $log\frac{f'(z_0)}{f(z_0)}$.

Let $\mathcal{R}$ denote the class of all analytic univalent functions $f(z)$ in the unit disk Δ with $f(0) = f'(0) = 1$ and $Re\frac{zF'(z)}{F(z)} > 0$, $z \in \Delta$ where $F(z) = \frac{zf'(z)}{f(z)}$. Let

$$P_f(z) = \frac{zF'(z)}{F(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \tag{1}$$

Clearly $P_f(0) = 1$. For $f \in \mathcal{R}$, we denote by $log\frac{f'(z_0)}{f(z_0)}$ the single valued branch of logarithm of $\frac{f'(z_0)}{f(z_0)}$. Hergoltz representation for analytic functions with positive real part in Δ shows that if $f \in \mathcal{R}$ then there exists a unique positive measure μ on $(-\pi, \pi]$such that

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t), \quad z \in \Delta$$

A computation gives

$$log\frac{f'(z_0)}{f(z_0)} = 2\int_{-\pi}^{\pi} log\left(\frac{1}{1-ze^{-it}}\right) d\mu(t)$$

For each fixed $z_0 \in \Delta$, the region of variability $V(z_0) = \left\{log \frac{f'(z_0)}{f(z_0)} : f \in \mathcal{R}\right\}$ coincides with the set $\{-2\log(1-z) : |z| \le |z_0|\}$. Let $\mathcal{B}_0$ denote the class of analytic functions ω in Δ suchthat $|\omega(z)| \le 1$ in Δ and $\omega(0) = 0$. Then for each $f \in \mathcal{R}$, there exist $\omega_f \in \mathcal{B}_0$ of the form

$$\omega_f(z) = \frac{P_f(z)-1}{P_f(z)+1}, \qquad z \in \Delta \tag{2}$$

and conversely. Clearly

$$P_f'(0) = 2\omega_f'(0) = f''(0) - 1 \tag{3}$$

If $f \in \mathcal{R}$ then a simple application of Schwarz lemma shows that

$$|P_f'(0)| = |f''(0) - 1| \le 2$$

Since $|\omega_f'(0)| \le 1$. This implies that $f''(0) = 2\lambda + 1$ for some $\lambda \in \bar{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$. For $\omega_f \in \mathcal{B}_0$ define $g : \Delta \longrightarrow \bar{\Delta}$ by

$$g(z) = \begin{cases} \dfrac{\frac{\omega_f(z)}{z} - \lambda}{1 - \frac{\bar{\lambda}\omega_f(z)}{z}} & if\ |\lambda| < 1 \\ 0 & if\ |\lambda| = 1 \end{cases} \tag{4}$$

Then

$$g'(0) = \begin{cases} \dfrac{\omega_f''(0)}{2(1-|\lambda|^2)} & if\ |\lambda| < 1 \\ 0 & if\ |\lambda| = 1 \end{cases}$$

For $|\lambda| < 1$,

$$\begin{aligned} |g'(0)| < 1 &\Leftrightarrow \frac{|\omega_f''(0)|}{2(1-|\lambda|^2)} \le 1 \\ &\Leftrightarrow \left|\frac{f'''(0) - 2\lambda - 2\lambda^2}{2(1-|\lambda|^2)}\right| \le 1 \\ &\Leftrightarrow f'''(0) = 2a(1-|\lambda|^2) + 2\lambda(\lambda+1) \end{aligned}$$

for some $a \in \bar{\Delta}$. For $\lambda \in \bar{\Delta}$ and a fixed $z_0 \in \Delta$, introduce

$$\mathcal{R}(\lambda) = \{f \in \mathcal{R} : f''(0) = 2\lambda + 1\}$$

and

$$V(z_0, \lambda) = \left\{log \frac{f'(z_0)}{f(z_0)} : f \in \mathcal{R}(\lambda)\right\}$$

The aim of this paper is to determine the region of variability $V(z_0, \lambda)$ of $log \frac{f'(z_0)}{f(z_0)}$ when f ranges over the class $\mathcal{R}(\lambda)$.

BASIC PROPERTIES OF $V(z_0, \lambda)$ AND THE MAIN RESULT

For a positive integer p, let

$$(S^*)^p = \{f = f_0^p : f \in S^*\}$$

We now recall the following result from [12] to prove our main theorem.

Lemma 2.1. Let f be an analytic function in Δ with $f(z) = z^p + . . .$. If

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \Delta \text{ then } f \in (S^*)^p.$$

Theorem 2.1. We have

(i) $V(z_0, \lambda)$ is a compact subset of $\mathbb{C}$.

(ii) $V(z_0, \lambda)$ is a convex subset of $\mathbb{C}$.

(iii) For $|\lambda| = 1$ or $z_0 = 0$, $V(z_0, \lambda) = \{-2\log(1 - \lambda z_0)\}$

(iv) For $|\lambda| < 1$ or $z_0 \in \Delta - \{0\}$, $V(z_0, \lambda)$ has $-2\log(1 - \lambda z_0)$ as an interior point.

(v) $V(e^{i\theta} z_0, \lambda) = V(z_0, e^{i\theta}\lambda)$ for $\theta \in \mathbb{R}$, the set of real numbers.

Proof :

(i) Since $\mathcal{R}(\lambda)$ is a compact subset of $\mathcal{H}(\Delta)$, it follows that $V(z_0, \lambda)$ is a compact subset of $\mathbb{C}$.

(ii) Let $f_1, f_2 \in \mathcal{R}(\lambda)$. Then for $0 \le t \le 1$, the function

$$f_t(z) = exp\left[\int_0^z \left(\frac{f_1'(\zeta)}{f_1(\zeta)}\right)^{1-t} \left(\frac{f_2'(\zeta)}{f_2(\zeta)}\right)^t d\zeta\right] \text{is in } \mathcal{R}(\lambda).$$

Since $log\frac{f_t'(z_0)}{f_t(z_0)} = (1-t)log\frac{f_1'(z_0)}{f_t(z_0)} + t\, log\frac{f_2'(z_0)}{f_2(z_0)}$, $V(z_0,\lambda)$ is a convex subset of $\mathbb{C}$.

(iii)If $z_0 = 0$ then the result holds trivially. If $|\lambda| = 1$ then $|\omega_f'(0)| = 1$.
By Schwarz lemma, $\omega_f(z) = \lambda z$which implies

$$P_f(z) = \frac{1+\lambda z}{1+\bar{\lambda}\, z}$$

A computation gives

$$log\frac{f'(z)}{f(z)} = -2\log(1-\lambda z)$$

which implies $V(z_0,\lambda) = \{-2\log(1-\lambda z_0)\}$.

(iv) For$\lambda \in \Delta$, $z_0 \in \Delta - \{0\}$ and $a \in \bar{\Delta}$ we define

$$\delta(z,\lambda) = \frac{z+\lambda}{1+\bar{\lambda}\, z}$$

$$F_{a,\lambda}(z) = exp\int_0^z \left[exp\int_0^{\zeta_2} \frac{2\delta(a\zeta_1,\lambda)}{1-\zeta_1\delta(a\zeta_1,\lambda)} d\zeta_1\right] d\,\zeta_2 \quad (5)$$

We prove that $F_{a,\lambda}$ satisfying (5)belong to the class $\mathcal{R}(\lambda)$.Note that

$$1 + \frac{zF''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{zF'_{a,\lambda}(z)}{F_{a,\lambda}(z)} = \frac{1+z\delta(az,\lambda)}{1-z\delta(az,\lambda)}$$

Since $\delta(az,\lambda)$ lies in the unit disk Δ , $F_{a,\lambda} \in \mathcal{R}(\lambda)$. Also note that

$$\omega_{F_{a,\lambda}}(z) = z\,\delta(az,\lambda) \quad (6)$$

The mapping $\Delta \ni a \longrightarrow log\frac{f'(z_0)}{f(z_0)}$ is a non-constant analytic function of a for each fixed $z_0 \in \Delta - \{0\}$and $\lambda \in \Delta$. To prove this we put

$$h(z) = \frac{2}{(1-|\lambda|^2)}\frac{\partial}{\partial a}\left\{log\frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)}\right\}_{a=0}$$

which gives $\qquad h(z) = \frac{2}{(1-|\lambda|^2)}\int_0^a \frac{\zeta}{(1-\lambda\zeta)^2} d\zeta = z^2 + \ldots$

from which it is easy to see that

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} = Re\left\{\frac{2}{1-\lambda z}\right\} > 0$$

By lemma (2.1), there is a function $h_0 \in S^*$ such that $h = h_0^2$. Since h_0 is univalent and $h(0) = 0$, we get $h(z_0) \neq 0$ for $z_0 \in \Delta - \{0\}$.

Thus the map $a \longrightarrow log\frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} = \int_0^z \frac{2\delta(a\zeta,\lambda)}{1-\zeta_1\delta(a\zeta,\lambda)} d\zeta$ is a non-constant analytic function of a and hence is an open mapping. Thus $V(z_0,\lambda)$ contains the open set $\left\{log\frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} : |a| < 1\right\}$.

In particular $log\frac{F_{0,\lambda}'(z_0)}{F_{0,\lambda}(z_0)} = -2\log(1-\lambda z_0)$ is an interior point of

$$\left\{log\frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} : |a| < 1\right\} \subset V(z_0,\lambda).$$

Since $V(z_0,\lambda)$ is a compact subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V(z_0,\lambda)$ is a Jordan curve and $V(z_0,\lambda)$ is the union of $\partial V(z_0,\lambda)$ and its interior domain.

(v) This follows from the fact that $e^{-i\theta} f(e^{i\theta} z) \in \mathcal{R}(\lambda)$ if and only if $f \in \mathcal{R}$.

We now state our main result and the proof will be presented in Section 3.

Theorem 2.2. For $\lambda \in \Delta$ and$z_0 \in \Delta - \{0\}$, the boundary $\partial V(z_0,\lambda)$ is the Jordan curve given by

$$(-\pi,\pi] \ni \theta \longrightarrow log\frac{F'_{e^{i\theta},\lambda}(z_0)}{F_{e^{i\theta},\lambda}(z_0)} = \int_0^{z_0} \frac{2\delta(e^{i\theta}\zeta,\lambda)}{1-\zeta\delta(e^{i\theta}\zeta,\lambda)} d\zeta.$$

If $\log\frac{f'(z_0)}{f(z_0)} = log\frac{F'_{e^{i\theta},\lambda}(z_0)}{F_{e^{i\theta},\lambda}(z_0)}$for some$f \in \mathcal{R}$then $f(z) = F_{e^{i\theta},\lambda}(z)$.

PROOF OF MAIN THEOREM

Theorem 3.1. For $f \in \mathcal{R}(\lambda), \lambda \in \Delta$ we have

$$\left|\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right) - c(z,\lambda)\right| \le r(z,\lambda) \ , z \in \Delta \qquad (7)$$

where

$c(z,\lambda) = \frac{2[\lambda(1-|z|^2)+\bar{z}(|z|^2-|\lambda|^2)]}{(1-|z|^2)(1-2Re(\lambda z)+|z|^2)}$ and $r(z,\lambda) = \frac{2|z|(1-|\lambda|^2)}{(1-|z|^2)(1-2Re(\lambda z)+|z|^2)}$

For each $z \in \Delta - \{z_0\}$ equality holds if and only if $f = F_{e^{i\theta},\lambda}$.

Proof : Let $f \in \mathcal{R}(\lambda)$. Then there exists a $\omega_f(z) \in \mathcal{B}_0$ satisfying $\omega_f(z) = \frac{P_f(Z)-1}{P_f(Z)+1}$ where $P_f(z)$ is defined by (1). Since $\omega_f'(0) = \lambda$, by Schwarz lemma,

$$\left|\frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \frac{\bar{\lambda}\omega_f(z)}{z}}\right| \le |z| \, , z \in \Delta \qquad (8)$$

which by definition of P_f is equivalent to

$$\left|\frac{\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right) - A(z,\lambda)}{\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right) + B(z,\lambda)}\right| \le |z||\tau(z,\lambda)| \qquad (9)$$

where $A(z,\lambda) = \frac{2\lambda}{1-\lambda z}$, $B(z,\lambda) = \frac{2}{z-\bar{\lambda}}$, $\tau(z,\lambda) = \frac{z-\bar{\lambda}}{1-\lambda z}$ (10)

A simple calculation shows that (9) is equivalent to

$$\left|\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right) - \frac{A(z,\lambda)+|z|^2|\tau(z,\lambda)|^2 B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}\right| \le \frac{|z||\tau(z,\lambda)||A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2} \qquad (11)$$

A computation gives

$$1 - |z|^2|\tau(z,\lambda)|^2 = \frac{(1-|z|^2)((1-2Re(\lambda z)+|z|^2)}{|1-\lambda z|^2} \qquad (12)$$

$$A(z,\lambda) + B(z,\lambda) = \frac{2(1-|\lambda|^2)}{(1-\lambda z)(z-\bar{\lambda})} \qquad (13)$$

$$A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda) = \frac{2[\lambda(1-|z|^2)+\bar{z}(|z|^2-|\lambda|^2)]}{|1-\lambda z|^2} \qquad (14)$$

Using the above equations we find that

$$\frac{A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2} = c(z,\lambda)$$

and

$$\frac{|z||\tau(z,\lambda)||A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2} = r(z,\lambda)$$

The inequality in (7) follows from these equalities and (11). The equality occurs in (7) for any $z \in \Delta$ only when $f = F_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$. Conversely if the equality occurs for some $z \in \Delta - \{0\}$ in (7) then the equality must hold in (8). By Schwarz lemma there exists a $\theta \in \mathbb{R}$ such that $\omega_f(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \Delta$. This implies $f = F_{e^{i\theta},\lambda}$.

When $\lambda = 0$ we have the following result.

Corollary 3.1. For $f \in \mathcal{R}(0)$

$$\left|\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right) - \frac{2\bar{z}|z|^2}{1-|z|^2}\right| \le \frac{2|z|}{1-|z|^2} \quad , \qquad z \in \bar{\Delta}$$

For each $z \in \Delta - \{0\}$, equality holds if and only if $f = F_{e^{i\theta},0}$ for some $\theta \in \mathbb{R}$. In particular

$$(1-|z|^2)\left|\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right| \le 2|z|$$

Corollary 3.2. Let $\gamma : z(t),\ 0 \le t \le 1$ be a C^1 - curve in Δ with $z(0) = 0$ and $z(1) = z_0$. Then

$$V(z_0,\lambda) \subset \{w \in \mathbb{C} : |w - C(\lambda,\gamma)| \le R(\lambda,\gamma)\}$$

where $C(\lambda,\gamma) = \int_0^1 c(z(t),\lambda)z'(t)dt$ and $R(\lambda,\gamma) = \int_0^1 r(z(t),\lambda)|z'(t)|\,dt$.

Proof. Let $f \in \mathcal{R}(\lambda)$. Then by theorem 3.1,

$$\left|log\frac{f'(z_0)}{f(z_0)} - C(\lambda,\gamma)\right| = \left|\int_0^1\left[\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right) - c(z(t),\lambda)\right]z'(t)dt\right|$$

$$\leq \int_0^1 r(z(t),\lambda)|z'(t)|\,dt$$
$$= R(\lambda,\gamma)$$

Since $log\frac{f'(z_0)}{f(z_0)} \in V(z_0,\lambda)$ was arbitrary, the result follows.

To prove our next result we need the following lemma [10].

Lemma 3.1. For $\theta \in \mathbb{R}$, $\lambda \in \Delta$, the function

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta}{\left[1 + \left(\bar{\lambda}e^{i\theta} - \lambda\right)\zeta - e^{i\theta}\zeta^2\right]^2}d\zeta \quad , \quad z \in \Delta$$

has a double zero at the origin and no zeros elsewhere in Δ. Furthermore, there exists a starlike univalent function G_0 in Δ such that$G = e^{i\theta}G_0^2$, $G_0(0) = G_0'(0) - 1 = 0$.

Theorem 3.2. Let $z_0 \in \Delta - \{0\}$. Then for $\theta \in (-\pi,\pi]$ we have $log\frac{F'_{e^{i\theta},\lambda}(z_0)}{F_{e^{i\theta},\lambda}(z_0)} \in \partial V(z_0,\lambda)$. Further if $log\frac{f'(z_0)}{f(z_0)} = log\frac{F'_{e^{i\theta},\lambda}(z_0)}{F_{e^{i\theta},\lambda}(z_0)}$ for some $f \in \mathcal{R}(\lambda)$ and $\theta \in (-\pi,\pi]$then $= F_{e^{i\theta},\lambda}$.

Proof .From (5)

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} = \frac{2\delta(az,\lambda)}{1 - z\delta(az,\lambda)} = \frac{-2(az+\lambda)}{az^2 + \left(\lambda - \bar{\lambda}\,a\right)z - 1}$$

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} - A(z,\lambda) = \frac{-2az(|\lambda|^2 - 1)}{(1-\lambda z)\left(az^2 + \left(\lambda - \bar{\lambda}\,a\right)z - 1\right)}$$

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} - B(z,\lambda) = \frac{2(|\lambda|^2 - 1)}{\left(z - \bar{\lambda}\right)\left(az^2 + \left(\lambda - \bar{\lambda}\,a\right)z - 1\right)}$$

and hence we obtain

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} - c(z,\lambda) = \frac{2(|\lambda|^2-1)\left(a(1-\overline{\lambda z})z - |z|^2(\bar{z} - \lambda)\right)}{(1-|z|^2)(1 - 2Re(\lambda z) + |z|^2)\left(az^2 + \left(\lambda - \bar{\lambda}\,a\right)z - 1\right)}$$

Substituting $a = e^{i\theta}$ we have

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} - c(z,\lambda) = r(z,\lambda)\frac{ze^{i\theta}}{|z|}\frac{\left|1 + \left(\bar{\lambda}e^{i\theta} - \lambda\right)z - e^{i\theta}z^2\right|^2}{\left(1 + \left(\bar{\lambda}e^{i\theta} - \lambda\right)z - e^{i\theta}z^2\right)^2}$$

Using above lemma,

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{F'_{a,\lambda}(z)}{F_{a,\lambda}(z)} - c(z,\lambda) = r(z,\lambda)\frac{G'(z)}{|G'(z)|} \tag{15}$$

Since G_0 is starlike, for any $z_0 \in \Delta - \{0\}$, the linear segment joining 0 and $G(z_0)$ lies entirely in $G_0(\Delta)$. Define γ_0 by

$$\gamma_0 : z(t) = G_0^{-1}\left(tG_0(z_0)\right) \ , 0 \leq t \leq 1 \tag{16}$$

Since $G\left(z(t)\right) = 2^{-1}e^{i\theta}G_0\left(z(t)\right)^2 = 2^{-1}e^{i\theta}G_0\left(z(t)\right)^2 = t^2G(z_0)$, we have

$$G'\left(z(t)\right)z'(t) = 2tG(z_0)\,,\ t \in [0,1] \tag{17}$$

Using (15) and (17) we have

$$log\frac{F'_{e^{i\theta},\lambda}(z)}{F_{e^{i\theta},\lambda}(z)} - C(\lambda,\gamma_0) = \int_0^1\left\{\frac{F''_{e^{i\theta},\lambda}\left(z(t)\right)}{F'_{e^{i\theta},\lambda}\left(z(t)\right)} - \frac{F'_{e^{i\theta},\lambda}\left(z(t)\right)}{F_{e^{i\theta},\lambda}\left(z(t)\right)} - c(z(t),\lambda)\right\}z'(t)dt$$

$$= \int_0^1 r(z(t),\lambda)\frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|}$$

$$= \frac{G(z_0)}{|G(z_0)|}R(\lambda,\gamma_0)$$

This implies that $log\frac{F'_{e^{i\theta},\lambda}(z_0)}{F_{e^{i\theta},\lambda}(z_0)} \in \partial\Delta(C(\lambda,\gamma_0),R(\lambda,\gamma_0))$. By Corollary (3.2) we have $log\frac{F'_{e^{i\theta},\lambda}(z_0)}{F_{e^{i\theta},\lambda}(z_0)} \in V(z_0,\lambda) \subset \bar{\Delta}\left(C(\lambda,\gamma_0),R(\lambda,\gamma_0)\right)$.

which implies $log \frac{F'_{e^{i\theta},\ \lambda}(z_0)}{F_{e^{i\theta}\ ,\ \lambda}(z_0)} \in \partial V(z_0,\lambda)$. For uniqueness, suppose

$$log \frac{f'(z_0)}{f(z_0)} = log \frac{F'_{e^{i\theta},\ \lambda}(z_0)}{F_{e^{i\theta}\ ,\ \lambda}(z_0)}$$

for some $f \in \mathcal{R}(\lambda)$. Introduce

$$h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\frac{f''(z(t))}{f'(z(t))} - \frac{f'(z(t))}{f(z(t))} - c(z(t),\lambda)\right\} z'(t)dt$$

where $\gamma_0 : z(t)\ ,\ 0 \le t \le 1$, then $h(t)$ is continuous function on $[0\ ,1]$ and satisfies

$$|h(t)| \le r(z(t),\lambda)|z(t)|'\ , \qquad 0 \le t \le 1$$

Also we have

$$\int_0^1 Reh(t)dt = \int_0^1 Re\left\{\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\frac{f''(z(t))}{f'(z(t))} - \frac{f'(z(t))}{f(z(t))} - c(z(t),\lambda)\right\} z'(t)\right\}dt$$

$$= Re\left\{\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{log \frac{f'(z_0)}{f(z_0)} - C(\lambda\ ,\gamma_0)\right\}\right\}$$

$$= Re\left\{\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{log \frac{F'_{e^{i\theta},\ \lambda}(z_0)}{F_{e^{i\theta}\ ,\ \lambda}(z_0)} - C(\lambda\ ,\gamma_0)\right\}\right\}$$

$$= \int_0^1 r(z(t),\lambda)|z'(t)|dt$$

which implies $h(t) = r(z(t),\lambda)|z'(t)|$for all $t \in [0,1]$. From (15) and (17) it follows that

$$\frac{f''}{f'} - \frac{f'}{f} = \frac{F''_{e^{i\theta},\lambda}(z(t))}{F'_{e^{i\theta},\lambda}(z(t))} - \frac{F'_{e^{i\theta}\ ,\ \lambda}}{F_{e^{i\theta}\ ,\ \lambda}}$$

on the curve γ_0. By normalization $f = F_{e^{i\theta}\ ,\ \lambda}$.

Proof of theorem 2.2

We need to prove that the closed curve

$$(-\pi\ ,\pi] \ni \theta \longrightarrow log \frac{F'_{e^{i\theta},\ \lambda}(z_0)}{F_{e^{i\theta}\ ,\ \lambda}(z_0)}$$

is simple. Suppose that $log \frac{F'_{e^{i\theta_1},\ \lambda}(z_0)}{F_{e^{i\theta_1}\ ,\ \lambda}(z_0)} = log \frac{F'_{e^{i\theta_2},\ \lambda}(z_0)}{F_{e^{i\theta_2}\ ,\ \lambda}(z_0)}$

for some $\theta_1\ ,\ \theta_2 \in (-\pi\ ,\pi]\ ,\ \theta_1 \neq \theta_2$. By Theorem (3.2) ,$F_{e^{i\theta_1}\ ,\ \lambda} = F_{e^{i\theta_2}\ ,\ \lambda}$

From (6) this gives a contradiction that

$$e^{i\theta_1} = \tau\left(\frac{\omega_{F_{e^{i\theta_1}\ ,\ \lambda}}}{z}\ ,\lambda\right) = \tau\left(\frac{\omega_{F_{e^{i\theta_2}\ ,\ \lambda}}}{z}\ ,\lambda\right) = e^{i\theta_2}$$

which implies the curve is simple. Since $V(z_0\ ,\lambda)$ is compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V(z_0\ ,\lambda)$ is a simple closed curve. By Theorem 3.1 , the curve contains the curve $(-\pi\ ,\pi] \ni \theta \longrightarrow log \frac{F'_{e^{i\theta},\ \lambda}(z_0)}{F_{e^{i\theta}\ ,\ \lambda}(z_0)}$. Since a simple closed curve cannot contain any simple closed curve other than itself, the result follows immediately.

REFERENCES

1. Bhowmik B, Ponnusamy S; Region of variability for concave univalent functions, Analysis., 2008; 28: 333 - 344.
2. Duren PL; Univalent Functions, (Grudlehren der mathematischen Wis- senschaften259, New York, Berlin, Heidel berg, Tokyo), Springer-Verlag1983.
3. Duren PL; Regions of variability for univalent functions, Trans. Amer. Math. Soc, 1986;295(1):119 - 126.
4. Goodman AW; Univalent Functions, Vols I and II, Mariner Publishing Co.Tampa, Florida, 1983.
5. Ponnusamy S, Vasudevarao A; Region of variability of two subclasses of univalent functions, J.Math.Anal.Appl., 2007; 332(2):1322-1333.
6. Ponnusamy S, Vasudevarao A, Yanagihara H; Region of variability of univalent functions f (z) for which z f ' (z) is spirallike, Houston J. Math.2008; 34(4):1037-1048.
7. Ponnusamy S, Vasudevarao A, Yanagihara H; Region of variability for close -to-convex functions, Comp. Var. and Ellip. Eqns.2008; 53 (8):709-716.
8. Ponnusamy S, Vasudevarao A, Yanagihara H; Region of variability for close-to-convex functions-II, Appld. Math. and. Comput., 2008;215:901-915.
9. Ponnusamy S, Vasudevarao A, Vuorinen M; Region of variability for spiral-like functions with respect to a

boundary point, Colloq. Math., 2009; 116 (1): 31-46.

10. Ponnusamy S, Vasudevarao A, Vuorinen M; Region of variability for exponentially convex functions, Comp .Anal. and Oper Theory., 2011; 5: 955- 966.
11. Yanagihara H; Regions of variability for functions with bounded derivatives, Kodai Math. J .2005; 452-462.
12. Yanagihara H; Regions of variability for convex functions, Math Nachr. 2006; 276: 1723-1730.
13. Yanagihara H; Variability regions for families of convex functions, Comp. Met. AndFunc. Theory.,2010; 10 (1):291- 302.