

Study on Properties of Monotone Function

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Abstract: Firstly, this paper gives the conditions of the existence of four kinds of unilateral limits and limit values for monotone function; points out the relationship between the existence of four kinds of unilateral limits and sequence limit, and weakens the conditions of Heine theorem. Secondly, it demonstrates that the discontinuity points of monotone functions are the discontinuity points of the first kind, and their number is countable at most within a interval. Finally, it gives the relationship between upper and lower limits of bounded sequence and monotone function and bounded sequence, and provides corresponding application examples.

Keywords: monotone function; limit; discontinuity point

PRELIMINARY KNOWLEDGE

Definition 1[1] Assume function f is defined within interval I . If $\forall x_1, x_2 \in I, x_1 < x_2$, and there exists eternally

$$f(x_1) \leq f(x_2),$$

then the function is called monotonic increasing function within interval I ; if $\forall x_1, x_2 \in I, x_1 < x_2$, and there exists eternally

$$f(x_1) \geq f(x_2),$$

then the function is called monotonic decreasing function within interval I . Monotonic increasing functions and monotonic decreasing functions are collectively referred to as the monotone functions.

Definition 2[1] Maximum accumulation point $\bar{A}$ and minimum accumulation point $\underline{A}$ of bounded sequence $\{x_n\}$ are called the upper limit and lower limit respectively, expressed as $\bar{A} = \lim_{n \rightarrow \infty} x_n, \underline{A} = \lim_{n \rightarrow \infty} x_n$.

PROPERTIES OF MONOTONE FUNCTION

Theorem 1[1] If f is a monotonic increasing (decreasing) function within $U^0(x_0)$, then both $f(x_0 - 0)$ and $f(x_0 + 0)$ exist, and

$$\begin{aligned} f(x_0 - 0) &= \sup_{x \in U_-^0(x_0)} f(x) \quad (f(x_0 - 0) = \inf_{x \in U_-^0(x_0)} f(x)); \\ f(x_0 + 0) &= \inf_{x \in U_+^0(x_0)} f(x) \quad (f(x_0 + 0) = \sup_{x \in U_+^0(x_0)} f(x)). \end{aligned}$$

Theorem 2 If f is a monotone function within $U_-^0(x_0)(U_+^0(x_0))$, then the necessary and sufficient conditions of $f(x_0 - 0) = A$ ($f(x_0 + 0) = A$) are that:

sequence $\{x_n\} \subset U_-^0(x_0)(U_+^0(x_0))$ exists, and $\lim_{n \rightarrow \infty} x_n = x_0$, making

$$\lim_{n \rightarrow \infty} f(x_n) = A.$$

Proof Assume f is a monotonic increasing function within $U_-^0(x_0)$. Its necessary condition holds obviously.

Sufficient condition: since $\lim_{n \rightarrow \infty} f(x_n) = A$, according to boundedness of convergent sequence, then $\exists M > 0$, then $f(x_n) \leq M$, $n = 1, 2, \dots$.

Select $x \in U_-(x_0)$ freely, since $\lim_{n \rightarrow \infty} x_n = x_0 > x$, then $\exists n_0 \in \mathbb{N}^+$, making $x_{n_0} > x$. Furthermore, since f is a monotonic increasing function within interval $U_-(x_0)$, then $f(x) \leq f(x_{n_0}) \leq M$. Namely f has upper bound within $U_-(x_0)$.

$$f(x_0 - 0) = \sup_{x \in U_-(x_0)} f(x) \text{ can be obtained according to Theorem 1.}$$

$$f(x_0 - 0) = \lim_{n \rightarrow \infty} f(x_n) = A. \text{ can be obtained according to Heine theorem.}$$

Theorem 3 is available for abnormal limit:

Theorem 3 If f is a monotonic increasing function within $U_-(x_0) (U_+(x_0))$, then the necessary and sufficient conditions of $f(x_0 - 0) = +\infty (f(x_0 + 0) = -\infty)$ are that:

sequence $\{x_n\} \subset U_-(x_0) (U_+(x_0))$ exists, and $\lim_{n \rightarrow \infty} x_n = x_0$, making

$$\lim_{n \rightarrow \infty} f(x_n) = +\infty (\lim_{n \rightarrow \infty} f(x_n) = -\infty).$$

If f is a monotonic decreasing function within $U_-(x_0) (U_+(x_0))$, then the necessary and sufficient conditions of $f(x_0 - 0) = -\infty (f(x_0 + 0) = +\infty)$ are that:

sequence $\{x_n\} \subset U_-(x_0) (U_+(x_0))$ exists, and $\lim_{n \rightarrow \infty} x_n = x_0$, making

$$\lim_{n \rightarrow \infty} f(x_n) = -\infty (\lim_{n \rightarrow \infty} f(x_n) = +\infty).$$

Similarly, theorem 4, 5 and 6 are available.

Theorem 4 If f is a monotonic increasing and bounded function within $(a, +\infty)((-\infty, b))$, then

$$f(+\infty) = \sup_{x \in (a, +\infty)} f(x) (f(-\infty) = \inf_{x \in (-\infty, b)} f(x));$$

If f is a monotonic decreasing and bounded function within interval $(a, +\infty)((-\infty, b))$, then

$$f(+\infty) = \inf_{x \in (a, +\infty)} f(x) (f(-\infty) = \sup_{x \in (-\infty, b)} f(x)).$$

Theorem 5 If f is a monotone function within interval $(a, +\infty)((-\infty, b))$, then the necessary and sufficient conditions of $f(+\infty) = A (f(-\infty) = A)$ are that: sequence $\{x_n\} \subset (a, +\infty)((-\infty, b))$ exists, and $\lim_{n \rightarrow \infty} x_n = +\infty(-\infty)$, making

$$\lim_{n \rightarrow \infty} f(x_n) = A.$$

Theorem 6 If f is a monotonic increasing function within interval $(a, +\infty)((-\infty, b))$, then the necessary and sufficient conditions of $f(+\infty) = +\infty (f(-\infty) = -\infty)$ are that:

sequence $\{x_n\} \subset (a, +\infty)((-\infty, b))$ exist, and $\lim_{n \rightarrow \infty} x_n = +\infty(-\infty)$, making

$$\lim_{n \rightarrow \infty} f(x_n) = +\infty (\lim_{n \rightarrow \infty} f(x_n) = -\infty).$$

If f is a monotonic decreasing function within interval $(a, +\infty)((-\infty, b))$, then the necessary and sufficient conditions of $f(+\infty) = -\infty (f(-\infty) = +\infty)$ are that:

sequence $\{x_n\} \subset (a, +\infty)((-\infty, b))$ exists, and $\lim_{n \rightarrow \infty} x_n = +\infty(-\infty)$, making

$$\lim_{n \rightarrow \infty} f(x_n) = -\infty (\lim_{n \rightarrow \infty} f(x_n) = +\infty).$$

Theorem 7 Assume f is a monotone function within interval I . If $x_0 \in (a, b)$ is a discontinuity point of f , then x_0 must be the discontinuity point of the first kind of f .

Proof Assume f is a monotonic increasing function within interval I .

If x_0 is not a endpoint of interval I , then neighborhood $U(x_0) \subset I$ of x_0 exists, making $f(x_0)$ be the lower bound of f within $U_+(x_0)$, and the upper bound of f within $U_-(x_0)$. According to Theorem 1, both $f(x_0 - 0)$ and $f(x_0 + 0)$ exist, and

$$f(x_0 - 0) \leq f(x_0) \leq f(x_0 + 0).$$

Among them, at most one equal sign holds. Therefore, x_0 is the jump discontinuity point of f .

If x_0 is a endpoint of interval I and assume x_0 is the left endpoint of interval I , then $f(x_0)$ is the lower bound of f within interval I . According to Theorem 1, $f(x_0 + 0)$ exists, and $f(x_0) < f(x_0 + 0)$; Therefore, x_0 is the discontinuity point of the first kind of f .

In short, x_0 is the discontinuity point of the first kind of f .

Theorem 8 Assume f is a monotonic increasing (decreasing) function within interval $[a, b]$. If its range is $[f(a), f(b)]$ ($[f(b), f(a)]$), then f is continuous within interval $[a, b]$.

Proof Assume f is a monotonic increasing function within $[a, b]$.

Select $x_0 \in [a, b]$ freely. According to Theorem 1: if $x_0 \in (a, b)$, then both $f(x_0 - 0)$ and $f(x_0 + 0)$ exist, and $f(x_0 - 0) \leq f(x_0) \leq f(x_0 + 0)$.

If $x_0 = a$, then $f(x_0 + 0)$ exists, and $f(x_0) \leq f(x_0 + 0)$.

If $x_0 = b$, then $f(x_0 - 0)$ exists, and $f(x_0 - 0) \leq f(x_0)$.

The proof is given as follow: if $x_0 \in (a, b)$, then $f(x_0 - 0) = f(x_0) = f(x_0 + 0)$.

Assume $f(x_0 - 0) < f(x_0)$, according to Theorem 1, $\forall x \in [a, x_0)$, $f(x) \leq \sup_{x \in [a, x_0)} f(x) = f(x_0 - 0)$. $\forall x \in [x_0, b]$, $f(x) \geq f(x_0)$. Bases on this, $\forall x \in [a, b]$, $f(x) \notin (f(x_0 - 0), f(x_0))$, but $(f(x_0 - 0), f(x_0)) \subset [f(a), f(b)]$, which contradicts the range $[f(a), f(b)]$ of f .

So $f(x_0 - 0) = f(x_0)$. Similarly, $f(x_0) = f(x_0 + 0)$. Therefore, f is continuous at x_0 .

Similarly, f is right continous at a and left continous at b . Therefore, f is continuous within $[a, b]$.

Theorem 9 below can be obtained according to Theorem 7 and 8.

Theorem 9 If f is a monotone function within interval I , then the number of discontinuity points of f within interval I is countable at most.

Theorem 10[2-4] Assume $\{x_n\}$ is a bounded sequence.

(1) If f is a monotonic increasing and continuous function, then

$$\overline{\lim}_{n \rightarrow \infty} f(x_n) = f(\overline{\lim}_{n \rightarrow \infty} x_n), \underline{\lim}_{n \rightarrow \infty} f(x_n) = f(\underline{\lim}_{n \rightarrow \infty} x_n).$$

(2) If f is a monotonic decreasing and continuous function, then

$$\overline{\lim}_{n \rightarrow \infty} f(x_n) = f(\underline{\lim}_{n \rightarrow \infty} x_n), \underline{\lim}_{n \rightarrow \infty} f(x_n) = f(\overline{\lim}_{n \rightarrow \infty} x_n).$$

For instance, for sequence $x_1 = 1$, $x_{n+1} = 1 + \frac{1}{1+x_n}$, $n = 1, 2, \dots$. since $1 \leq x_n \leq 2$, then $\overline{\lim}_{n \rightarrow \infty} x_n = \beta$ and

$\lim_{n \rightarrow \infty} x_n = \alpha$ exist. Assume $f(x) = 1 + \frac{1}{1+x}$, then f monotonically decreases and is continuous within $(0, +\infty)$, according to Theorem 8,

$$\beta = 1 + \frac{1}{1+\alpha}, \alpha = 1 + \frac{1}{1+\beta},$$

then $\alpha = \beta$. Therefore, $\lim_{n \rightarrow \infty} x_n$ exists.

Assume $\lim_{n \rightarrow \infty} x_n = L$, according to $\lim_{n \rightarrow \infty} x_{n+1} = 1 + \frac{1}{1 + \lim_{n \rightarrow \infty} x_n}$, then $L = 1 + \frac{1}{1+L}$, $L = \sqrt{2}$.

CONCLUSION

In monotone function, normal unilateral limit exists at each point within a interval; either normal unilateral limit or abnormal unilateral limit exists at finite endpoints within a interval. If f is bounded within $(a, +\infty)((-\infty, b))$, then there exists normal limit $f(+\infty)(f(-\infty))$; if f is unbounded, then there exists abnormal limit $f(+\infty)(f(-\infty))$. In addition, conditions of Heine theorem in the case of existence of unilateral limit can be weakened. Discontinuity points of monotone function must be the discontinuity points of the first kind, and their number is countable at most. Relationship between monotone function and bounded sequence can be used to demonstrate the existence of sequence limit.

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