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Invariant submanifolds of (ε,δ)-trans-Sasakian manifolds

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Abstract: The object of present paper is to find necessary and sufficient conditions for invariant submanifolds of (ε,δ)-trans-Sasakian manifolds to be totally geodesic.

Keywords: (ε,δ)Trans-Sasakian manifold, second fundamental form, invariant submanifold, totally geodesic, semi parallel, pseudo-parallel.

INTRODUCTION

Invariant submanifolds of a contact manifold have been a major area of research for long time since the concept was borrowed from complex geometry.It helps us to understand several important topics of applied mathematics; for example, in studying non-linear autonomous systems the idea of invariant submanifolds plays an important role [1]. A submanifold of a contact manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. The concept of (ε)-Sasakian manifolds was introduced by A.Bejancu and K.L.Duggal [2] and further investigation was taken up by Xufend and Xiaoli[3] and Rakesh kumar et al.[4]. De and Sarkar [5] introduced and studied conformally flat,Weyl semisymmetric, ϕ-recurrent (ε)-Kenmotsu manifolds. In [1], the authors obtained Riemannian curvature tensor of (ε)-Sasakian manifolds and established relations among different curvatures. H.G.Nagraja et al.[6] have studied (ε,δ)-trans-Sasakian structures which generalizes both (ε)-Sasakian manifolds and (ε)-Kenmotsu manifolds.

PRELIMINARIES

Let $(\overline{M},g)$ be an almost contact metric manifold of dimension $(2n+1)$ equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2=-I+\eta\otimes\xi, \tag{2}$$

$$\eta(\xi)=1, \tag{3}$$

$$\phi\xi=0,\eta\circ\phi=0. \tag{4}$$

An almost contact metric manifold $\overline{M}$ is called an (ε)-almost contact metric manifold if

$$g(\xi,\xi)=\varepsilon, \tag{5}$$

$$\eta(X)=\varepsilon g(X,\xi), \tag{6}$$

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y),\forall X,Y\in TM, \tag{7}$$

where $\varepsilon=g(\xi,\xi)=\pm1$. An (ε)-almost contact metric manifold $\overline{M}$ is called an (ε,δ)-trans-Sasakian manifold if

$$(\overline{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \varepsilon\eta(Y)X] + \beta[g(\phi X,Y)\xi - \delta\eta(Y)\phi X], \quad (8)$$

holds for some smooth functions α and β on $\overline{M}$ and $\varepsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$, $\alpha = 1$, an (ε, δ)-trans-Sasakian manifold reduces to an (ε)-Sasakian and for $\alpha = 0, \beta = 1$ it reduces to a (δ)-Kenmotsu manifold.

Let $(\overline{M}, g)$ be a (ε, δ)-trans-Sasakian manifold. Then from (8), it is easy to see that

$$(\overline{\nabla}_X \xi) = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \quad (9)$$

$$(\overline{\nabla}_X \eta)Y = -\alpha g(Y, \phi X) + \varepsilon\delta\beta g(\phi X, \phi Y). \quad (10)$$

In an (ε, δ)-trans-Sasakian manifold $\overline{M}$, the curvature R Ricci tensor S satisfies [6]

$$\begin{aligned} R(X,Y)\xi &= \varepsilon((Y\alpha)\phi X - (X\alpha)\phi Y) + (\beta^2 - \alpha^2)(\eta(X)Y - \eta(Y)X) \\ &\quad - \delta((X\beta)\phi^2 Y - (Y\beta)\phi^2 X) + 2\varepsilon\delta\alpha\beta(\eta(Y)\phi X \\ &\quad - \eta(X)\phi Y) + 2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi. \end{aligned} \quad (11)$$

$$S(X,\xi) = -(\phi X)\alpha + ((n-1)(\varepsilon\alpha^2 - \beta^2\delta) - (\xi\beta))\eta(X) - (2n-1)(X\beta) \quad (12)$$

SUBMANIFOLDS OF AN ALMOST CONTACT METRIC MANIFOLD

Let M be a submanifold of a contact manifold $\overline{M}$. We denote ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$ respectively, and $T^{\perp}(M)$ the normal bundle of M. Then Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (13)$$

$$\overline{\nabla}_X N = \nabla_X^{\perp} N - A_N X \quad (14)$$

for any $X, Y \in TM$. $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X,Y), N) = g(A_N X, Y). \quad (15)$$

From (13) we have

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X,\xi). \quad (16)$$

Lemma 1. *Let M be a invariant submanifold of (ε, δ) trans-Sasakian manifold $\overline{M}$ then we have*

$$h(X,\xi) = 0 \quad (17)$$

$$h(\phi X, Y) = \phi(h(X,Y)) = h(X, \phi Y) \quad (18)$$

$$h(\phi X, \phi Y) = -h(X,Y) \quad (19)$$

$$(\nabla_X h)(Y,\xi) = -h(Y, \nabla_X \xi) \quad (20)$$

Proof . By straight forward calculations we will get the above results.

Theorem 1. *Let M be a invariant submanifold of (ε,δ)-trans-Sasakian manifold $\overline{M}$ then*

$$(\nabla_X h)(Y,\xi) = h(Y,\xi) = h(Y,\varepsilon\alpha\phi X) + h(Y,\beta\delta\phi^2 X) \tag{21}$$

for any $X,Y \in TM$.

Proof. By using (20), we get

$$\begin{aligned}(\nabla_X h)(Y,\xi) &= -h(Y,\nabla_X \xi) = -h(Y,-\varepsilon\alpha\phi X - \beta\delta\phi^2 X) \\ &= h(Y,\varepsilon\alpha\phi X) + h(Y,\beta\delta\phi^2 X).\end{aligned} \tag{22}$$

Corollary 1. *Let M be a invariant submanifold of (ε,δ)-trans-Sasakian manifold $\overline{M}$ then*

$$(\nabla_X h)(Y,\xi) = \varepsilon\alpha h(Y,\phi X) - \beta\delta h(Y,X) \tag{23}$$

for any $X,Y \in TM$.

Proof. By using (21), we get

$$\begin{aligned}(\nabla_X h)(Y,\xi) = \quad & h(Y,\varepsilon\alpha\phi X) + h(Y,\beta\delta\phi^2 X) \\ &= \varepsilon\alpha h(Y,\phi X) + \beta\delta h(Y,-X+\eta(X)\xi) \\ &= \varepsilon\alpha h(Y,\phi X) - \beta\delta h(Y,X).\end{aligned} \tag{24}$$

Theorem 2. *Let M be a invariant submanifold of (ε,δ)- trans-Sasakian manifold $\overline{M}$ then h is parallel if and only if M is totally geodesic.*

Proof. Suppose that h is parallel. For each $X,Y \in TM$ and using (20) we get

$$(\nabla_X h)(Y,\xi) = 0 \Rightarrow h(Y,\nabla_X \xi) = 0 \tag{25}$$

or

$$h(Y,-\varepsilon\alpha\phi X - \beta\delta\phi^2 X) = 0. \tag{26}$$

Hence

$$-\varepsilon\alpha h(Y,\phi X) - \beta\delta h(Y,\phi^2 X) = 0. \tag{27}$$

Since M is an invariant submanifold of $\overline{M}$, we have,

$$\phi(h(X,Y)) = 0. \tag{28}$$

From (18) it follows that

$$\phi(h(X,Y)) = h(Y,\phi X) = 0. \tag{29}$$

Then we get

$$\beta\delta h(Y,\phi^2 X) = 0. \tag{30}$$

Hence it follows that

$$h(Y,-X+\eta(X)\xi) = 0. \tag{31}$$

So

$$h(Y,X) = 0. \tag{32}$$

Vice versa, let M is totally geodesic. Then $h = 0$. For all $X,Y,Z \in TM$,

$$(\nabla_X h)(Y,Z) = \nabla_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) = 0. \tag{33}$$

Thus we have $\nabla h = 0$.

Theorem 3. *An invariant submanifold of (ε,δ)-trans-Sasakian manifold $\overline{M}$ is totally geodesic if and only if its second fundamental form is Ricci generalized pseudo-parallel, provided* $[(\alpha^2-\beta^2)+2nf(\varepsilon\alpha^2-\delta\beta^2)]\neq 0$.

Proof . Since the submanifold is Ricci generalized pseudo-parallel, we have

$$(R(X,Y).h)(U,V) = fQ(S,h)(X,Y,U,V). \tag{34}$$

So,

$$\begin{aligned} &R(X,Y)h(U,V)-h(R(X,Y)U,V)-h(U,R(X,Y)V)\\ &= f(-S(V,X)h(U,Y)-S(V,Y)h(X,U)+S(U,Y)h(X,V)). \end{aligned} \tag{35}$$

Putting $Y=V=\xi$ and applying (17) we obtain

$$-h(U,R(X,\xi)\xi) = -fS(\xi,\xi)h(X,U) \tag{36}$$

By using (11) and (12), we obtain

$$[(\alpha^2-\beta^2)+2nf(\varepsilon\alpha^2-\delta\beta^2)]h(X,U)=0 \tag{37}$$

Hence the submanifold is totally geodesic. The converse holds trivially.

Theorem 4. *An invariant submanifold of (ε,δ)-trans-Sasakian manifold $\overline{M}$ is totally geodesic if and only if its second fundamental form is 2-semi-parallel , provided* $[(\alpha^2-\beta^2)(\alpha^2+\beta^2)]\neq 0$.

Proof. Since, the second fundamental form is 2-semi-parallel, we have

$$(R(X,Y)\cdot(\nabla_U h)(Z,W)=0, \tag{38}$$

which implies

$$(R^{\perp}(X,Y)(\nabla_U h)(Z,W)-(\nabla_U h)(R(X,Y)Z,W)-(\nabla_U h)(Z,R(X,Y)W)=0. \tag{39}$$

Now,

$$R^{\perp}(X,Y)(\nabla_U h)(\xi,\xi)=0 \tag{40}$$

Therefore,

$$\begin{aligned} (\nabla_U h)(R(X,\xi)\xi,\xi) &= (\nabla_U h)((\alpha^2-\beta^2)(X-\eta(X)\xi),\xi)\\ &= -h((\alpha^2-\beta^2)(X-\eta(X)\xi),\nabla_U\xi)\\ &= \varepsilon\alpha(\alpha^2-\beta^2)h(X,\phi U)+(\alpha^2-\beta^2)\beta\delta h(X,U). \end{aligned} \tag{41}$$

Similarly,

$$(\nabla_U h)(\xi,R(X,\xi)\xi) = \varepsilon\alpha(\alpha^2-\beta^2)h(X,\phi U)+(\alpha^2-\beta^2)\beta\delta h(X,U). \tag{42}$$

Therefore, we have

$$\varepsilon\alpha(\alpha^2-\beta^2)\phi h(X,U)+(\alpha^2-\beta^2)\beta\delta h(X,U)=0. \tag{43}$$

Applying ϕ on both sides of (43) we get

$$(\alpha^2-\beta^2)\beta\delta\phi h(X,U)=\varepsilon\alpha(\alpha^2-\beta^2)h(X,U) \tag{44}$$

From (43) and (44), we have

$$[(\alpha^2-\beta^2)(\alpha^2+\beta^2)]h(X,U)=0. \tag{45}$$

Hence the submanifold is totally geodesic. The converse holds trivially.

REFERENCES

1. Guojing Z, Jianguo W; Invariant submanifolds and modes of non-linear autonomous system, Appl.Math,1998;19:587-693.
2. Bejancu A, Duggal KL; Real hypersurfaces of idefinite Kahler manifolds. Int. J. Math. Math. sci., 1993; 16(3):545-556.
3. Xufeng X, Xiaoli C; Two theorems on ε -Sasakian manifolds, Int. J. Math. Math. sci., 1998; 21(2):249-254
4. Kumar R, Rani R, Nagaich RK; On sectional curvature of (ε)-Sasakian manifolds, Int. J. Math. Math. sci., 2007;93562.
5. De UC, Sarkar A; On (ε)-Kenmotsu manifolds. Hadronic journal, 2009;32:231-242.
6. Nagaraja HG, Premalatha RC, Somashekara G; On an (ε, δ)-trans-Sasakian structure, Proceedings of the Estonian Academy of sciences, 2012; 61(1):20-28.
7. Nagaraja HG, Somashekhara; Special Trans-Sasakian manifolds and curvature conditions, International journal of pure and applied Mathematics,2012; 81(3):411-420.
8. Sarkar A, Sen M; On invariant submanifolds of trans-Sasakian manifolds, Proc.Estonian Acad.Sci., 2012; 61:29-37.
9. De UC, Guha N, Kamilya D; On generalized Ricci-recurrent manifolds, Tensor,N.S.,1995;56:312-317.