**Scholars Journal of Physics, Mathematics and Statistics**
Sch. J. Phys. Math. Stat. 2015; 2(3A):308-309

**ISSN 2393-8056 (Print)**
**ISSN 2393-8064 (Online)**

# A Note on the Identity Element in a Function Space

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**Abstract:** This note demonstrates that the identity element in appropriately defined function spaces is weakly compact, but not compact; and bounded, but not weakly compact.
**Keywords:** Bounded, Compact, Weakly Compact

## DISCUSSION

Consider the family of all bounded continuous linear functions from a Banach space X into a Banach space Y [2,3,4]. Denote this function space $\mathcal{L}$(X, Y). If f ε $\mathcal{L}$(X, Y), then

i. $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1$ and $x_2$ ε X,
ii. f(ax) = af(x) for all x ε X and a ε **R**, where, **R** denotes real numbers.
iii. $\|f\| = \sup_{|x| \le 1} |f(x)| < \infty$ for x ε X

The symbol ||.|| denotes the norm in $\mathcal{L}$ (X, Y) and |.| denotes the norms in X and Y.

Elements of the $\mathcal{L}$(X, Y), which are often discussed in the functional analysis literature, are characterized in the following definitions. Let S = {xεX| |x| ≤ 1}:

Definition 1: A mapping f ϵ $\mathcal{L}$(X, Y) is weakly compact if the weak closure of f(S) is compact in the weak topology of Y.

Definition 2: A mapping f ϵ $\mathcal{L}$ (X, Y) is compact if the strong closure of f(S) is compact in the strong topology of Y.

One question regarding the robustness of the above definitions concerns the existence of elements of $\mathcal{L}$(X,Y) that are:

i. Weakly compact, but not compact
ii. Bounded, but not weakly compact.

The purpose of this note is to show that the identity element in the appropriately defined function space satisfies (i) and (ii) above.

EXAMPLE I: A Hilbert space H is a Banach spaces over the field of complex numbers **C** together with a complex function (x,y) on H x H which satisfies the following properties:

a) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$
   (ax,y) = a(x,y) for a ϵ **C**
b) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$
   $(x, ay) = \bar{a}(x,y)$ for a ϵ **C** and where $\bar{a}$ denotes the conjugate of a
c) $(x,y) = \overline{(y,x)}$
d) (x,x) ≥ 0, equality only for x=0.

Under these conditions $\|x\| = \sqrt{(x,x)}$ is the norm. Assume H is an infinite dimensional Hilbert space and consider $\mathcal{L}$ (H,H). Let i ϵ $\mathcal{L}$ (H, H) denote the identity map. The following three theorems are well known results in a functional analysis, so the proof is omitted.

*Theorem 1:* A normed linear space is finite dimensional if and only if its closed unit ball is compact.
Proof: Chapter 4 of Dunford and Schwartz [1].

Definition 3: The dual of a Banach space X is the function space of real-valued continuous linear functions on that space, denoted $X^* = \mathcal{L}(X, \mathbf{R})$.

Definition 4: Let X be a normed linear space, and X** the dual of the Banach space X*. The mapping $k: x \rightarrow \hat{x}$ of X into X**, defined by $\hat{x}$, x* is called the natural embedding of X into X**.

Definition 5: A Banach space X is reflexive if the natural embedding k maps X onto X**.

*Theorem 2:* Any Hilbert space is reflective.
Proof: Chapter 4 of Dunford and Schwartz [1].

*Theorem 3:* If either X or Y is reflexive, then every mapping in $\mathcal{L}(X, Y)$ is weakly compact.
Proof: Chapter 6 of Dunford and Schwartz [1].

From Theorem 1, we know $i: H \rightarrow H$ is not a compact map. From Theorem 2 and Theorem 3, we conclude that $i: H \rightarrow H$ is weakly compact.

EXAMPLE II: The following results are needed to construct the second example. These theorems are also well known, so the proofs are omitted.

*Theorem 4:* A Banach space X is reflexive if and only if its closed unit ball is compact in the weak topology.
Proof: Chapter 5 of Dunford and Schwartz [1].

Definition 6: A function f is essentially bounded if there exists a constant K such that $f(x) \leq K$ almost everywhere.

Definition 7: $L\infty = \{f|f$ is an essentially bounded function$\}$ and $||f||_\infty = \inf \{K|f(x) \leq K$ almost everywhere$\}$; that is, $|f| \leq ||f||_\infty$ almost everywhere.

We know $L^\infty$ is a Banach space that is not reflexive. Therefore, $\mathcal{L}(L^\infty, L^\infty)$ is also a Banach space. If we let i denote the identity mapping on $L^\infty$, $i \in \mathcal{L}(L^\infty, L^\infty)$, then i is bounded. Furthermore, $L^\infty$ is not reflexive. From Theorem 4, the closure of i(S) = S is not weakly compact. Therefore, the mapping i is bounded, but not weakly compact.

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