Scholars Journal of Physics, Mathematics and Statistics

OPEN ACCESS

Abbreviated Key Title: Sch J Phys Math Stat
ISSN 2393-8056 (Print) | ISSN 2393-8064 (Online)
Journal homepage: https://saspublishers.com

Topological Entropy of Regular Chaotic Quadratic Maps

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DOI: https://doi.org/10.36347/sjpms.2025.v12i03.002 | **Received:** 04.02.2025 | **Accepted:** 10.03.2025 | **Published:** 13.03.2025

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Abstract **Review Article**

The complex quadratic map as iterated complex curvature is discussed with respect to topological entropy and physical implications. The general quadratic map is a Hermite-Tschirnhaus relation of cubic invariants which can be casted into a binary linear substitution. It serves as a quadratic transformation of the elliptic integral of Friedmann equations. For invariant substitutions of holomorphic functions, a charge definition is proposed for the Riemann Xi-function and the Dirichlet L-function in form of a two-dimensional Poisson equation.

Keyword: One-Dimensional Quadratic Map, Topological Entropy, Bi Spinor, Feigenbaum Renormalization, Spacetime Curvature, Charge Definition.

1. INTRODUCTION

A fractal information universe with dimensionless coupling constants is capable to grasp hundreds of orders of magnitude while experiments only can capture about 10 orders of magnitude. Period doubling of a regular chaotic one-dimensional complex map is set in context to doubly-periodic iterated complex curvature. In two dimensions curvature is a complex generation rate. Complex curvature is a cubic invariant of hyperelliptic/elliptic curves generating algebraic units [1-3]. Zoomed bulbs and cardioids superimposed by optimal regulator conditions constitute a 3-parameter Huygen-Fresnel principle. Iterated quadratic maps of complex curvature has significance for unified physical fields and surrounds matter as an eternal binary process within a fractal zeta universe (FZU) [4, 5]. Besides periods ν_{Sh} due to the theorem of Sharkovskii vanishing Gaussian periods describe a self-consistent optimal regulator in the L- function (see Section 2 and Appendix 1). Mass m is defined as a tower of generators as roots of unity within periods of the interval. Bifurcation lines map line bundles moving in a lattice of algebraic units. Inflection tangents of elliptic curves are the stability axis in a Feigenbaum analysis. Iterates of units of a bicubic subfield cause fluctuating invariants and periods in elliptic theta. Topological entropy is discussed for algebraic units and power integral bases of optimal units related to a time-thermal rate. Nontrivial zeros of the Riemann zeta function and a Dirichlet L-function for a given iterated field yield a two-dimensional Poisson equation giving a definition of charge. The origin of charge and mass in the universe is explained by Feigenbaum renormalization extending Hieb's hypothesis within FZU [6, 7]. FZU uses Hieb' conjecture $2\pi\delta_F^2 \simeq \alpha_f^{-1}$ with Feigenbaum constant δ_F, fine structure constant α_f giving already an accuracy of $9.12\cdot10^{-4}$. Similarly, the fine structure constant α_f can be refined by optimizing the information density on a surface of area $4\pi R^2 \simeq g_1^{\vdots g_n}$ of a sphere of radius R for a constant sum $g_1+g_2+\ldots+g_n$ of generators which is in accord with FZU [8, 9]. Exactly solvable chaos consists in mapping of universal covering space [10, 11]. Elliptic curves are attractors in a Lattés map on a two-sphere [12]. However, iterated functions of a Lattés map have higher degree and do not exhibit ν_{Sh}. In distinction, a Hermite-Tschirnhausen map $\gamma(\phi_3)$ of a cubic invariant ϕ_3 is a linear and quadratic conjugate with 2-power bases. $\gamma(\phi_3)$ forces complex multiplication (CM) by addition on elliptic curves. Periods ν_{Sh} may be $\gamma(\phi_3)$ fixpoints. A Riemann zeta function $\zeta(z)$ scanned by $\gamma(\phi_3)$ for extensions z of a cubic field $\mathbb{K}[\partial]$ allows to start from holomorphic $\xi(z)$ and holomorphic Dirichlet L-function [13].

$$\frac{\zeta(z,\mathbb{K})}{\zeta(z)} = \frac{\Gamma(z/2)z(z-1)\zeta(z,\mathbb{K})}{2\pi^{z/2}\xi(z)} = \mathrm{L}(z,\chi). \qquad (1.1)$$

The $z\rightarrow1$ limit $L(1,\chi)\simeq H_\Delta R_\Delta$ is proportional to a regulator $R_\Delta = \ln_b E$ with fundamental unit E, base b of extensions of $\mathbb{K}[\partial]$ and to class number H_Δ of a cubic normal field with discriminant Δ. Optimal units are feasible solutions Subsequent optimal solutions (2.1) for the regulator R_Δ=l=ln$_b$E=Ω_w-$\mathscr{L}$ with $\mathscr{L}$=$\mu_2E^2\zeta(l_s,m_s,l)+½\mu_2Nm(E)\zeta'(l_s,m_s,l)$ yield an eternal clock rate $R_{k,qq'}$ on complex plane. The binary process consists in surrounding zeta zeros on irreducible four-component elliptic curves λ_μ. Accordingly, the complex L-function behaves as an action functional. Cardioid-like oscillations around centers on complex plane z yield a Dirichlet L- function as a candidate for a Lovelock-like action functional in $\mathbb{C}^w$. Then a quadratic map $\gamma(\phi_3)$ is capable to link nontrivial zeros z_{nt} and the pole at z=1. This map can be written as a mass operator

$$F(t,z) = \phi_3(t)/(t-z) - ⅓\phi'_3(t) = \gamma(\phi_3(t)) ⊡ z \simeq V_\mu G_{ss''}\gamma^\mu_{s''s'''}G_{s'''s'} - ⅓\frac{\delta V_\mu}{\delta(G_{ss''}\gamma^\mu_{s''s'''}G_{s'''s'})} \quad (1.2)$$

Within space of quartic roots $G_{ss'}$ is equivalent to a Green's function with indices s=1,2,3,4 relating cubic roots to quartic roots. One quartic root is shifted to ±∞,±i∞ in a quadruple of steps q={k,k+1,k+2, k+3} for a simplest cycle quadruples q: {k+3∊{k,k+1,k+2}} [14, 15]. The complex map generates a general Riemann surface as layers of bifurcating cardioids and bulbs. In Section 2 a three-parameter superposition of z and lnz is called a feasible solution in eq. (2.1) which yields w=1,2,3,4,5 layer. The Hermite map is discussed in Section 3. For fluctuating lattice periods modular units are discussed as invariants for an iterated lattice of algebraic units in Section 4. Inflection tangents of curves are set in context to stability axes needed for a Feigenbaum diagram technique in Section 4. Whereas the Lebesgue measure is discussed in Section 6 invariant relations for the Legendre module are discussed on Section 7. Feigenbaum renormalization for simplest cycles in Section 8 is used for the definition of charged states in Section 9.

2. COMPLEX GYRO-TWIST HYPERSURFACE

N Mandelbrot cardioid z- planes yield N distinguishable spheres. But only up to five spheres are independent in space: a w-dimensional complex space offers w(w+1)/2 -3w+3 independent parameter [16]. Division of N>w cardioids into w balls yields indistinguishable permutations of identical particles. This generalized Riemann surface with $\binom{w-2}{2}$ independent complex parameter is w spheres centered around poles driven by a clock rate $\Phi_2^k(z)$. The generalized Riemann surface denoted by $\mathbb{C}^w$ is like an organic constantly changing plant embedded into five atmospheric layer as nested spheres. On interval [0,1] a non-wandering set around the point ½ yields a black hole entropy h_t=ln2 [17]. On interval [0,1] w unit spheres for extremal h_t and optimal units have a non-wandering set of the quadratic map $\gamma(\phi_3)$ with topological entropy $h_t=H_{CS}$ or $H(l_s,m_s)$ in [0,1] [18]. Maximum information probability expected at a critical point F'(t,z)=0 at ½ is shifted to a circle of radius $H(l_s,m_s) = ln_{l_s}m_s$. A non-wandering set of k-components in $\zeta(l_s,m_s,z_k)$ has poles on $z = H(l_s,m_s)\left(1+\frac{2\pi in}{\ln m_s}\right)$. Around the pole e.g. of the Cantor set $z = H_{CS}\left(1+\frac{2\pi in}{\ln 2}\right) \notin \mathbb{QL}_w$ on unit sphere in $C^w[\mathbb{L}_w]$ iterates z_k perform doubly-periodic (atmospheric) waves of world points $X(z_k)$ realizing vibrations of fractal strings [19]. The number of z_k- clouds in w-spheres performs non-dissipative independent regular complex fluctuations resolvable by nested spheres $C^w[\mathbb{L}_w]$ w=1,2,3,4,5. The theory is understandable superposing z and l=lnz as an optimal complex curvature $\mathbf{R}_{\mu\upsilon}$. Stable orbits z_k encircling with radius $H(l_s,m_s)$ are complex $\phi(g_1) = \sum_{i=0,\ldots,g_\infty-2} a_i g_1{}^{g_2^i}$ [20] [21]. Highest information densities correspond to two or three exponentiation levels in $g_1^{\cdot^{\cdot^{\cdot g_l}}} \simeq g_1{}^{g_2{}^{g_3}}$ which is optimal as a feasible solution [8, 22]. The Kepler singularity is treated as a $\zeta(z)$- pole, Coulomb singularities are complex-conjugated poles in f(ω). A thin layer of a sphere of diameter $2H(l_s,m_s)$ has an altitude temperature gradient (pressure) and lateral waves. If $1+H(l_s,m_s)$ is a root of unity combinations $w!H^w(l_s,m_s)$ can be regarded as Gaussian periods of complex roots $\phi(g_1)$ see §112 in [21]. The coupling constant G_w can be given the following explanation in terms of rates. First the logarithm of a unit l=lnE is equivalent to curvature $e^{\mathbf{R}}=\exp(\mathbf{R}_{\mu\upsilon}[\gamma_\mu,\gamma_\nu]_-)$. In two dimensions curvature and stress-energy is a count rate which as a current proportional to a coupling constant. A scaling by $H(l_s,m_s)$ requires projective spheres which are invariant with quadratic birational transformations. In homogenous coordinates a sphere $\mathbb{S}^2(\mathbf{a_i}, p_i) = S_i$ is five-component and depends on position $\mathbf{a}_i$ and a power of origin p_i with respect to sphere. Maximal five independent spheres $\mathbb{S}^2(\mathbf{a_i}, p_i)$ i=1,..,5, at $\mathbf{a}_i \in\mathbb{R}^3$ correspond to stereographically projected five Gaussian planes. The minimum of a quadratic form $\sum(\mu_1 l^2+ \mu_2 l)$ is written as a map $l_w \rightarrow l_{w+1}$ of logarithms.

$$\prod_w \sum_{iq}(2\mu_1 l_{w+1} + \mu_2 + 2\mu_2 e^{-2l_w}\zeta(l_s,m_s,l_w) + \mu_3(\textstyle\sum_{(q)} E_q^{-2})\zeta'(l_s,m_s,l_w)) = 0 \quad (2.1)$$

of curvatures $l=\log E_q \simeq \mathbf{R}_{\mu\upsilon}$ in sphere w. For $\zeta'\rightarrow 0$, $\mu_2\rightarrow 0$, $\mu_3\rightarrow 1$ and $\oint\zeta(l_s,m_s,l_w) = H(l_s,m_s)$ as a mean over a tower of roots of unity the map reads $l_{w+1} \leftarrow 2wH(l_s,m_s)l_w$ with a w-fold rate for w spheres. Then $l_w = w!2^wH^w$ which gives $l_{w+1} = 2\mu_3 G_w{}^2\zeta(l_s,m_s,l_w)$ with a coupling constant $G_w=\exp(w!2^wH^w(l_s,m_s))$. Topological entropy $h_t(f) = \lim_{n\rightarrow\infty}\frac{1}{n}\ln l_\omega(f^n)$ as a mean over laps $\delta_k \prod \delta_{l_\omega}$ is the product of generators. The total count rate of onion-shaped shells is the product of partial count rate probabilities. As a result, inner shells have higher periods which are proportional to masses. The most general Riemann surface within $\mathbb{C}^w$ implies pseudo-congruence which reads $2^{2^k} = G_w^{-1}$ which is an image of an infinite set on a finite set of possible points in $\mathbb{C}^w$. At the same time the number of crossing points of ν_{Sh} lines of the logistic map windows is of the order 10^1 as shown in the Figure 1.

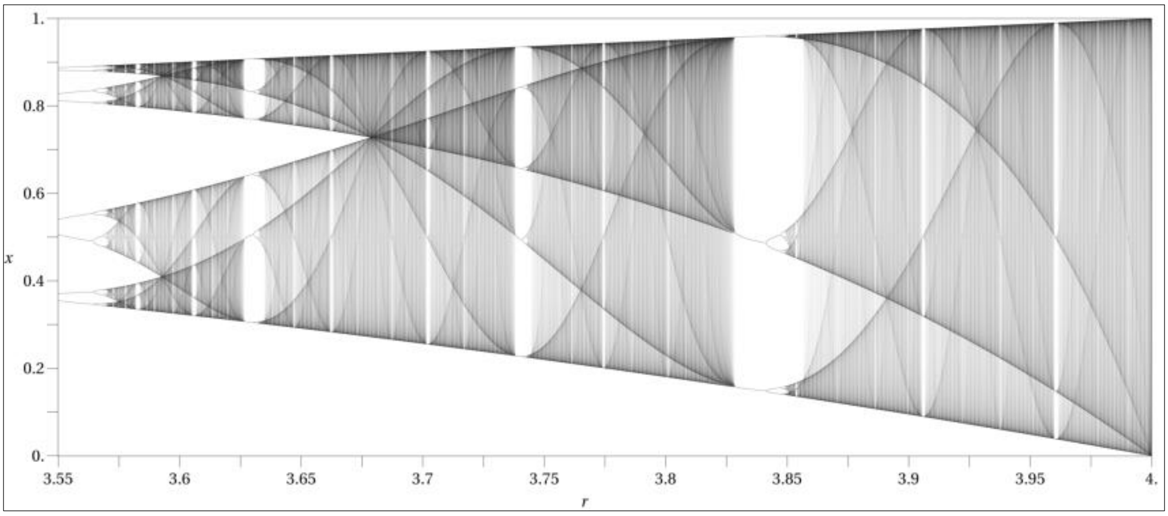

Figure 1: High resolution subsection of the bifurcation diagram of the logistic map From [Wikipedia 'logistic map]

General relativistic coordinates result from a one-to-one relation between the Hermite map $\gamma(\phi_3)$, a Moebius map γ_M and a two-by-two Lorentz-transformation γ_L in the special case of a vanishing discriminant of a quadratic field Δ_2. A cubic behavior of $f(\omega)$ is a presupposition for a regular chaotic map. The invariant $f(\omega) = 1^{\frac{-1}{48}}\frac{\eta\left(\frac{\omega+1}{2}\right)}{\eta(\omega)}$ depends on the Dedekind eta function $\eta(\omega)=⅓\vartheta_1$ (⅓, ⅓ω) and the Jacobi zeta function $\zeta_1(u,\omega)$ giving

$$f(\omega) = \sqrt[-48]{1}\exp\left(⅓ \int_\varepsilon^1 dv \left(\zeta_1\left(⅓v,\frac{\omega+3}{6}\right)-\zeta_1\left(⅓v,\frac{\omega}{3}\right)\right)\right) \qquad (2.2)$$

laps of ω_k iterates tend to differentials $v_{k+1}-v_k=a(\omega_{k+1}-\omega_k)\to dv$ around z_λ. Due to asymmetry of $\gamma(\phi_3)$ in eq. (3.2) around ½ highest probabilities and critical points with zero first F(t,z)- derivative are expected at inflection tangents of the pencil $\mu_1\phi_3+\mu_2H(\phi_3)=0$ of ϕ_3 with its Hessian , i.e. zero second derivative [23]. A geometric zeta function $\zeta(l_s,m_s,l_k)$ is a fractal vibration of l_k at z_λ. A line $dv \simeq a\gamma(\phi_3)\omega_k \simeq G\cdot G\ \omega_k$ splits within a bicubic Kummer extension field $\mathbb{K}\,[\partial, g_1^{\cdot\cdot\cdot g_l}]$ as a fourth-order product in ψ_s. Second and first order shifts $\delta_k\delta_k$ and δ_k of k-components are finite generation terms.

$$\sum_{k,w}\delta_k^2\int_0^1 dt\zeta(tv,\mathbb{L}_k) = \sum_{k,qq'}\int_0^1 dt(\delta_k v)R_{k,qq'}\zeta(l,m,R_{k,qq'})\big(\delta_k\zeta(tv,\mathbb{L}_k)\big). \qquad (2.3)$$

A time-thermal rate $R_{k,qq'}\simeq\ln E_{k,qq'}$ at z_λ of lines F(C) in eq. (A3.7)

$$R_{k,qq'}(v,\mathbb{L}_k) = \varepsilon(\omega)\bar{\varepsilon}(\omega)\chi\big(v,\zeta(v,\mathbb{L}_k)\big)\chi^{-1}\big(v,\zeta(v,\mathbb{L}_k)\big)\theta(\delta_k h_t) \qquad (2.4)$$

is set with susceptibility $\chi\big(v,\zeta(v,\mathbb{L}_k)\big) = 1 + \frac{\delta_k\ln\delta_k v}{\delta_k\ln\zeta(v,\mathbb{L}_k)}$,a pair of complex-conjugated units ε and $\bar{\varepsilon}$ of the normal field $\mathbb{N}[\sqrt{\Delta}]$, $\varepsilon\bar{\varepsilon} = ½f^3\big(\sqrt{\Delta}\big)$,$E\varepsilon\bar{\varepsilon}=1$. The step function Θ(x) where $\lambda=1-\delta_k h_t$ denotes a non-ergodic time arrow in a time-thermal rate $R_{k,qq'}$ of Δ_k fluctuations where $R_{k,qq'}\simeq l_{k,qq'}\simeq\ln_l m$ yields a root of unity $e^{\delta_k h_t}$. Expression $\chi\big(v,\zeta(v,\mathbb{L}_k)\big)\chi^{-1}\big(v,\zeta(v,\mathbb{L}_k)\big)$ is a generator g_l in a cyclic polynomial $\phi(g_1)$. Susceptibility χ has a superconducting part of space as non-intersecting lines governed by the map $\gamma(\phi_3)$. The geometric zeta function $\zeta(l_s,m_s,R)$ for a circular rate R ≃l with k-component generator $g_1^{\cdot\cdot\cdot g_l} \simeq g_1^{\,g_2^{\,g_3}}$ on a circle area $dk\wedge d\bar{k}$ on $\mathbb{C}^w$ is viewed as an z_λ-point occupation number. Shells C^w scale to unit circles by $\ln_l m$ which justifies a normalization to ω=1 used in [24]. In the spirit of ω-fluctuations the rational factor $a\in\mathbb{Q}^2$ in modular units g(u=aγω) is inert and Lorentz-invariant for $\gamma\simeq\gamma_M\simeq\gamma_L$. Periods ν_{Sh} contain cyclotomic units in modular units on. Multiplicity m_s and string length l_s of the geometric zeta function $\zeta(l_s,m_s,z)$ in eq. (2.1) correspond to a pseudo-congruent minimum of eq. (2.1) for Fermat number transform F_t having base 2 and base 3 for the first prime F_t. In distinction to a hyperbolic tessellation γ are superposed pseudo-random bulb-cardioid lines on spheres like a Huygens-Fresnel principle. Diffusive paths u=aω create a thermal conductivity $\sigma_T\simeq\omega$ and a

Gaussian kernel with standard deviation 1/ω. Chaotic period changes $\delta_k\omega\simeq\delta_F$ yield a standard deviation of the Gaussian kernel if $2\pi\delta_F^2 \simeq\alpha_f^{-1}$. Thus, a chaotic map around z_{nt} is capable to describe charges. Time standard as the number of completed orbits is independent on the actual value of variable ω. Simplest cycles of X(f(ω)), Y(f(ω))- iterates are embedded into two hyperelliptic Riemann surfaces of universal covering $u_\pm$, $u'_\pm \to u_\pm$ bifurcating into two tori with universal covering $u_\pm$. The regular fluctuation of X(f(ω)) and Y(f(ω)) leaves the differential $|\gamma|^2(ch)^4D_h^2$ with $c=(c_+,c_-)$, $h=(h_+,h_-)\simeq(f_k,f_{k+1})$ and $D_h=D_+h_++D_-h_-$, $D_\pm = \frac{\partial}{\partial u_\pm} - \frac{\partial}{\partial u'_\pm}$ invariant [25]. Then $D_h\sigma(u)\sigma(u') = \tilde{\zeta}(a\boldsymbol{\omega})\sigma(u)\sigma(u')$ giving in the limit u'→u a vector potential equivalent to $\sum\zeta(a\omega_k,\omega_k)$. A Lattés map u→au + b for a Hurwitz automorphism $\sum_{i=1}^{r} 1 - r_i^{-1} = 2$ between sphere and torus implies four modular units aω with $\omega = 1^{1/r_i}$. The solution prescribes four branch point sets r_i= {2,2,2,2}, {2,3,6}, {2,4,4}, {3,3,3} [26]. This branched covering covers a quadruple q of simplest cycles on iterated tori. The complex z-plane $\mathbb{C}(\mathbf{n})$ of normal vector **n** projects to a sphere $\mathbb{S}^2(\mathbf{a_i}, p_i)$ in space. A Mandelbrot zoom cardioid can be embedded into planes in space without intersections. The Weber- Schottky problem $\mathbb{L}_w$ allows for genus w=4 or w=5 one or three independent parameter e.g. 3w-3 rotation angles of w-1 nested spheres. The difference $\binom{w-2}{2} = \binom{w+1}{2} - 3w + 3$ yields 1 or 3 complex parameters for genus surfaces w=4 and 5. These parameters are identified with the number of physical units in measurement systems, necessary for measurement (e.g., CGS or Planck system). A simple geometric explanation is in terms of arbitrary spheres $\mathbb{S}^2\{x^2+\mathbf{x}_i\ \mathbf{a}_i=p_i\}$. Maximal five spheres $\mathbb{S}^2(\mathbf{a_i}, p_i)$ are independent if det $(a_i, pi,1)=0$ which yields a determinant of fourth order

$$\det A[\delta a,\delta p] = \prod_{\substack{i\neq j\\ \pi(i,j)}} det(a_i - a_j, p_i - p_j) = 0 \qquad (2.5)$$

with permutation π(i,j). The number w of different paths connecting spheres $\mathbb{S}^2(\mathbf{a_i}, p_i)$ is identified with interaction w of nested spheres with faster inner rotating gyro-twist shells where cyclotomic fields in {2,2,2,2}, {2,3,6}, {2,4,4}, {3,3,3} are consistent with 2^{2^k} congruences of iterated Weber invariant $f(\sqrt{\Delta})$. Here a base b=2 and b=3 in a number transform $e^{h_t} mod\left(2^{2^k}-1\right)$ for topological entropy $h_t = 2^{2^k}\ln 2$ contains the generator $b^{2^k} = 1 mod\left(2^{2^k}-1\right)$ [27]

3. HERMITE SUBSTITUTION

Hermite substitutions relate power integral bases (PIB) x^i, t^i in polynomial $F(t,z) = \frac{\Phi_n(t)}{t-z} - \frac{1}{n}\Phi_n'(t)$ rational to polynomial roots t,z of $\Phi_n(z) = \sum_{i=0,\dots,n}\binom{n}{i} a_i z^{n-i} = 0$. PIB $t^i \to t_i$ is regarded as symbolic n+1 dimensional vector. Iterated fractional substitutions $z_{k+1} \leftarrow \prod_{i=0,\dots,k}(z_i - z_{i+})(z_i - z_{i-})$ are binary invariant in homogeneous variables $z = \frac{z_1}{z_2}$.

Hyperelliptic details of ϕ_{2^k} are $\prod_{i=k,k+1,k+2}(z_i - z_{i+})(z_i - z_{i-})$ as simplest cycles. The fundamental hyperelliptic addition theorem on Kummer surface K(X) and Weddle surface W(Y) reads [28].

$$s_{+[gh]}(u,v)s_{-[gh]}(u,v)=X(u)jX(v) \qquad (3.1)$$

where $s_{+[gh]}(u,v)=\vartheta_{[gh]}(u+v)/\vartheta_{[gh]}^2(u)$, $s_{-[gh]}(u,v)=\vartheta_{[gh]}(u-v)/\vartheta_{[gh]}^2(v)$, $j^2=-1$. For a quadruple q of steps k variable z→f(ω) relates $X(f)=(\wp_{\pm\pm},1)=(1,-f,f^2,1)$ and $Y(f)= \wp_{\pm\pm\pm}=(1,-f,f^2,-f^3)$ to four points $u_\pm$ and $u'_\pm$ on the universal covering of K(X) and W(Y). Quartic roots x_i capture K(X) and W(Y) and correspond to cubic roots e_i as follows $(x_i x_j)(x_k x_l) = e_k - e_l$ ∀ i, j, k, l= {1,2,3,4}. For $\Phi_3(x_l)= m_l=(lj)(lk)(li)$, $\sum e_i=0$, $(ze_1e_2e_3\pm\infty\pm i\infty) \to (xx_ix_jx_kx_l)$ for root $x_l \to\pm\infty\pm i\infty$ one gets the Hermite substitution (1.2) where $\phi_3(z) = \frac{m_l^2}{(x-x_l)^4}\Phi_4(x)$ where [29]. The 1:2 relation $\sqrt{(\Delta e)}\backsim\Delta x$ is written in terms of quadruples q≃s and spin indices relates a linear map to a bi spinor ψ_s in which $\gamma(\phi_3)$ is quadratic in fermion Greens functions

$$\gamma(\phi_3(t)) = \begin{vmatrix} \frac{1}{3}\phi_3'(t) & \phi_3(t) - \frac{t}{3}\phi_3'(t) \\ -1 & t \end{vmatrix}. \qquad (3.2)$$

F(t, z) yield $z \leftarrow F(t,z) = t_0F_1 + t_1F_0$ where $(F_0,F_1) = (a_0z+a_1, a_0z^2+a_1z+2a_2)$.
For t- dependent $a(t) = t_0a_0$, $b(t) = t_0a_1 + t_1a_0$, $c(t) = 2t_0a_2 + t_1a_1$

$$z_{k+1} \leftarrow F(t,z_k) = a(t)z_k^2 + b(t)z_k + c(t) \qquad (3.3)$$

with peculiarity

(1) if $\Phi_3(z_{k+1})=0$ then deg$z_{k+1}=3$
(2) if $\Phi_3(z_{k+1}) \neq 0$ deg $z_{k+1}=2^k$

or in terms of elliptic units g(aω)

(1) if $z_{k+1}-z_k \in g(a\omega)$ then deg $z_{k+1}=3$
(2) if $z_{k+1}-z_k \notin g(a\omega)$ then deg $z_{k+1}=2^k$.

Chaotic period-doubling on hyperelliptic $\phi_n(z)$ is that z is a doubly-periodic polynomial invariant f(ω). Modular units g(a**ω**) depend on f(ω) and periods ω are are self-consistently connected with periods ν_{Sh}. Conjugates of linear- quadratic $\gamma(\phi_3)$ is a Mandelbrot map M_c : $z_{k+1} \leftarrow z_k^2 + c$ which results from (3.3) for parameter a_0=1, a_1= t_1=0 , $2a_2$ =c , t_0=1,$\forall a_3$ leading a harmonic cross-ratio λ=-1 and $x_{k+1} \leftarrow \frac{x_{k+1}}{a}$, $x_k \leftarrow x_k + \frac{b}{2a}$ and $c \leftarrow -\frac{\Delta(t)}{4a^2(t)}$ where $\Delta(t) \leftarrow b^2(t) - 4a(t)c(t)$. The logistic map $x_{k+1} \leftarrow t_0 x_k(1 - x_k)$ results from (3.3) for parameter a_0=-1, a_1= a_2 =0,$\forall a_3$ on a line t_1=--t_0 for an equianharmonic cross-ratio λ =$1^{⅓}$. Logistic map $x_k \leftarrow t_0\left(\frac{1}{2} - x_k\right)$ and Mandelbrot map $c = \frac{t_0}{2}\left(1 - \frac{t_0}{2}\right)$ are conjugate to equianharmonic and harmonic elliptic curves. Hermite substitutions $\gamma(\Phi_3(t))$ hold for a finite region t_0, t_1 on complex plane with axes t_0 and t_1 with Hessian $H(\phi_3)=H(t_0, t_1) \simeq \gamma_2(\omega)$ which vanishes quadratically. A Hermite- transform- solution of cubic roots is equivalent to Cardano's method [29]. $\gamma(\Phi_3(t))$ enforces CM as endomorphism of bundle $\mathbb{L}[E_\lambda]$ over a subfield $\mathbb{K}[\partial]$. A Hermite substitution $\gamma(\phi_3)$ induces a conductor det$\gamma(\phi_3)$ =$\phi_3(t)$ for discriminant Δ_3. For cubic residues $\left[\frac{\gamma(\phi_3(t))}{p}\right]_3 = 1$ $\gamma(\phi_3) \in SL(2,\mathbb{Z})$ yields equivalent periods. $\gamma(\phi_3)$ generates the Tschirnhausen resolvent of the elliptic invariant j(ω) ←j(γ(ϕ_3(t))∘ω). For class number one h_Δ=1 the linear j- resolvent reduces the degree of polynomial j(f(ω)) from 72 to 3. One addition step yields a quartic polynomial (Appendix 3) reduced to ϕ_3. This defines $\binom{4}{3}$ =4 layers from torus to sphere $\mathbb{S}^2$ up to a choice of normalization or calibration known as the Weierstrass ℘-function. For a definite fractional substitution $\gamma[\phi_3(e_i)]$ of second order one has $e_i = \wp(\omega_i)$ and

$$\wp(u,\omega) = \gamma[\phi_3(e_i)]\vartheta^2_{\left[\substack{g\\h}\right]}(u,\omega) \qquad (3.4)$$

A more general theta function s(u) is a substituted Jacobi function sn(u,√λ) for a quartic polynomial ϕ_4

$$s^2(u) = \begin{pmatrix} 1 & 0 \\ 1 + e_3 - e_1 & e_1 - e_3 \end{pmatrix} sn^2\left(\sqrt{e_1 - e_3}u, \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}\right) \qquad (3.5)$$

which depends on the Weierstrass ℘-function [11] [10], e.g.

$$s^2(u) = \begin{pmatrix} -1 & -⅓(e_2 - e_3) \\ 1 & ⅓(e_3 - e_2) - (e_3 - e_1)(e_1 - e_2) \end{pmatrix} \wp(u) \qquad (3.6)$$

Homogeneous projective coordinates and Lorentz-coordinates enter as an eternal reduction process. The complicated problem to reduce a hyperelliptic theta function $\vartheta_{[gh]}(u)=\vartheta_{[gh]}(u_\pm)$ is treated on four world-points on $K(X^\mu{}_{[gh]}(f))=(\wp_{[gh]}(u_\pm),1)=(1,-f,f^2,1)$ as a function of the invariant f=f(ω). A quadruple of four elliptic ℘-functions belongs to quartic roots shifted to s≃q≃±∞,±i∞ which yields $\wp_q \simeq \wp_s$. This four-component quantity is set equal to $\wp_s \simeq \gamma_\mu M(\mathbf{a}) X^\mu{}_{[gh]}(f) = e^{S(\mathbf{A},\mathbf{a})} X^\mu{}_{[gh]}(f)$ on four points μ=1,2,3,4. Accordingly, a reduced hyperelliptic function yields four points X^μ, μ=1,2,3,4 as an irreducible tidal point. Hyperelliptic

$$\vartheta_{[gh]}(u_\pm) \to \vartheta^\mu{}_{[gh]}(u_\pm)$$

$$\wp^\mu{}_{[gh]}(u_\pm) = \wp^\mu{}_{[gh]}(u_+,u_-) = \gamma(\phi_3(f))\wp^\mu(u)$$

substituted elliptic ℘- functions have components μ and s≃q as SE(3) parameter. A chaotic map $\gamma(\phi_3(f))$ has simplest cycles $f_q \simeq f_s$. Thus, a theory of a point is self-similar where a complex plane $\mathbb{C}$ is an envelope of up to five planes $\mathbb{C}^w$.

4. MODULAR UNITS

Modular units (4.1) are Lorentz-invariant g(aγω) for a subset $\gamma \simeq \gamma_L$ despite fluctuating periods ω. Cyclotomic-like units $\varepsilon_{ij}(\omega) = \varepsilon(\omega_i)\bar{\varepsilon}(\omega_j) = \frac{\eta(\omega_i)\bar{\eta}(\omega_i)}{\eta(\omega_j)\bar{\eta}(\omega_j)}$ depend on modular units $g(a\omega) = \tilde{k}_a(\omega)\eta^2(\omega)$ for ω_i, ω_j -congruences [30] [31] [32] [33]. Klein functions $k_a(\omega)$ are normalized sigma functions. For fluctuating periods ω generalized Klein functions $\tilde{k}_a(\omega)=\exp(-½a\tilde{\eta}[\zeta]a\omega)\sigma(a\omega))$ differ by independent periods $\tilde{\eta}[\zeta]$ where $\tilde{\eta}[\zeta]$= ζ(u+ω,ω)-ζ(u,ω)→ η[ζ] and $\tilde{k}_a(\omega) \to k_a(\omega)$ of Weierstrass zeta functions ζ(u,ω). Then g(aω) depend on two Lorentz-invariant points $\gamma(\phi_3(f_{k+1})) \simeq \gamma_{L,k+1}$ and $\gamma(\phi_3(f_k)) \simeq \gamma_{L,k}$.. A simplest cycle quadruple q≃s imposes lattices ω[$\mathbb{K}[\partial, g_1^{\dots g_l}]$] and ν_{Sh} as congruences in ω_k. A cubic fundamental unit ε_{ij} with conjugates $e^{i\varphi}/\sqrt{\varepsilon_{ij}(\omega)}, e^{-i\varphi}/\sqrt{\bar{\varepsilon}_{ij}(\omega)}$ changes into one of a cyclotomic fields $1\text{-}1^z$ with class number →∞. For simplest cycles Euclidean norms $(\sum_{(q)}E_q^{-2})=\psi_s\bar{\psi}_s$ are accepted quantum statistical ones tending to a unified thermodynamic one $\simeq b^{2\Omega-2\mathscr{L}}$. In distinction to quantum statistics, low values of the real algebraic unit $\varepsilon_{ij}\to 0$ and a vanishing regulator index R_Δ for developing cyclotomic extensions question $\psi_s\bar{\psi}_s=1$ which is only valid if $\varepsilon_{ij}\simeq 1$. The correct unified bi spinor ψ_s depends via ε_{ij} on a coupling constant G_w. Whereas Feynman diagram series remain valid non-radiative exchange scattering contains the factor 10^{-167}. The quantum statistical expectation value is about 10^{-167} lower, i.e. the entropy current 10^{167} higher which solves the cosmological constant problem. Replacing the Dedekind eta function η(ω) in $\varepsilon_{ij}(\omega)$ by $1\text{-}1^z$ one gets eqs. (1.2-4). Hyperelliptic points $X_{[gh]}$ on K(X) with $s_{+[gh]}(u,v)s_{-[gh]}(u,v)=X(f_k)jX(f_k)$ in Section 3 suffering simplest cycle quadruples q are written near iterated Legendre modules λ_k and nontrivial zeros $z_{ntk} \simeq \lambda_k \simeq \lambda_{mk}/m_k+½$. The cubic congruence yields eq. (7.6) where ψ_s contains iterated quadruples γ,γ∘γ,γ∘γ∘γ,γ∘γ∘γ∘γ. Shifting subsequently cubic/quartic roots ={e_k, e_{k+1}, e_{k+2},±∞ or ±i∞} the

Euclidean norm of a quadruple $Nm(E_q)$ of real algebraic units E_q with $E_q e' e''=1$ exhibits invariances. The Euclidean norm $Nm(E_q)$ differs only by a rational factor from the Bezoutian or Hessian $B(t,z)=(F(t,z),F(t,z))=-\frac{2}{3}H(\phi_3)$ [29]. World-points $X^\mu(f)=(1,-f,f^2,1)=X_{[gh]}(f)$ depend on 6·16 hyperelliptic theta functions $\alpha_{[gh]}(u+v)\tilde{\Gamma}\beta_{[gh]}(u-v),\gamma_{[gh]}(u+v)\tilde{\Gamma}\delta_{[gh]}(u-v)$ $e_{[gh]}(u+v)\tilde{\Gamma}f_{[gh]}(u-v)$ in space of characteristics [gh] [34]. Hyperelliptic orthogonal substitutions $\alpha_{[gh]}(u+v)\tilde{\Gamma}\beta_{[gh]}(u-v)$ are four-component linear forms ($\alpha\beta$) with 16 matrix elements $\tilde{\Gamma}_{ij,i'j'}=\pm1$ where five linear forms ($\alpha\beta$) also constitute general elliptic theta functions $\vartheta_{[gh]}(u)$. In FZU inert characteristics [gh]={0,1} are iteratively resolved in $\mathbb{K}[\partial]$. Units $\varepsilon(\omega,\sqrt{\Delta}) = e^{\frac{5\pi i}{24}(\omega\bar{\omega}+1)-\frac{1}{2}\ln 2}\frac{\eta(\frac{1}{2}\omega)}{\eta(2\omega)}$ in $\mathbb{N}[\sqrt{\Delta}]$ where $E^{h_\Delta} = \frac{1}{2}f^3(\sqrt{\Delta})$ are independent on the actual period ω where $Nm(f(\sqrt{\Delta}),\mathbb{K})=2$ for $h_\Delta=1$. Besides modular units g(aω) sixteen hyperelliptic half-periods [gh] are inert which determine mass ratios in a quadratic equation for K(X). This f-parametrized hyperelliptic equation is $X(f_k)jX(f_k)=s_{+[gh]}(u,v)s_{-[gh]}(u,v)=0$ which is set in context to nontrivial zeros z_{nt}. A γ-invariant derivative $\exp(|\gamma|^2(ch)^4D_h^2)\vartheta(u)\vartheta(u')=0$ is set in context to an amplitude representation $\xi(z)=\exp(\int ds\mathbb{A}(s))$ of the Riemann zeta function ζ(z) [35]. Poles of the amplitude are zeros of ξ(z). For f(ω)-points c and h in $\phi_3(c)=0$ and $\phi_3(h)=0$ and $(ch)^4\simeq G(\psi_s)G(\psi_s)\Gamma^{(ren)}[\gamma_\mu\gamma_\nu D_{\mu\nu}]G(\psi_s)G(\psi_s)$ in invariant derivative a quadratic map implies to replace $u_\pm$-derivatives by Dirac matrices γ_μ. Instead of a Lattés map αu+b which in case of α=2 yields a complicated quartic map only Poncelet involution with α=±1,±i is regarded. The invariant derivative D_h in $|\gamma|^2(ch)^4D_h^2$ is related to certain values of the Weierstrass zeta function $\tilde{\zeta}$ as shown below. Then a square of the invariant derivative D_h can be captured by a vertex function $\Gamma^{(ren)}[\gamma_\mu\gamma_\nu D_{\mu\nu}]$ giving S-matrix-like $\exp(|\gamma|^2(ch)^4D_h^2)\simeq\exp(\int ds\mathbb{A}(s))$ with boson-propagator-like $D_{\mu\nu}=[\zeta,\zeta]$ is. The poles of $\exp(|\gamma|^2(ch)^4D_h^2)$ are zeros of X(f)jX(f) giving points on $K(\wp_{\pm\pm},1)$ and $W(\wp_{\pm\pm\pm})$. The exponent is a time-thermal rate $R_{kqq'}$ of γ-orbits

$$f_s(\sqrt{\Delta_{ik}})\leftarrow\gamma(\phi_3)f_s(\sqrt{\Delta_{ik}})$$

where simplest cycles q≃s are tidal motions of four points crossing two lines. Then rational values a in g(aω) are inert whereas periods ω fluctuate like a hyperelliptic period matrix as a variable elliptic period. Modular units g(aω) are modular-invariant Weierstrass sigma functions known from quantum Hall functions [36] [30]

$$g(a\omega) = \Delta^{1/12}e^{-a\tilde{\eta}[\zeta]\cdot a\omega}\sigma(a\omega,\mathbb{L}) = \Delta^{1/12}e^{-\int_\varepsilon^{a\omega}dv\tilde{\zeta}(u,\mathbb{L})} \quad (4.1)$$

with ε→0, independent $\tilde{\eta}[\zeta]=\zeta(\omega/2,\omega)$, half-periods $\omega=(\omega_1,\omega_2)$, $a\omega= a_1\omega_1+ a_2\omega_2$, $\tilde{\zeta}(v,\omega) = \zeta(v,\omega) - \omega\tilde{\eta} - \frac{1}{v}$. Denoting Klein functions $\tilde{k}_a(\omega)$ by (a_1a_2) a square of the Dedekind eta function $\eta^2(\omega)\simeq\omega$, e.g.

$$\eta^2(\omega) = \frac{(\frac{1}{2}0)(\frac{1}{2}0)(\frac{1}{2}0)}{(\frac{1}{3}0)(0\frac{1}{3})(\frac{1}{3}\frac{1}{3})(\frac{1}{3}-\frac{1}{3})} \quad (4.2)$$

leads to a number of ω representations in terms of the modular group Γ(N) [30] [31] [32] [33] [37]. Iterates cumulate a congruent memory of universal covering of $\sum_{k=1}^{2^8} B_{\mu,k}\omega_k$ of a group $G_{2^{2^8}}$ of the order 2^{2^8} also for hyperelliptic characteristics [gh] (see Appendix 2). This is in accord to a maximal number of fermions states embedded into a congruence $2^{2^9}, 2^{2^{10}} \simeq G_5^{-1}$ [1] [38]. For iterated periods ω a chosen value $a\in\mathbb{Q}^2$ depends on values of the differential operator $\frac{\partial}{\partial u}$ on a hyperelliptic surface like periods ν_{Sh}. For a prime $p \approx 1 + \frac{g}{N}$ one has $p^N\simeq e^g$ for g≪N ≫1 with prime-counting function $\pi(p)=(1+g/N)g/N$ quadratic in g/N and $\varepsilon(g,p^N) = \left(1-1^{\frac{g}{p^N}}\right) \simeq ge^{-g}$ independent on N→∞. The concept is to search cyclotomic units within the elliptic addition theorem $\wp_u - \wp_v = \frac{\sigma_{u+v}\sigma_{u-v}}{\sigma_u^2\sigma_v^2}$. Three cases $r_\Delta=1$ (Δ≠-3,-4), $r_\Delta=2$(Δ=-4), $r_\Delta=3$(Δ=-3) and $\varepsilon(1)=1^{\frac{1}{2}}\cdot1^{\frac{1}{3}}$, $\varepsilon(2)=1^{\frac{1}{2}}$, $\varepsilon(2)=1^{\frac{1}{3}}$ yield invariant λg^2 relations with ℘- function $P(u) = \left(\varepsilon(r_\Delta)\frac{\wp(u)}{\sqrt[6]{\Delta(r_\Delta)}}\right)^{r_\Delta}$ [14] [39]. Quadratic (r=2) and hexagonal (r=3) yield $P\simeq\wp^2$ and $P\simeq\wp^3$.The elliptic analog of the correspondence η(ω) and $1\text{-}1^z$ consists in replacing $\varepsilon(g,g_\infty)$ by

$$\varepsilon(u,v) = \frac{g(u+v)g(u-v)}{g^2(v)} \quad (4.3)$$

leading to the λg^2 relation

$$P(u) - P(v) = \prod_{r=0}^{r_\Delta-1}(-\varepsilon(r))\frac{\varepsilon(u,1^{r/r_\Delta}v)}{g^2(u)} \simeq \Theta \quad (4.4)$$

which generates a cyclic field with PIB 1, θ, θ², … where $\lambda \simeq \prod_r\frac{\wp(u)-\wp(v)}{\wp(u)-\wp(v)} \simeq \prod_{r,r'}\frac{\varepsilon(u(r),v(r))}{\varepsilon(u(r'),v(r'))}$.

5. AXES OF STABILITY ANALYSIS

Stable orbiting laps are defined in terms of a stability analysis around axes e.g. $y_F=x_F$ in a Feigenbaum diagram. For a definite elliptic curve x_F, y_F should be a γ-invariant for e.g. vanishing Hessian $H(\phi_3)=0$. The 2x2 matrix of four points in universal covering space $u_\mu\text{-}u_{\mu=0}$ (μ=0,1,2,3) reads for inflection tangents $\delta\zeta'_{21}\delta\wp'_{31} =$

$\delta\zeta'_{31}\delta\wp'_{21}$. The inflection tangent dF(x_F,y_F) is a constant tangent Δx/Δy in x-y-plane of three points 1,2,3. Simplest cycles of γ(ϕ_3) around a given inflection tangent dF(x_F,y_F) yield a bi spinor ψ_s. But $\binom{4}{3} = 4$ possible elliptic curves of a hyperelliptic quartic yield 4 possible stability axes dF$_\mu$(x_F,y_F) between pairs (1,1'),(1,2'),(2,1'),(2,2') of tidal points 1,2,1',2'. For Dirac matrices γ^μ the Dirac-like equation reads $dF_{\mu i i'}\gamma^{\mu}_{ss'}\psi_{i's'} = m\delta_{ii'}\psi_{is}$. Laps and k-component dynamics X(γ∘f), Y(γ∘f) in $\mathbb{K}[\partial^{1/2},\{1^{1/m}\}]$ on Kummer and Weddle surfaces capture also different axes (x_F,y_F) as a 2^{2^k} -polar ball of fusilli-like invariant strings in space. Inflection tangent quadruples (δ_ku, du, dζ, d℘) are singular 4x4 matrices suffering SE(3) steps, unimodular aperiodic quaternary continued fractions (M(**a**)dF$_{\mu,ii'}$ →dF$_{\mu,ii'}$, 3·3 aperiodic bifurcating continued fractions and 2·2 periodic continued fractions (see Appendix 3). Cyclic bifurcation in the cyclic regulator field allow a fast composite 2^{2^k} number theoretic transform for quaternary unimodular collineations. One-periodic ν_{Sh} and Abelian quadratic fields of continued fractions are compatible whereas Hermite's problem of expressing cubic irrationalities treated by ternary steps is non- unique and open [40]. Continued fractions are capable to create a tower g(aω) →$e^{g(a\omega)}$ of modular units and Legendre modules λ→ e^λ in inflection tangents dF$_{\mu,ii}$=(δ_ku, du, dζ, d℘). Permuting first and third row in quaternary collineations M(a)dF$_{\mu,ii'}$→dF$_{\mu,ii'}$

$$M(\vec{\boldsymbol{a}}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -(N+1) \\ 0 & 1 & 0 & N \\ 0 & 0 & 1 & a_3 \end{pmatrix} \quad (5.1)$$

one gets continued fractions $M\left(a_3^{(i)}\right)$ mod N. Periods ν_{Sh} are set in context to 2·2-dimensional k collineations

$$\prod_{i=1,\dots,2k} M(\boldsymbol{a_i}) = \begin{pmatrix} 1 & \sum_{i=1,\dots,k}(-1)\cdot\prod_{j=2i,\dots,2k} M\left(a_3^{(i)}\right) \\ 0 & \prod_{i=1,\dots,2k} M\left(a_3^{(i)}\right) \end{pmatrix} \quad (5.2)$$

Tangents F$_{\mu,ii}$ of simplest cycles $F_k \leftarrow (F_{k+1}-F_{k+}) - N\delta F + a_3F_{k+4} = (\lambda_k - N)\delta F + a_3F_{k+4}$ constitute γ-invariants $\lambda = \frac{F_{k+1}-F_{k+2}}{F_{k+2}-F_{k+3}}$ giving an approximative collineation $M(a) \to (\lambda-1)g^2(u) \oplus \begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix}$

The tangent dF(x_F,y_F) is capable for a quadratic stability analysis of steps around z_{k+1}, z_k

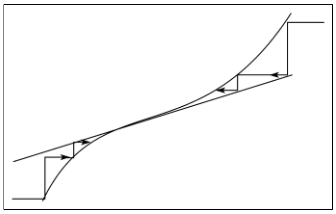

Figure 2: Iterates γ as steps z_{k+1}, z_k around an axis dF(x_F,y_F) of a given inflection tangent

Periodic collineations of continued fractions and fixed points of Hermite maps are mutually dependent. Collineations of {u- ζ}, {℘-ζ}, {u-℘} pairs as periodic continued fractions (CF) $\Pi\begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix}$ correspond to γ-fixpoints. Hermite's problem of expressing a cubic irrationality ∂ in terms of periodic ternary continued fractions is unsolved. Iterates of $\lambda = \lambda_m/m + 1/2$ converge to λ_m is ψ_s- and mass m- dependent. Period-3 addition is congruent to 3u=0. Equianharmonic elliptic curves estimate δ_F for points aω in $\lambda = 1^{1/3}$ with periods ⅓{(1,0), (1,1), (1,0), (0,1)} (ω_1, ω_2)= ⅓{(1,0), (1,-2), (1,2), (0,1)}(ω_1, ω_2) or ⅓{(1,1), (1,-1), (1,0), (0,1)}(ω_1, ω_2)= ⅓{(1,1), (1,2), (1,0), (0,1)}(ω_1, ω_2) leading to a=⅓ (∓1, ∓2) (ω_1, ω_2) in terms of half- periods ω_1 and ω_2 [41]. The cross ratio $\lambda = 1^{1/3}$ depends on 8 nontrivial combinations of ω_1 and ω_2 fractions for four parameters u_s. A Cantor set results from a_{k+1}=T_3(a_k) on I_0=[0,½] and I_1=[½,1] and initial condition a_k= ζ(2,3,k) from tent map T_c [42].

$$T_c(x) = cx : x \in I_0, c(1-x) : x \in I_1 \quad (5.3)$$

and Cantor string zeta function

$$\zeta(2,3,l) = \sum_{n\in\mathbb{N}} 2^n\, 3^{-(n+1)l} \quad (5.4)$$

Congruences modulo 2^{2^k} yield a circulant matrix as a polynomial with cyclotomic coefficients a(l) in the series $\sum_l \zeta_{CS}(l)\,a(l) = H_{CS}\sum_{l'=H_{CS}l}\frac{a(l'/H_{CS})}{2^{l'}-2}$. This is like a time-thermal rate $R_{kk'}$ with occupation number $\frac{1}{2^l-2}$ of states on a circle where coefficients a(l'/H_{CS}) consist of a bilinear expansion of ψ_s.

6. INTERVAL DIFFEOMORPHISM

The hyperelliptic Lebesgue measure μ(dx) for a quartic polynomial is [11]

$$\mu_h(dx) = \frac{dx}{2K\sqrt{\Phi_4(x)}}. \quad (6.1)$$

compares to that for the logistic map

$$\mu_l(dx) = \frac{dx}{\pi\sqrt{x(1-x)}} \quad (6.2)$$

A hyperelliptic measure with deg Φ(x)= 4 is directly related to Jacobi theta functions ϑ(u, 𝕃) for quarter periods K and K'. The Lebesgue measure μ_h and μ_l differ by K/π with quarter period K. A map on unit interval x∈I=[0,1] as exactly solvable chaos is [10].

$$\int_0^x \mu_h(dx) \simeq \int \frac{du}{K} \quad (6.3)$$

A general Riemann surface with cardiods is reducible to flat space. Accordingly, the Euler- Poincaré characteristic vanishes $\chi(\mathbb{L}_w) = \frac{1}{w!}\int_{\delta G}\wedge^w c_1(\mathbb{L}_w) = 0$ with first Chern class $c_1(\mathbb{L}_w) = \sum_{i=1}^{w} \mathrm{d}_i\, du_i \wedge d\bar{u}_i$ leading to the alternating w-form of a line bundle of type $\{d_1,\ldots,d_w\}$ of a lattice with periods ω_w in w-dimensions [43].

$$\chi(\mathbb{L}_w) = (-1)^w d_1\ldots d_w du_1 \wedge\ldots\wedge du_w \wedge d\bar{u}_1 \wedge\ldots\wedge d\bar{u}_w \to 0 \quad (6.4).$$

The general Riemann surface is a bizarre ensemble of plants, trees and coast lines of infinite length but of vanishing volume. Accordingly, one hast to resort to Hausdorff measure and Hausdorff dimension.

7. INVARIANT RELATIONS FOR LEGENDRE MODULES

The claim of convergence from λ to δ_F for infinite k-components yields a complex charged current for w<4 $\lambda=\lambda_m/m+½$ with $\lambda_m = \bar{\Psi}_s\lambda_{mss'}\Psi_{s'}$

$$\phi_N = 1 - \mu r_N(f(\omega)) \quad (7.1)$$

which justifies iterated Hermite maps $\gamma(\phi_N)$ for certain parameter μ which have been calculated in [44]. ϕ_N changes with $(2\pi\lambda^2-\alpha_f^{-1})\to 0$ and a real current for w> 3 where λ→0 due to a map γ(Φ)∘f(ω). Fluctuating line bundles $\mathbb{L}_w$ for quadratic Hermite substitutions $\gamma(\phi_3)$ in fields 𝕂[∂] and 𝕂[$\partial^{½}$] change f(ω) by the trend $4\boldsymbol{q}^{2^{k-1}} = \lambda_k$ for doubling ω→2ω which is $f^{24}\simeq 2^4/\lambda_k\to\infty$. This explains complex values of the Legendre module λ near Nm(f(ω))=2 for h_Δ=1. Invariant equations $j^{⅓}(\omega) = \gamma_2(\omega) = \frac{f^{24}(\omega)-2^4}{f^8(\omega)}$ recover cubic equations $r_N^3-\gamma_2 r_N-2^4=0$ for certain degrees $r_N=f^N(\omega)$ of f(ω) e.g. for N=3,8,24. Then a cubic equation exists for

$$\delta_F(N) = \frac{\mu_K-\mu_{K+1}}{\mu_{K+1}-\mu_{K+2}} \quad \text{and} \quad \alpha_F(N)=-z_k/z_{2k}.$$

Computed values $\delta_F(N)$ and $\alpha_F(N)$ from work [44], are compared to the product $\delta_F\alpha_F^2$

Table 1: Relation $\delta_F\alpha_F^2$ for transformation degrees ={2-8,10} from [44]

N	δ_F	α_F	$\delta_F\alpha_F^2$
2	4.67	2(2.5)	29.19
3	6.08	1.93	22,64
4	7.29	1.69	20,82
5	8.35	1.56	20,32
6	9.3	1.47	20,1
7	10.2	1.41	20,28
8	10.95	1.36	20,25
10	12.37	1.29	20,58

The invariant product $\delta_F\alpha_F^2$ is set in context to an invariant λg^2 as a CM property [45, 39]. Being invariant, it stands for an interaction-independent mean vacuum density in the universe [1]. A regular map as a quadratic test of a cubic equation is possible for nine h_Δ=1-fields. Then appearing periods ν_{Sh} enable nine elementary fermions. Period-doubling coefficients c_k in $z\leftarrow z^2+c$ and $\delta_F=1-\delta_k\ln\delta_k c_k$ correspond to cubic roots $e_k\in$ [∂], 𝕂[$\partial^{½}$] of a sextic polynomial on K(X) and W(Y). Period-doubling relates to doubly-periodic addition. Treating cubic roots as addition steps of quadruples roots at k→k+1,k+2 are $\gamma(\Phi)\circ x_k\to x_{k+1}$, $\gamma(\Phi)\circ x_{k+1}\to x_{k+2}$. The equianharmonic value $\exp(\omega_e)\simeq 4{,}61$ with omega-2 constant ω_e is close to $\delta_F \cong 4{,}669$ where

$$\omega_e = \frac{\Gamma^3\left(\frac{1}{3}\right)}{4\pi} \cong 1.5299 \tag{7.2}$$

A quartic polynomial for adding points u,v,u±v for symbolic invariant $A_1^3 = A_2^3$ corresponds to an equianharmonic conjugate logistic map. The expression of λ in terms of theta constants is

$$\lambda = -\frac{\vartheta^4[10]}{\vartheta^4[01]} \tag{7.3}$$

Being $1^{1/3}$ for equianharmonic curves λ can be rewritten in terms of modular units g(aω). For period- doubling $\{a\}=\{a_{10},a_{01},a_{11}\}=½\{(1,0),(0,1),(1,1)\}$ it is a quadratic transformation of ϑ(u). Then $\lambda[g(a\omega)]$ with $a=(a_1,a_2)\in\mathbb{Q}^2$ yields

$$\lambda[g(a\omega)] \simeq e^{4(a_{10}\tilde{\eta}[\zeta]a_{10}\omega - a_{01}\tilde{\eta}[\zeta]a_{01}\omega)} \simeq e^{\tilde{\eta}[\zeta]\omega_e} \tag{7.4}$$

which yields 4,61 for $\tilde{\eta}[\zeta]\simeq 1$ Powers of the Dedekind eta function $\eta^{-2N}(\omega)=\Pi_{(a)}\tilde{k}_a(\omega)$ are Γ(N) modular invariant [33]. The modular group Γ(N) is set in context to a one-periodic $\lambda_k[\lambda_{k'}]$ behaviour related by periods ν_{Sh}. Computationally one has $\mu_k = c_0 + c_1 e^{kc_2}$ for the logistic map c_2=1.530 which is roughly $c_2\simeq\omega_e$ [46]. The one-dimensional map induces world-point dynamics $M(\mathbf{a})X(f)= e^{S(\mathbf{a})}X(f)\to X(f)$ in terms of SE(3) steps in $\mathbb{C}^w$ which ranges from diffusion-like to a robot-like discrete potential flow. Zeros of X(f)jX(f) are linked to the fundamental theorem of hyperelliptic addition [28]. The poles of the amplitude $\mathbb{A}(s)$ function in $\exp(|\gamma|^2(ch)^4D_h^2)\simeq\exp(\int ds\mathbb{A}(s))$ depend on simplest cycles quadruples q≃s. The bi-spinor ψ_s is defined as an eigenstate or stationary state of the shift- operator quadruple $[1,\delta_k,\delta_k\delta_k,\delta_k\delta_k\delta_k]$. Whereas equilibrium is defined as elastic matter which minimizes action $\mathscr{L}$ by $\delta_k\mathscr{L}=0$ a bi spinor describes eternal non-equilibrium states where matter is generated. The fundamental theorem for a quadruple of steps q describes a number of δ_k correlations $\sum_q \delta_k s_+(u,v)s_-(u,v) = 0$ as well orthogonal in characteristics [gh] substituted products

$$\sum_q \delta_k(\alpha_{[gh]}(u+v)\tilde{\Gamma}\beta_{[gh]}(u-v))s_+(u,v)s_-(u,v) = \sum_q \delta_k(X_{[gh]}(u)jX_{[gh]}(v)) = 0. \tag{7.5}$$

Throughout equality signs are understood as the assignment operator ≔. 16 hyperelliptic characteristics$[gh]$, g,h= ½{0,1} as well ⅓ characteristics as n=3 period transforms of simplest cycles raise the fundamental hyperelliptic theorem to a quadratic equation of mass ratios of elementary particles [47]. For δ_k shifted states $\psi_s\simeq\{f_q\}\simeq\{f_k,f_{k+1},f_{k+2},f_{k+3}\}$ result from SE (3) joints $M(\mathbf{a})X(f)= e^{S(\mathbf{A},\mathbf{a})}X(f)$ where $S_{ii'\mu}(\mathbf{A},\mathbf{a})$ projects a quadratic term onto (f_k,f_{k+1},f_{k+2}) around inflection tangent axis $dF_{\mu ii'}$(μ=1,2,3,4, s=1,2,3,4)

$$dF_{\mu ii'}\gamma^{\mu}_{ss'}\psi_{i's'} = m\delta_{ii'}\psi_{is}\; S(\boldsymbol{A},\boldsymbol{a})_{ii',\mu}\gamma^{\mu}_{ss'}\psi_{i's'} = m\delta_{ii'}\psi_{i's} \tag{7.6}$$

Like a quadruple four rotational SE(3) steps are equivalent to one collineation M(**a**) (see Appendix 4).

8. RENORMALIZATION

Spacetime is fluctuating doubly-periodic lattices under dense lattices of cyclotomic units where a quotient of elliptic units $\prod_{N,N'}\frac{\varepsilon(u_N,v_N)g^2(u_{N'})}{\varepsilon(u_{N'},v_{N'})g^2(u_N)}$ as a product over modular groups Γ(N) tends to a invariant $\lambda(C)g^2 \simeq \delta_F\alpha_F^2$ with composite generator $g^2 = \frac{\prod_N g^2(u_{N'})}{\prod_{N'} g^2(u_N)}$. A Poncelet polygon contour $C[\prod\delta\wp]$ in space has $\lambda(C[\prod\delta\wp]) =\prod\lambda(C[\delta\wp])$ [48]. Next to scaling $z_k=-\alpha_F z_{2k}$ which leads to the renormalized equation (8.8) a linear complex relation $\sum_{(q)}c_q z_q=0$ exists for quadruples q. For a simplest cycle it reduces to $c_k z_k+c_{k+1}z_{k+1}+c_{k+2}z_{k+2}=0$ with Feynman diagram series of $z_{k+2}[z_k]$. In the limit of infinite cyclotomic degrees N→∞ one recovers a dimensionless vacuum energy density $\rho_{vac} \simeq \frac{H_w^2}{8\pi G_w}$ with a Hubble-like parameter $H_w=\delta_k \ln g(a\omega)$. The k-component coupling constant G_w scales with generator as 2^{2^k}. For k=3 one gets first periods ν_{Sh} as a fraction $6\cdot 10^4$ of ρ_{vac} which is equivalent to the microwave background. For k=9 one gets the cosmological constant problem [1]. An expected congruence at k=9 yields the experimental value of ρ_{vac}. Fixed points of maps $\gamma(\phi)\circ\ldots\circ\gamma(\phi)$ are generators which replace a Dedekind eta function η(z) in units ε(u,v) by roots of unity 1^z. With $\varepsilon(u,v)\simeq g^2(u_N)(\wp(u_N)-\wp(v_N))$ one gets [45].

$$\prod_{N,N'}\frac{\varepsilon(u_N,v_N)}{\varepsilon(u_{N'},v_{N'})} \simeq \prod_{N,N'}\frac{g^2(u_N)}{g^2(u_{N'})}\frac{\left(\wp(u_N)-\wp(v_N)\right)}{\left(\wp(u_{N'})-\wp(v_{N'})\right)} \simeq g^2\,\lambda[\delta\wp] \tag{8.1}$$

where $\wp(u_N=\omega)=e_i$. The γ-map is a shift of the difference $\delta_k e_i$. For k^{th} shifts $\delta\wp^{\circ k}=\gamma^{\circ k}(\phi)\delta\wp=\gamma(\phi)\circ\ldots\circ\gamma(\phi)\delta\wp$ the Legendre module reads $\lambda=\lambda(C[\delta\wp^{\circ k}])\simeq g^2\lambda(C[\delta\wp])$ on contour $C[\delta\wp]$ of a Poncelet polygon. The g^2 factor creates a topological entropy h_t. By means of the symbolic map the hyperelliptic addition is linearized in the complex variable $z\simeq f(\omega)$. The exact quadratic relation between ϑ^2 and $\wp$ or $\delta\wp$ where the elliptic $\wp$-function is extracted from the hyperelliptic $X(f)=(\wp_{\pm\pm},1)$ by a rotation of $(1,-f,f^2)$ allows to set $z_k=\vartheta_1\simeq g(a\omega)$ where $z_{k+1}\simeq\wp$. Values $z_k\simeq f(\sqrt{\Delta_k})\simeq g(a_k\omega_k)$ on universal covering $u_k=a_k\omega_k$ relate to simplest cycles most likely to a four-point contour around an inflection tangent. Already the Legendre modular function as a cross ratio depends on a 13-segment contour C. Central to FZU and unified fields is a linear relation $c_q z_q=0$ of this strongly- nonlinear problem. For linear expanded modular units $g(a_q\omega_q)\simeq f(\sqrt{\Delta_q})$ the product of $q=\binom{4}{3}=4$ factors of simplest cycles is called plaquette $U_\square$

$$U_\square=\prod_q e^{c_4}e^{S(a_q)}g^{n_q}(a_q\omega_q) \qquad (8.2)$$

where for cross-ratio powers $n_q=(1,1,-2,-2)$ describe a 8- segment contour of $\lambda(C)$ which reduces for simplest cycles of four points $z_k=z_{k+3}$ or $z_{k+1}<z_k<z_{k+2}$ or $z_{k+2}<z_k<z_{k+1}$ of interval $I(z,z')$ where $z_k=\wp(u_k)$ which is equivalent to three points z_k, z_{k+1}, z_{k+2} on different sites of inflection tangents. The 4·4 matrix $M(\boldsymbol{a})=e^{S(\mathbf{A},\boldsymbol{a})}$ in the condition for stability (inflection tangent) $\det dF_{ii'}\Pi M(\boldsymbol{a})$ contains modular units $g(a)\simeq\frac{\prod_a g(a)}{\prod_{a'}g(a')}$ with $dF_{\mu ii'}=(\delta u, du, d\zeta, d\wp)$ collineations $dF_{\mu ii'}\leftarrow dF_{\mu ii'}e^{c_4}e^{S(\mathbf{A},\boldsymbol{a})}\leftarrow dF_{\mu ii'}M(\boldsymbol{a}_k)$ (Appendix 4). q=4-components $g(u_q)\leftarrow\sigma(u_{q'})M_{kk'}(\boldsymbol{a})\leftarrow e^{\int dv\zeta(v)}e^{c_4}e^{S(\boldsymbol{a})}$ get non- commutative in

$$\det dF_{ii'}\simeq\frac{\prod_a g(a)M(\boldsymbol{a})}{\prod_{a'}g(a')M(\boldsymbol{a'})}=\frac{\prod_k g(a_k)e^{t_4}e^{S(a_k)}}{\prod_k g(a'_k)e^{t_4}e^{S(a'_k)}} \qquad (8.3)$$

with cyclic matrix t_4 for an infinite number of ν_{Sh}. Steps $\gamma(\phi_3)$ as additions with nearly invariant $\lambda\Pi g^2$ operate on a Riemann surface with zero self-intersection number and zero Euler-Poincare characteristic $\chi(\mathbb{L}_w)$. This w-dimensional generalized Riemann surface formerly of genus w=1, …,5 consists of non-intersecting lines. Linearized world-points $X(f)\simeq\psi_s$ yield Kirchhoff equations for discrete $\mathbf{X}_k$ lines of a potential flow of an ideal fluid. Having doubly-periodic cycles of entropy h_t and temperature T the liquid is a universal superfluid state. Periods ω_k split in a sextic number field $[\partial^{½}]$. Minima of the Legendre modular function $\lambda=1-\delta_k h_t$ are shifts of topological entropy h_t. For cubic roots $(e_1, e_2, e_3)=(\wp_k, \wp_{k+1}, \wp_{k+2})$ the Legendre module reads

$$\lambda\simeq 1+\langle\frac{\wp_{k+1}-\wp_k-\wp_{k+2}+\wp_{k+1}}{\wp_{k+2}-\wp_{k+1}}\rangle\simeq 1+\delta_k\ln\delta_k\wp \qquad (8.4)$$

where

$$\delta e_{k+N}=\gamma^{\circ N}(\phi_3)\circ\delta\wp(\omega_k)=\gamma^{\circ N}(\phi_3)\circ\delta e_k$$

are related to topological entropy

$$h_t(\gamma^{\circ N}(\phi_3)=\sup_{d\mu(\gamma(\phi_3))}h_\mu(\gamma^{\circ N}(\phi_3))$$

as a supremum over Lebesgue measure (6.1) of metrical entropy h_μ

$$\lambda_L(\gamma^{\circ N}(\phi_3))=h_\mu(\gamma^{\circ N}(\phi_3))=\frac{1}{N}\sum_{k=0}^{N-1}\ln\delta_k\wp(\omega_k) \qquad (8.5)$$

which is related to the Lyapunov exponent λ_L. A chain rule in differential form is valid also over discrete k-shifts. Writing a finite shift $\delta_k\wp_k$ as $\delta_k\wp_k=\delta_k(\gamma^{\circ N}(\phi_3)\circ\wp_{k-N})$ and $\delta\wp^{\circ N}=\gamma^{\circ N}(\phi_3)\circ\delta\wp$ one gets [49]

$h_t\left(\gamma^{\circ g^k}(\phi)\right)=g^k h_t(\gamma_\Phi)$ and $h_t(\gamma_\Phi)=\frac{1}{N}\sum_{k=0}^{N-1}\ln\delta_k\wp(\omega_k)=\frac{1}{N}\sum_{k=0}^{N-1}\ln\delta_k e$ which confirms

$$\lambda\left(C[\gamma^{\circ g^k}(\phi_3)]\right)=1-\delta_k h_t\left(\gamma^{\circ g^k}(\phi_3)\right) \qquad (8.6)$$

Topological entropy h_t is optimized for a tower of $g(a\omega)$ $\gamma^{\circ N}(\phi_3)\circ\delta\wp\simeq g(a_1\omega)^{g(a_2\omega)^{\cdot^{\cdot^{N}}}}$. The k-shift of the topological entropy h_t classifies an information current as density of generators and modular units. Optimized values of $\lambda(h_t)$ and $G_w(h_t)$ belong to the second exponential level of generators: A one-dimensional supremum of $N=g_k^{g_{k+1}}$ is reached at constant $(g_k+g_{k+1})(g_k-g_{k+1})$ where $g_k=g(a_k\omega)$. The k-process with $N=e^{g_k^2\ln g_k}$ offers to attach the fine structure constant [8]. An optimal quadratic map consists in two steps. A quadratic map of degree z^{2^k} embedded into bases $z_1\ldots z_{2^k}$ in 2^k complex planes is projectable onto w independent Riemann spheres. In distinction, an optimal map searches a minimum of $z_{k+1}+\log_{g_k}z_k$ which yields a power tower $z_k\rightarrow g_k^{z_{k+1}}$. Both maps are compatible for a 2-power bases with generator $g_k=2$ or $g_k=3$ if Fermat number congruences exits. Either g^k powers of g=2 or g^k powers of g=3 are roots of unity for the first four prime Fermat number F_t. Congruences are expected above |M| which is the number of elements of a Monster group M with respect to the elliptic invariant $j(\omega)$. Subsequent maps $\gamma^{\circ N}(\phi_3)$ change the integral base $(1,\partial,\partial^2)$ of the pure cubic field $\mathbb{K}[\partial]$ or $\mathbb{K}[\partial^{½}]$ with discriminant $\Delta_{k+1}=\phi_3^2\Delta_k$ [29]. Then the invariant polynomial ϕ_3 represents the index form $I(\phi_3)$ where $\det\gamma(\phi_3(t))=\phi_3=1$ covers possible integral bases [50]. Feigenbaum renormalization consists in the assumption of scaling $z_k=-\alpha_F z_{2k}$ of the doubling component where $z_k=z^{\circ k}$ and $z^{\circ k}=-\alpha_F z^{\circ 2k}$ yields $-\alpha_F g(g(-z/\alpha_F))=g(z)$

[51]. This conjecture is equivalent to a a cyclic shift of $z_{k+1}=gz_k$ and $z_{2k}=g^k z_k$ for a definite generator g as a root of unity. This paper relates the second Feigenbaum constant α_F to a generator as a universal congruence. Periods ν_{Sh} appear if $[1,\delta_k,\delta_k\delta_k,\delta_k\delta_k\delta_k]$.contains a linear relation.

$$c_k z_k + c_{k+1} z_{k+1} + c_{k+2} z_{k+2} = 0 \quad (8.7)$$

between $1,\delta_k$ and $\delta_k\delta_k$. Accordingly, a linear relation between three functionals is irreducible, e.g. for the Dyson equation of G,G_0,Σ of Greens function G and mass operator Σ or for the Bethe-Salpeter equation of P,P_0 and Ξ for polarization function P and vertex part Ξ where eq. (1.2) indicates the importance of γ for quantum statistics

$$\gamma^{(ren)} = \gamma + \gamma \circ \Gamma^{(ren)} \circ \gamma^{(ren)} \quad (8.8)$$

Consistent with spinor definition by $\delta e=(\delta x)^2$ a renormalized invariant $f(\omega)\rightarrow g(z)$ obeys quartic polynomial.

9. CURRENT DENSITY AND VACUUM ENERGY DENSITY

The proof that zeros z_{nt} describe charge and mass is simple. For $\xi(z_{nt})=0$, $\lambda=z_{nt}$ and $\Delta_h\xi(z)=0$ with $\Delta_h=y^2\Delta_{xy}=$ $\mathrm{Im}\lambda^2\Delta_{xy}$ one gets the screened Poisson equation $\Delta_{xy}\xi(z)+\mu_1\xi(z)=\mu_2(\mathrm{Im}\lambda-m_n)$ where the hyperbolic Laplacian defines m_n slices $\mathrm{Im}z_{nt}=\mathrm{Im}\lambda=m_n$. Subsequent $\gamma(\phi_3)$- maps yield $\xi(z)=0$ $=L(z,\chi)$ $\xi(z)=0$ and

$$\Delta_{xy}(L(z,\chi)\xi(z))+\mu_s L(z,\chi)\xi(z)=\mu_c(\mathrm{Im}\lambda-m_n) \quad (9.1)$$

A phase transition where the susceptibility χ vanishes yields poles in the amplitude $\mathbb{A}$ giving $\xi(z_{nt})=0$. where the exponent can be

$$\sum_{k,w}\int_C d\nu\zeta(\nu) = \sum_{k,w}\int_C dt\varepsilon\bar{\varepsilon}j(\nu)\chi(\nu,\zeta,k)\frac{\partial\zeta(\nu)}{\partial k} \quad (9.2)$$

Lagrange parameter μ_1 describe screening and μ_2 charges for a continuum of maps γz. But a continuum of γ-maps is a regular chaotic map. For proportional shifts $\delta_k,\simeq\delta_k\delta_k$ of periods one gets $\delta_F\simeq 1-\delta_k \ln\delta_k\omega$ in case of CM where $\omega\simeq f^2(\omega)$. Accordingly, $\omega \simeq\delta_k\omega$ leading to a Gaussian distribution with standard deviation of δ_F. Therefore, Hieb's conjecture is related to the definition of charges. The invariant relation for Legendre module λg^2 is viewed as a mean density of additions on elliptic curves as a vacuum energy density for various densities of $1/R_\Delta$ algebraic units. In the limit $1/R_\Delta\rightarrow\infty$, i.e. $R_\Delta\rightarrow 0$ the number of roots on circles is sufficient in order to define a point. Fermat congruences define w imaginary units for rotations. Faster inner shells end up in in a resting bowl which defines a point. The module $\lambda_\mu=\lambda_{\mu m}/m+½$ is a Dirac-like current density where $\lambda\lambda'=2^4/f^{24}(\omega)$ or $(\lambda_{\mu m}/m)^2-¼=2^4/f^{24}$ and $\lambda_{\mu m} = \overline{\Psi}_q\, \lambda_{\mu m q q'}\Psi_{q'}$ and a linear relation $c_q z_q=0$ transmits to $c_q\lambda_q=0$ for simplest cycles quadruples. Poncelet involution $i(u)\simeq=\alpha u+\beta$ with $\alpha=\pm1,\pm i$ is transmitted to an involution $i(\lambda)$ on inscribed and circumscribed contour in $\lambda\big(C(ijkl)\big) = \exp\left(\int_{C(k)-C(k')} d\nu\zeta(\nu)\right) = \prod_{ijkl}(ijkl)$. The product over cross-ratios (ijkl) is independent on the path between contour endpoints and is capable to describe a potential flow. Indices are $i\in\mathbb{P}^1$, $\mathbb{P}^3$, $\mu,\nu\in\mathbb{R}^{3,1}$, s=1,2,3,4. A stability to iterate near four tangents $dF_{\mu,ii'}(x_F,y_F)$ around four points $K(X_\mu)$ is supposed. A quadruple of steps q=k,k+1,k+2,k+3 in $\delta_q\prod\delta_{l_\omega}$ is supposed to be stationary frozen for all laps l_ω. Then shifts δ_k are independent fields in S-matrix $\lambda(\delta C) \simeq e^{\int_0^1 dtS(t)} = e^{\sum_{k,w}\delta_k^2\int_0^1 dt\zeta(t\nu,\omega_k)}$ with susceptibility

$$\chi\big(\nu,\zeta(\nu,\omega_k)\big) = 1 + \frac{\delta_k \ln\delta_k \nu}{\delta_k \ln\zeta(\nu,\omega_k)} = 1 + \frac{\lambda_\nu}{\delta_k\ln\zeta(\nu,\omega_k)}$$

$$S(t) = \sum_{k,qq'}(\delta_k t\nu)R_{k,qq'}\zeta\big(l,m,R_{k,qq'}\big)\big(\delta_k\zeta(t\nu,\omega_k)\big)$$

with ε and $\bar{\varepsilon}$ in $N[\sqrt{\Delta}]$. Like in quantum statistics one gets a mutual current density-vector potential and vector potential- current density dependence which allows an action functional representation of λ

$$\lambda = \prod_i \exp\left(\int_{C_i} d\nu\zeta\big(u(\nu,\omega_k)\big) + \mu_1 \mathrm{det}dF_{\mu ii'} + \mu_2(1-U_\square) + \mu_3\chi(\mathbb{L}_w)\right) \quad (9.3).$$

Four stability axes in the ψ_s - symmetrized determinant (9.3) $\mu_1 \mathrm{det}dF_{\mu ii'}=0$ is equivalent to a scattering amplitude of four potentials and 8 bi spinor states. A plaquette $U_\square$ term μ_2 contains doubly-periodic ω with one-periodic ν_{Sh}. It is claimed that Sharkovskii periods are locally roots of unity $\exp(i\nu_{Sh}z)$ which holds for four stability axes of four-dimensional superlattices $\exp(i\nu_\mu z_\mu)$ of lattices of algebraic units. A vanishing self-intersection number by Lagrange parameter μ_3 can possibly be arranged as a potential flow in the absence of radiation and dissipation which is one condition of superfluidity. Next a phase transition in the exact $K(\lambda)$-equation [14]

$$\frac{d}{d\lambda}\lambda\lambda'\frac{dK}{d\lambda} = \lambda\lambda'\frac{d^2K}{d\lambda^2} + (1-2\lambda)\frac{dK}{d\lambda} = \frac{K}{4} + \mu_1\phi_3(K,K') \quad (9.4)$$

by CM is discussed which is a cubic term $\phi_3(K,K')$ of quarter periods. CM implies a fractal behavior where quarter periods $K, K'\rightarrow\vartheta_1$ get itself theta variables which occurs in a superconducting phase transition [24]. Subsequent self-similar iterates K, $K'\rightarrow\vartheta_1$ of tori within tori by means of a map $\gamma(\phi_3)$ on $K(X(f))$ and $W(Y(f))$ satisfy the $\gamma(\phi_3)$- invariant hyperelliptic addition $X(f_k)jX(f_k)=$ $s_{+[gh]}(u,v)s_{-[gh]}(u,v)$

where $X(f_k)=d^2\vartheta(u+v)$. This is because $\gamma(\phi_3)$ forces CM. Zeros of bifurcated hyperelliptic theta function $\vartheta(u\pm v)$ are set in context to non-trivial zeros z_{nt} of the Riemann zeta function $\zeta(z)$. Because λ relates to a current $d\lambda$ in (9.3) is also related to coordinates. A phase transition to cylindrical coordinates around a center $\lambda\simeq z_{nt}$ yields a winding number $N_\hbar=\Pi F_t$ between w-shells of $\hbar \simeq N_\hbar^2 \simeq 1.84\cdot 10^{19}$ being the definition of Planck constant $\hbar$. Surface tension σ_s is related to topological entropy $h_t\simeq \ln 2$. The winding concerns laps l_ω of orbits in $\delta_k \prod \delta_{l_\omega}$ of a non-turbulent potential flow near fixed points of $\gamma(\phi_3)$. Formally, a Heuman lambda function $\Lambda^*(u,\sqrt{\lambda'})=\partial/\partial u\vartheta_2(iu,\sqrt{\lambda})=\pi u/KK'+\partial/\partial u\vartheta_4(u,\sqrt{\lambda})$ exhibits a plateau for the classical value $K\to 0$ where the discriminant vanishes for Jacobi theta functions $\vartheta_i(u,\sqrt{\lambda})$. Self-similar steps $K(\lambda)$, $K'(\lambda)\to\vartheta_1$ are self-reproducing cubic terms of (9.4) modulo a quartic $\phi_4(K,K')$ where a prime denotes quarter periods K,K' or a λ- derivative where [24].

$$K_w'' - \sigma_s^2(1-B^2\lambda^2)K_w = 0 \qquad (9.5)$$

Derivatives $\frac{d}{d\lambda^{-1}(C^w)} = \frac{iN_\hbar}{\sigma_s}\frac{d}{d\lambda^{-1}(C^{w+1})} + \Lambda(u,\mathbb{L}_{w+1})$ reproduce a Schrödinger equation. Inserting the second derivative K'' of (9.5) into (9.4) one gets

$$(1-2\lambda)K' + \sigma_s^2\lambda\lambda'(1-B^2\lambda^2)K - \mu_1\lambda\lambda'\phi_3(K,K') - ¼K = 0 \qquad (9.6)$$

Polynomials of (9.6) of up to degree 11 in λ are reduced by a quartic polynomial of the Jacobi theta function and by $\phi_4(\lambda)$. The iterated equation displays modulo $\phi_4(K,K')$ a thermodynamic potential $\phi_3(K,K')\bmod\phi_4(K,K')\simeq j_\mu A_\mu\simeq\Omega_k$ and w gyro twist shells $\Omega_w=\int d\sigma_5 j_w A_w$

10. CONCLUSION

Optimal variables of the quadratic map are superpositions of z and lnz, are cardioids and superposed bulbs which depend on three Lagrange parameters, thus explaining the Huygens-Fresnel principle. Subsequent optimization yields a tower $g_1^{\cdots g_l}$ of generators which competes with periods ν_{Sh} which requires a maximal general Riemann surface $\mathbb{C}^5$ to search for pseudo-congruences. Analogously, periods ν_{Sh} are treated as general complex number in a Kummer theory which only locally allow roots of unity $\exp(i\nu_{Sh}z)$ [20]. An optimal regulator R_Δ (2.1) in the L- function consists in cyclotomic units in a number field extension of a cubic field. Modular units $g(a\omega,\omega)$ combined with Poncelet involution $i(u)=\alpha u+$ with $\alpha^2=\pm1$ by multiplication αu with $\alpha=\pm1,\pm i$ survive period fluctuations on universal covering $u=a\omega$. Poncelet involution is an addition step and a Poncelet polygon consists of simplest triangles u,v,u±v. A fractal set of quarter periods $K\simeq\vartheta_1$ is implemented in every elliptic curve. It is equivalent to a phase transition for a cubic/quartic $K(\lambda)$. The normalization $K\simeq\delta_F$ stands for a charge or flux quantization in quantum statistics in [24]. Forced CM by eq. (1.2) is equivalent to addition as an exactly iterated chaos. The second Feigenbaum constant α_F is equivalent to a generator $g(a\omega)$ as a mean cyclic shift. The relation $\delta_F\alpha_F^2$ =invariant relates to a stationary addition process on fluctuating elliptic curves where $\lambda(C_N)g^2(a\omega)$ =invariant holds as an invariant dimensionless energy density. Iterated curvature forces complex multiplication of elliptic curves. A tower of modular units $g(a\omega)$ shapes gyro twist-like hypersurfaces ($w\leq5$) in w-dimensional complex space $\mathbb{C}^w$. Cycles on $\mathbb{C}^w$ yield invariances of the product of both Feigenbaum constants in agreement with computation. The linear map $\gamma(\phi_3)$ in a bicubic number field predicts a bicubic norm of a bi spinor field which is a quadratic norm multiplied by a coupling constant valid for all interactions.

Appendix 1: Period-Doubling as Doubly-Periodic CM Points

A general quadratic map F(t,z) in Section 3 is 4-parameter-dependent whereas a Mandelbrot map $z_{\kappa+1}\leftarrow z_\kappa^2+c_\kappa$ on period-doubling c_κ depends on a single root $c_\kappa=e_i=\wp(\omega_i)$. Optimal units in L-function (1.1) and regulator (2.1) behave as a tower $g_1^{\cdots g_l}\simeq g_1^{g_2^{g_3}}$ where already a base 2 exponentiation is ultrafast. Only for a square number $g_2^{g_3}$ series of vanishing Gaussian periods exists, i.e. for k=2,4,6,8,10 in 2^{2^k} which are related to interactions. In distinction, periods ν_{Sh} appear as slow 2^k components related to particles. Local curvature z of a self-similar bifurcating spacetime tree is encapsulated by L-functions as a Lovelock-like Lagrangian of elastic spacetime in nontrivial zeros of the zeta function. Legendre module $\lambda(C)=\frac{e_k-e_{k+1}}{e_{k+1}-e_{k+2}}$ and Feigenbaum constant $\delta_F=\frac{c_\kappa-c_{\kappa+1}}{c_{\kappa+1}-c_{\kappa+2}}$ are formally equivalent comparing period- doubling Mandelbrot parameter c_κ with cubic root e_k. Elliptic curves cycles obey a zoom property because the square of the Dedekind eta function $\eta(\omega)\backsim\sqrt{\det\gamma}\left(\frac{\Delta(\gamma\tau)}{\Delta(\tau)}\right)^{1/24}$ in $\omega_{k+1}\leftarrow\Pi\eta^2(\omega_k)\simeq f^2(\omega_k)$ is modular and $\gamma(\phi_3)$ is scale-invariant [52]. Then units $\varepsilon(\omega_k)$ in $\mathbb{N}[\sqrt{\Delta}]\leftarrow\mathbb{K}[\partial]\mathbb{K}`[\partial]\mathbb{K}``[\partial]$ of simplest cycles (bi spinor) have a four-component complex bicubic norm $\mathbb{K}[\partial^{1/2}]$ for power integral bases of periods $\omega\simeq(f(\omega), f'(\omega), f''(\omega))$ in $\mathbb{K}[\partial]$ or $\mathbb{K}[\partial^{1/2}]$ of four δ_k iterates. [53] [54]. For Hermite maps $\gamma(\phi_3)\circ f(\omega)$ the index form is $I(\phi_3)=\phi_3^2$. Equivalent number fields $\mathbb{K}[\partial]$ or $\mathbb{K}[\partial^{-1}]$ demand $\phi_3(f(\omega))=1$ for index form [50]. The aim of the paper is to explain a bi spinor state as a four-component state of simplest cycles of quadratic maps within a sextic number field $\mathbb{K}[\partial^{1/2}]$ with a cubic subfield $\mathbb{K}[\partial]$ of $\sqrt{K(X)}$. Monogenic fields exist for index forms $I(\phi)=\mp1$ [50]. Periods ω result from a stationary folding process $\omega\leftarrow\delta_k\omega=(1-\lambda)\delta_k^2\omega$. The shift-operator $\delta_k\omega=\omega_{k+1}-\omega_k$

is nonlinear. The Lattés maps as u- doublings yield a quartic with higher complexity

$$\wp(2u)=\frac{(\wp_u^2+¼g_2)^2+2g_3\wp_u}{4\wp_u^3-g_2\wp_u-g_3} \quad (A1.1)$$

even if ℘(2u) [℘(u)] is rational for $g_2=4$, $g_3=0$ or $g_2=0$, $g_3=4$ [26] [18]. For $h_\Delta = 1$ invariants $\gamma_2=\sqrt[3]{j}=\frac{f^{24}(\omega)-2^4}{f^8(\omega)}\in\mathbb{N}$ are reducible to a cubic polynomial [52]

$$\phi_3\big(f(\sqrt{\Delta})\big)=f^3\big(\sqrt{\Delta}\big)+2Ef^2\big(\sqrt{\Delta}\big)+2Ff\big(\sqrt{\Delta}\big)+2=0 \quad (A1.2)$$

with integers (E F) = (0,0), (1,1), (0, -1), (1,0), (1, -1),(3,2) which results from $\phi_4=f^4(\sqrt{\Delta})$.

$$\phi_4^6+\gamma_2\left(\frac{-3+\sqrt{\Delta}}{2}\right)\phi_4^4-2^8=(\phi_4^3-4A\phi_4^2+8B\phi_4-2^4)(\phi_4^3+4A\phi_4^2+8B\phi_4+2^4).$$

Periods $\omega_{k+1}\simeq\eta^2(\omega_k)=\frac{\prod g(a\boldsymbol{\omega}_k)}{\prod g(a'\boldsymbol{\omega}_k)}$ are invariant for certain modular units (4.1) [30] [31] [33] [32]. The folding process $\omega_{k+1}\leftarrow\omega_k$ creates a 2-power tower $f^{2^{\cdot^{\cdot}}}$. Cycles contain generators as roots of unity with $f^{b^{2^k}}(\omega)=1$. Formally, the iteration index k can be set complex k. A generator b^{2^k} exists for first four Fermat number congruences $b=3^k=2$ with geometric zeta function as a Cantor string ζ(2,3, z) [55] [27]. This power tower corresponds to the nome [14]

$$q=1^{\frac{iK'}{2K}}=e^{2^{-k}\ln\lambda_k+2^{2-k}\ln 2} \quad (A1.3)$$

Rational points on Kummer surface K(X) and W(Y) are calculated by a folding and taping process which is equivalent to quadratic iterates. Iterates $K_w[\lambda_w]$ on C^w and $K_{w-1}[\lambda_{w-1}]$ on C^{w-1} yield a string sequence $u[K_w[\lambda_w[g[u, \mathbb{L}_w[K_{w-1}[\lambda_{w-1}[\dots]]]]]]]$ with cycles expressed by modular units g(u=aω) ($a\in\mathbb{Q}^2$) which depend on fluctuating lattices of units {l}. The generalized principal ideal theorem of CM predicts an equivalence $\lambda_N g^2(a,\mathbb{L})\simeq\delta_F\alpha_F^2$ if a generator or a number theoretic transform exists for a tower of modular units.

Appendix 2: A Cumulated Group for Weierstrass Relations

Weierstrass relations are connected to spheres which transmits also to hyperelliptic addition. A six-component spherical triangle of arc length a, b, c and angle α, β, γ can be addressed by three four-component parameter u_μ,v_μ,w_μ (μ=1,2,3,4) where $\tan(½m_0a_\mu)=i\exp(v_\mu)$ and $\tan(½m_0\alpha_\mu)=i\exp(w_\mu)$ with $a_\mu=(2\pi,a,b,c)$, $\alpha_\mu=(2\pi,\alpha,\beta,\gamma)$ and matrices S and S_0

where.

$S=diag(-1,1,1,1)S_0$ is a third root $S^3=1$ and S_0 is a second root $S_0^2=1$ of

$$m_0=½\begin{vmatrix}1&-1&-1&-1\\0&-1&1&1\\0&1&-1&1\\0&1&1&-1\end{vmatrix},S_0=\frac{1}{2}\begin{vmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{vmatrix} \quad (A2.1)$$

Rotations of 3·4 parameter $(u,v,w)=(1,S,S^2)u\cong(X,Y,Z)$ with matrices. A sphere $\mathbb{S}^2$ is related to three four-component variable (X, Y, Z) in Weierstrass relations s(u)+s(v)+s(w)=0 of elliptic theta $s(u)=\sum\vartheta(u_1)\vartheta(u_2)\vartheta(u_3)\vartheta(u_4)$ [56]. 120 possible products of hyperelliptic theta $\vartheta_{[g]}(u)$ of characteristics $[g]\Rightarrow\begin{bmatrix}g\\g'\end{bmatrix}\in F_2^4$

$$2^4s[g](v)=\sum_{[h]}(-1)^{h\wedge g}s[h](u) \quad (A2.2)$$

where

$$s[g](u)=(-1)^{(h+l)g'}\vartheta_{[g]}(u_1)\vartheta_{[g+h]}(u_2)\vartheta_{[g+l]}(u_3)\vartheta_{[g-h-l]}(u_4) \quad (A2.3)$$

appear in 15 groups of 8 species [57]. Within one-periodic units (v_{Sh}) of doubly-periodic fluctuating congruences ω_k the groups of characteristics [gh] have the highest degree of inertia. Inert [gh] groups are associated with particles in [47]. Invariances of (A2.1) with respect to adding $B\omega=B_1\omega_1+B_2\omega_2$ in $(u_\mu,v_\mu,w_\mu)\rightarrow u+½(1, S, S^2)B_\mu\omega$ yield 2^8 possible substitutions $B_\mu\in\{0,1\}$ as elements of a group G_{256} [58]. Iteration $f(\omega_{k+1})\leftarrow\gamma(\phi_3)\circ f(\omega_k)$ changes universal covering $(u_{\mu k},v_{\mu k},w_{\mu k})$ because of CM $\omega_{k+1}\leftarrow\prod\eta^2(\omega_k)\simeq f^2(\omega_k)$. After 2^8 steps elements of the group of order G_{256} are traversed. Iterates cumulate a congruent memory of universal covering of $\sum_{k=1}^{2^8}B_{\mu,k}\omega_k$ of a group $G_{2^{2^8}}$ of the order 2^{2^8} for characteristics [gh].

Appendix 3: Elliptic Addition

Three points of $z_k=p(u)$, $z_{k+1}=p(v)$ and $z_{k+2}=p(u\mp v)$ where

$$\wp'^2(u)=\Phi_3(z)=\sum_{i=0}^{3}\binom{3}{i}a_iz^{3-i} \quad (A3.1)$$

yield [41]

$$a_0(z_k+z_{k+1}+z_{k+2}) + 3a_1 = \left(\frac{\sqrt{\Phi_3(z_k)}\pm\sqrt{\Phi_3(z_{k+1})}}{z_k-z_{k+1}}\right)^2 \quad \text{(A3.2)}$$

For $3s_1(z_k,z_{k+1},z_{k+2})=\sum z_k$, $3s_2(z_k,z_{k+1},z_{k+2})=\sum z_k z_{k'}$ and $s_3(z_k,z_{k+1},z_{k+2})= z_k z_{k+1} z_{k+2}$

$$4[a_0s_3+a_3][3a_0s_1+3a_1] - [3a_0s_2-3a_2]^2 = 0 \quad \text{(A3.3)}$$

Roots k,k+1,k+2 belong to simplest cycles q of $\gamma(\Phi_3)z_k$ in polynomial $\Phi_q(z) = \sum_{i=0}^{3}\binom{3}{i} s_i(z_k,z_{k+1},z_{k+2})\, z^{3-i}$. For $s_4 = 4s_1s_3 - 3s_2^2$, s_0=1 and symbolic $s_i \to \sum_1^{4-i}\sum_2^{i}$ and $a_i = A_1^{4-i}A_2^{i}$ one gets

$$\Phi_q(z) = \sum_{i=0}^{4}\binom{4}{i} a_i\, s_i = \sum_{i=0}^{4}\binom{4}{i} A_1^{4-i}A_2^{i}\textstyle\sum_1^{4-i}\sum_2^{i} = 0 \quad \text{(A3.4)}$$

which is equianharmonic a_4 if

$$a_0a_4 - 4a_1a_3 + 3a_2^2 = 0 \quad \text{(A3.5)}$$

For σ_2=0 leading to σ_4=0 one has $y = \frac{A_2\Sigma_2}{A_1\Sigma_1} = (-4)^{\frac{1}{3}}$ $\lambda=1^{1/3}$ with

$$4A_1\Sigma_1A_2^3\Sigma_2^3 + A_2^4\Sigma_2^4 = 0 \quad \text{(A3.6)}$$

The inflection tangent condition reads [59]

$$F(C) = \begin{vmatrix} \zeta_2' - \zeta_1' & \zeta_3' - \zeta_1' \\ \zeta_2'' - \zeta_1'' & \zeta_3'' - \zeta_1'' \end{vmatrix} = \begin{vmatrix} 1 & \zeta_1' & \wp_1' \\ 1 & \zeta_2' & \wp_2' \\ 1 & \zeta_3' & \wp_3' \end{vmatrix} = \frac{\vartheta_0'^3\vartheta_s\vartheta_{12}\vartheta_{23}\vartheta_{13}}{\vartheta_{01}^3\vartheta_{02}^3\vartheta_{03}^3} = 0 \quad \text{(A3.7)}$$

where $\zeta_i=\zeta(u_0-u_i)$, $\wp_i=\wp(u_0-u_i)$, $\vartheta_{ij}=\vartheta_1(u_i-u_j)$, $\vartheta_s=\vartheta_1(3u_0-u_1-u_2-u_3)$, For finite differentials of differences with u_0, ζ_0, $\wp_0$ one gets for a simplest cycle q

$$dF_{q,ii'} \simeq \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & du_k & d\zeta_k & d\wp_k \\ 1 & du_{k+1} & d\zeta_{k+1} & d\wp_{k+1} \\ 1 & du_{k+2} & d\zeta_{k+2} & d\wp_{k+2} \end{vmatrix} = 0$$

or

$$dF_{q,ii'} \simeq \begin{vmatrix} 1 & du_0 & d\zeta_0 & d\wp_0 \\ 1 & du_k & d\zeta_k & d\wp_k \\ 1 & du_{k+1} & d\zeta_{k+1} & d\wp_{k+1} \\ 1 & du_{k+2} & d\zeta_{k+2} & d\wp_{k+2} \end{vmatrix}=0$$

In $\mathbb{N}[\sqrt{\Delta}]$ units ε_q homogeneous $\wp(u)$- and $\zeta(u)$ functions $(u,\omega)\to(\varepsilon_q u,\varepsilon_q\omega)$ yield for column 1 replaced by $M_q du_q$

$$dF_{q,ii'} \simeq \begin{vmatrix} M_0du_0 & \varepsilon_0du_0 & \varepsilon_0^{-1}d\zeta_0 & \varepsilon_0^{-2}d\wp_0 \\ M_kdu_k & \varepsilon_kdu_k & \varepsilon_k^{-1}d\zeta_k & \varepsilon_k^{-2}d\wp_k \\ M_{k+1}du_{k+1} & \varepsilon_{k+1}du_{k+1} & \varepsilon_{k+1}^{-1}d\zeta_{k+1} & \varepsilon_{k+1}^{-2}d\wp_{k+1} \\ M_{k+2}du_{k+2} & \varepsilon_{k+2}du_{k+2} & \varepsilon_{k+2}^{-1}d\zeta_{k+2} & \varepsilon_{k+2}^{-2}d\wp_{k+2} \end{vmatrix} = 0 \quad \text{(A3.8)}$$

(A3.7) vanishes if units ε_q satisfy a cubic equation $M_q + a_1\varepsilon_q + a_2\varepsilon_q^{-1} + a_3\varepsilon_q^{-1} = 0$ for various tangents a=(du,dζ,d℘) where CM parameter $M_q \in \mathbb{C}$ is forced by a cubic z and ω equation due to $\gamma(\phi_3)$. (A3.7) allows SE(3) transformations and M(a) collineations for a quadruple of units ε_q.

$$\mathrm{dF}(x_F, y_F) \to \det F_{q,ii'}\textstyle\prod M(\boldsymbol{a}) = 0 \quad \text{(A3.9)}$$

An orthogonal hyperelliptic substitution $\alpha_{[gh]}(u+v)\tilde{\Gamma}\beta_{[gh]}(u-v) \to \frac{1+F(c,\beta_{[gh]})}{1-F(\alpha_{[gh]},\beta_{[gh]})} \in [0,\pm 1]$ is SE(3) related e.g. by a Cayley transform with skew symmetric matrix F(α, β) in [gh] space if $\det\alpha\tilde{\Gamma}\beta=1$. The stability axis $dF(x_F,y_F)\to dF_{[gh]}(x_F,y_F)$ in (A3.7) depends on hyperelliptic $\vartheta_{[gh]}(u_\pm)$ and $\wp_{\pm\pm} = \wp_{\pm\pm[gh]}$, $f_{[gh]}(\omega)$. The SE(3)- rotation matrix S(**A**,a) depends on 16 combinations of characteristics [gh].

$$\mathrm{dF}(x_F, y_F) \to \mathrm{d}F_{[gh]}(x_F, y_F)\, \alpha_{[gh]}(u+v)\Gamma\beta_{[gh]}(u-v) \quad \text{(A3.10)}$$

Again, the quadratic map of (A3.8) as a continued fraction of 2x2 minors is capable for fixed points where $M_q \in \mathbb{Q}[\sqrt{\Delta}]$ (q=1,2,3,4).

Appendix 4: Transition from Robot Dynamics to Chaotic Dynamics

SE(3) steps of $X=(\wp_{\pm\pm},1)=(1,-f,f^2,1)$ include unimodular collineations M(a) as continued fractions (n=1), bifurcating continued fractions (n=2) and quaternary continued fractions (n=3)

$$M(a_n) = C_{n+1} + \begin{pmatrix} 0_{1,n} & 0 \\ 0_{n,n} & a_n^T \end{pmatrix} \quad \text{(A4.1)}$$

with zero matrix $0_{n,m}$ of n rows and m columns, a vector $a_n=(a_1,\ldots,a_n)$, a cyclic matrix C_{n+1} of order n+1 with $C_{n+1}^{n+1} = 1$, $C_{n+1}C_{n+1}^T = 1$ and its exponential map $C_4 = e^{c_4}$. One has [60]

$$C_{n+1}^T M(a_n) = \begin{pmatrix} 1_{n,n} & a_n^T \\ 0_{1,n} & 1 \end{pmatrix} = e^{S(1,a_n)}. \quad \text{(A4.2)}$$

for $S(\boldsymbol{A}, a_n) = \begin{pmatrix} \boldsymbol{A} & a_n \\ 0_{n,1} & 0 \end{pmatrix}$ satisfying $S^4 - \boldsymbol{A}^2 S^2 = 0$ where **A** is a rotation matrix in $M(\boldsymbol{A}, \boldsymbol{a}) = e^{S(\boldsymbol{A},\boldsymbol{a})}$.

With nilpotent $N_0 = \frac{1}{\boldsymbol{A}^2} S(\boldsymbol{A}^2 - S^2)$ ($N_0^2 = 0$) and idempotent $P_0 = 1 - \frac{S^2}{\boldsymbol{A}^2}$, $P_\pm = \frac{1}{2\boldsymbol{A}^3} S^2(|\boldsymbol{A}| \mp S)$ the exponential reads $e^S = P_0 + N_0 + e^{-|\boldsymbol{A}|} P_+ + e^{|\boldsymbol{A}|} P_-$ where $(P_\varrho^2 = P_\varrho)$ for ρ=0, ∓. A collineation M(a) consists of four SE(3) steps $e^{S(\mathbf{A},\mathbf{a})}$.

$$\prod_{k=1}^{4} \begin{pmatrix} \boldsymbol{A}_k & \boldsymbol{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\boldsymbol{A}_4 & \boldsymbol{A}_1\boldsymbol{A}_2(\boldsymbol{A}_3+1)\boldsymbol{a} + (\boldsymbol{A}_1+1)\boldsymbol{a} \\ 0 & 1 \end{pmatrix} = \prod_{k=1}^{4} \begin{pmatrix} \boldsymbol{A}_k & \boldsymbol{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^4 & (\boldsymbol{A}^2+1)(\boldsymbol{A}+1)a \\ 0 & 1 \end{pmatrix}$$

for a single skew rotation A_k=A where $\mathbf{A}^2<0$.

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