Scholars Journal of Physics, Mathematics and Statistics
Sch. J. Phys. Math. Stat. 2015; 2(4):341-348
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(An International Publisher for Academic and Scientific Resources)

ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

On The Ternary Quadratic Diophantine Equation $X^2+Y^2=37Z^2$

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Abstract: The Ternary Quadratic Diophantine Equation given by $X^2+Y^2=37Z^2$ is analyzed for its patterns of non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented.

Keywords: Ternary quadratic, integer solutions, figurate numbers, polygonal number, Pyramidal numbers.

NOTATIONS USED:

1. POLYGONAL NUMBER of rank n with size m

$$T_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

2. PYRAMIDAL NUMBER of rank n with size m

$$P_n^m=\frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$$

3. PRONIC NUMBER of rank n

$$\Pr_n=n(n+1)$$

4. OCTAHEDRAL NUMBER of rank n

$$OH_n=\frac{1}{3}\left[n(2n^2+1)\right]$$

INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-2]. In particular, one may refer [3-16] for quadratic equation with three unknowns. This communication concerns with yet another interesting equation $X^2+Y^2=37Z^2$ representing non-homogeneous cubic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

METHOD OF ANALYSIS:

The quadratic equation to be solved for its non-zero integer solutions is

$$X^2+Y^2=37Z^2 \tag{1}$$

Assume $Z=Z(a,b)=a^2+b^2, a,b>0$ (2)

We present below different patterns of integral solutions to (1)

PATTERN:1

Write 37 as $37=(6+i)(6-i)$ (3)

Substituting (2) and (3) in (1), we get

$$X^2+Y^2=(6+i)(6-i)(a^2+b^2)^2$$

Employing the method of factorization

$$(X+iY)(X-iY)=(6+i)(6-i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative factors, we get

$$X + iY = (6+i)(a+ib)^2 \quad (4)$$

$$X - iY = (6-i)(a-ib)^2 \quad (5)$$

Equating the real and imaginary parts in either (4) or (5), we get

$$X = X(a,b) = 6a^2 - 6b^2 - 2ab \quad (6)$$

$$Y = Y(a,b) = a^2 - b^2 + 12ab \quad (7)$$

Thus (2),(6) and(7) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $37^2 Z^2(A,B) - (6X(A,B) + Y(A,B))^2$ is a perfect square
2. $Y(A,1) - T_{4,A} \equiv -1 \pmod{12}$
3. $Y(2A,2) - T_{10,A} \equiv -4 \pmod{51}$
4. $X[A, A(A+1)] - 6T_{4,A} + 4P_A^5$ is a nasty number
5. $X(A,1) - T_{14,A} + 6 = 3A$

PATTERN:2

Write 37 as $37 = (1+6i)(1-6i)$ (8)

Substituting (2) and (8) in (1), we get

$$X^2 + Y^2 = (1+6i)(1-6i)(a^2+b^2)^2$$

Employing the method of factorization

$$(X+iY)(X-iY) = (1+6i)(1-6i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative factors, we get

$$X + iY = (1+6i)(a+ib)^2 \quad (9)$$

$$X - iY = (1-6i)(a-ib)^2 \quad (10)$$

Equating the real and imaginary parts in either (9) or (10), we get

$$X = X(a,b) = a^2 - b^2 - 12ab \quad (11)$$

$$Y = Y(a,b) = 6a^2 - 6b^2 + 2ab \quad (12)$$

Thus (2), (11) and (12) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $37^2 Z^2(A,B) - (X(A,B) + 6Y(A,B))^2$ is a perfect square
2. $X(A,1) - T_{4,A} \equiv -1 \pmod{12}$
3. $41Z(A,1) - X(A,1) - T_{82,A} \equiv 41 \pmod{51}$
4. $T_{18,A} - X(A,1) - Y(A,1) - Z(A,1) \equiv 0 \pmod{3}$

PATTERN:3

The ternary quadratic equation (1) can be written as

$$X^2 - Z^2 = 36Z^2 - Y^2 \quad (13)$$

Factorizing (13), we have

$$(X+Z)(X-Z) = (6Z+Y)(6Z-Y) \quad (14)$$

$$\Rightarrow \frac{X+Z}{6Z+Y} = \frac{6Z-Y}{X-Z} = \frac{A}{B}, \quad B \neq 0 \quad (15)$$

This is equivalent to the following two equations

$$BX - AY + Z(B-6A) = 0 \quad (16)$$

$$AX + BY - Z(A+6B) = 0 \tag{17}$$

Applying the method of cross multiplication, we get

$$X = X(A,B) = A^2 - B^2 - 12AB \tag{18}$$

$$Y = Y(A,B) = -6A^2 + 6B^2 + 2AB \tag{19}$$

$$Z = Z(A,B) = A^2 + B^2 \tag{20}$$

Thus (18), (19) and (20) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $X(A, A+1) - T_{26,A} \equiv -1(\bmod 21)$
2. $Y(3,2A) - T_{50,A} \equiv -19(\bmod 35)$
3. $X(A,2A^2+1) + 4T_{4,A}^2 + 3T_{4,A} - 36OH_A \equiv -1$
4. $X(A, A+1) - 12\Pr_A + 1 = -2A$
5. $X(A+2, A+2) - T_{26,A} \equiv 48(\bmod 59)$

PATTERN:4

Equation (14) is expressed as

$$\frac{X-Z}{6Z+Y} = \frac{6Z-Y}{X+Z} = \frac{A}{B}, B \neq 0 \tag{21}$$

This is equivalent to the following two equations

$$BX - AY - Z(B+6A) = 0 \tag{22}$$

$$AX + BY + Z(A-6B) = 0 \tag{23}$$

Applying the method of cross multiplication, we get

$$X = X(A,B) = A^2 - B^2 - 12AB \tag{24}$$

$$Y = Y(A,B) = 6A^2 - 6B^2 + 2AB \tag{25}$$

$$Z = Z(A,B) = -A^2 - B^2 \tag{26}$$

Thus (24),(25) and(26) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $X(A,1) - T_{4,A} + 1 = -12A$
2. $Y(A,1) - T_{14,A} \equiv -6(\bmod 7)$
3. $Y(A,2) - T_{14,A} \equiv -6(\bmod 9)$
4. $Y(A,-2) - 14T_{14,A} + 24 = A$
5. $Y(A, A+1) - 2\Pr_A + 6 = -12A$
6. $X(3B,2) - T_{20,B} + 4 = -64A$

PATTERN:5

The ternary quadratic equation (1) can be written as

$$X^2 = 37Z^2 - Y^2 \tag{27}$$

Assume $X(A,B) = 37A^2 - B^2$, $A, B > 0$ (28)

Employing the method of factorization

$$(\sqrt{37}A + B)^2(\sqrt{37}A - B)^2 = (\sqrt{37}Z + Y)(\sqrt{37}Z - Y)$$

Equating the positive and negative factors, we get

$$(\sqrt{37}A + B)^2 = (\sqrt{37}Z + Y) \tag{29}$$

$$(\sqrt{37}A - B)^2 = (\sqrt{37}Z - Y) \tag{30}$$

Comparing the rational and irrational factors, we get

$$Y = Y(A,B) = 37A^2 + B^2 \quad (31)$$

$$Z = Z(A,B) = 2AB \quad (32)$$

Thus (28), (31) and (32) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $X(A,B) + Y(A,B) = Z(37A, A)$
2. $X(A,B) + Y(A,B) = 74T_{4,A}$
3. $Y(1,2B) - 2Z(1,B) - 4T_{4,B} \equiv 1(\text{mod } 4)$
4. $Y(A, A+1) - X(A,2) - T_{4,A} \equiv 1(\text{mod } 2)$
5. $Z(20A, A) - Y(A,A) = 2T_{4,A}$

PATTERN:6

The ternary quadratic equation (1) can be written as

$$37Z^2 - Y^2 = X^2 \times 1 \quad (33)$$

Write 1 as

$$1 = (\sqrt{37} + 6)(\sqrt{37} - 6) \quad (34)$$

Using (28), (34) in (33) and employing the method of factorization, we have

$$(\sqrt{37}Z + Y)(\sqrt{37}Z - Y) = (\sqrt{37}A + B)^2(\sqrt{37}A - B)^2(\sqrt{37} + 6)(\sqrt{37} - 6)$$

Equating the positive and negative factors, we get

$$(\sqrt{37}Z + Y) = (\sqrt{37}A + B)^2(\sqrt{37} + 6) \quad (35)$$

$$(\sqrt{37}Z - Y) = (\sqrt{37}A - B)^2(\sqrt{37} - 6) \quad (36)$$

Comparing the rational and irrational factors, we get

$$Y = Y(A,B) = 222A^2 + 6B^2 + 74AB \quad (37)$$

$$Z = Z(A,B) = 37A^2 + B^2 + 12AB \quad (38)$$

Thus (28),(37) and(38) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $Z(2A,4) - T_{298,A} \equiv 16(\text{mod } 243)$
2. $Z(A,4) - T_{76,A} \equiv 4(\text{mod } 21)$
3. $X(1,2B) - 4T_{4,B} = 37$
4. $Z(2A+1, 2A) - T_{402,A} \equiv 37(\text{mod } 371)$
5. $Y(2B+1, 2B) - T_{2322,B} \equiv 222(\text{mod } 2195)$
6. $X(4A+1,2) - Z(2A+1,2) - T_{886,A} + 32 = 543A$

REMARKABLE OBSERVATION:

A: If the non-zero integer triple (X_0, Y_0, Z_0) is any solution of (1), then each of the following three triples also satisfies (1).

TRIPLE:1 (X_0, Y_n, Z_n)

Let the first solution of (1) be

$$\left.\begin{aligned} X_1 &= X_0 \\ Y_1 &= Y_0 + 6h \\ Z_1 &= Z_0 + h \end{aligned}\right\} \quad (39)$$

Substituting (39) in (1), we get

$$X_0^2 + (Y_0 + 6h)^2 = 37(Z_0 + h)^2$$

$$\Rightarrow X_0^2 + Y_0^2 + 36h^2 + 12Y_0 h = 37Z_0^2 + 37h^2 + 74Z_0 h$$

$$\therefore h = 12Y_0 - 74Z_0$$

Substituting the value of 'h' in (39), we get

$$X_1 = X_0$$

$$Y_1 = 73Y_0 - 444Z_0$$

$$Z_1 = 12Y_0 - 73Z_0$$

Hence, the matrix representation of above solution is

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} 73 & -444 \\ 12 & -73 \end{pmatrix} \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}$$

Let A be $\begin{pmatrix} 73 & -444 \\ 12 & -73 \end{pmatrix}$

To find the eigen values of A:

Consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 73-\lambda & -444 \\ 12 & -73-\lambda \end{vmatrix} = 0$$

Therefore, the eigen values of A are 1 and -1

Take $\alpha = 1, \beta = -1$

To find A^n, we use the following formula.

$$A^n = \frac{\alpha^n}{\alpha - \beta}[A - \beta I] + \frac{\beta^n}{\beta - \alpha}[A - \alpha I] \quad (40)$$

Substituting the values of α, β and A in (40), we get

$$A^n = \frac{1^n}{2}\begin{bmatrix} 74 & -444 \\ 12 & -72 \end{bmatrix} - \frac{(-1)^n}{2}\begin{bmatrix} 72 & -444 \\ 12 & -74 \end{bmatrix}$$

Thus,

$$\begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = A^n \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}$$

$\therefore$ We get the n^{th} solution of (1) as given below

$$X_n = X_0$$

$$Y_n = \frac{1}{2}\left[\left(74\alpha^n - 72\beta^n\right)Y_0 - 444\left(\alpha^n - \beta^n\right)Z_0\right]$$

$$Z_n = \frac{1}{2}\left[12\left(\alpha^n - \beta^n\right)Y_0 + \left(-72\alpha^n + 74\beta^n\right)Z_0\right]$$

TRIPLE:2 (X_n, Y_0, Z_n)

Let the first solution of (1) be

$$\left.\begin{aligned} X_1 &= 2X_0 + 6h \\ Y_1 &= 2Y_0 \\ Z_1 &= 2Z_0 + h \end{aligned}\right\} \quad (41)$$

Substituting (41) in (1), we get

$$(2X_0+6h)^2+(2Y_0)^2=37(2Z_0+h)^2$$

$$\therefore h=24X_0-148Z_0$$

Substituting the value of 'h' in (41), we get

$$X_1=146X_0-888Z_0$$
$$Y_1=2Y_0$$
$$Z_1=24X_0-146Z_0$$

Hence, the matrix representation of above solution is

$$\begin{pmatrix}X_1\\Z_1\end{pmatrix}=\begin{pmatrix}146 & -888\\24 & -146\end{pmatrix}\begin{pmatrix}X_0\\Z_0\end{pmatrix}$$

Let A be $\begin{pmatrix}146 & -888\\24 & -146\end{pmatrix}$

To find the eigen values of A:

Consider $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}146-\lambda & -888\\24 & -146-\lambda\end{vmatrix}=0$$

Therefore, the eigen values of A are 2 and -2

Take $\alpha=2,\beta=-2$

To find A^n, we use the following formula.

$$A^n=\frac{\alpha^n}{\alpha-\beta}[A-\beta I]+\frac{\beta^n}{\beta-\alpha}[A-\alpha I] \quad (42)$$

Substituting the values of α,β and A in(42), we get

$$A^n=\frac{2^n}{4}\begin{bmatrix}148 & -888\\24 & -144\end{bmatrix}-\frac{(-2)^n}{4}\begin{bmatrix}144 & -888\\24 & -148\end{bmatrix}$$

$$\begin{pmatrix}X_n\\Z_n\end{pmatrix}=A^n\begin{pmatrix}X_0\\Z_0\end{pmatrix}$$

$\therefore$ We get the n^{th} solution of (1)as below

$$X_n=\frac{1}{4}\left[\left(148\alpha^n-144\beta^n\right)X_0-888\left(\alpha^n-\beta^n\right)Z_0\right]$$
$$Y_n=2^nY_0$$
$$Z_n=\frac{1}{4}\left[24\left(\alpha^n-\beta^n\right)X_0+\left(-144\alpha^n+148\beta^n\right)Z_0\right]$$

TRIPLE:3 (X_n,Y_n,Z_0)

Let the first solution of (1) be

$$\left.\begin{aligned}X_1&=2X_0-2h\\Y_1&=2Y_0-2h\\Z_1&=2Z_0\end{aligned}\right\} \quad (43)$$

Substituting (43) in (1), we get

$$(2X_0-2h)^2+(2Y_0-2h)^2=37(2Z_0)^2$$

$$\therefore h=X_0+Y_0$$

Substituting the value of 'h' in (43), we get

$$X_1 = -2Y_0, Y_1 = -2X_0, Z_1 = 2Z_0$$

Similarly we have the second ,third ,fourth…… .solution are as given below.

SECOND SOLUTION:

$$X_2 = -2Y_1 = 4X_0, Y_2 = -2X_1 = 4Y_0, Z_2 = 2^2 Z_0$$

THIRD SOLUTION:

$$X_3 = -2Y_2 = -8Y_0, Y_3 = -2X_2 = -8X_0, Z_3 = 2^3 Z_0$$

FOURTH SOLUTION:

$$X_4 = -2Y_3 = 16X_0, Y_4 = -2X_3 = 16Y_0, Z_4 = 2^4 Z_0$$

FIFTH SOLUTION:

$$X_5 = -2Y_4 = -32Y_0, Y_5 = -2X_4 = -32X_0, Z_5 = 2^5 Z_0$$

Thus we have the n^{th} solution as follows:

$$X_{2n-1} = -2^{2n-1} Y_0, Y_{2n-1} = -2^{2n-1} X_0, Z_{2n-1} = 2^{2n-1} Z_0$$

$$X_{2n} = 4^n X_0, Y_{2n} = 4^n Y_0, Z_{2n} = 2^{2n} Z_0$$

B: Employing the solution (X, Y, Z) of (1), each of the following expression among the special polygonal and pyramidal numbers are perfect square.

1. $\frac{1}{37}\left\{\left(\frac{3P_{X-2}^3}{T_{3,X-2}}\right)^2 + \left(\frac{P_Y^5}{T_{3,Y}}\right)^2\right\}$
2. $37\left(\frac{P_Z^5}{T_{3,Z}}\right)^2 - \left(\frac{3P_{X-2}^3}{T_{3,X-2}}\right)^2$
3. $37\left(\frac{3P_{Z-2}^3}{T_{3,Z-2}}\right)^2 - \left(\frac{P_X^5}{T_{3,Z}}\right)^2$

It is worth to note that, on multiplying each of the above observation by 6 , it represents a nasty number.

CONCLUSION:

In this paper, we have presented six different of patterns non-zero distinct integer solutions of the homogeneous equations given by $X^2 + Y^2 = 37Z^2$ .To conclude one may search for other patterns of solutions and their corresponding properties.

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