

Relativistic Harmonic Oscillator

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Abstract: This is a straightforward approach to the system, namely through the solution of the concerned differential equation. In this paper, an expression for the time-period of the motion as measured by a stationary observer, and 'proper time-period', is obtained. The expressions for the time period in both cases are left as unsolved integrals.

Keywords: Harmonic Oscillator, Time-Period, Special Relativity.

INTRODUCTION

The Harmonic Oscillator is an important topic in Physics and its applications are vast. Therefore, the relativistic case has been discussed extensively in the past. Several analyses of Relativistic Harmonic Oscillators have been conducted through the use of sophisticated mathematical techniques[1]. However, the objective of this paper is precise: the time-period of the motion.

A body executing simple harmonic motion in one dimension will have its motion described by a solution to the equation:

$$\frac{dp}{dt} + kx = 0 \quad (1)$$

where k is an arbitrary constant.

In the classical case, this can be simply reduced to the differential equation:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \quad (2)$$

where $\omega_0 = \sqrt{k/m_0}$ and m_0 is the rest mass of the body. The solution to this equation is well known.

In the relativistic case, however, \vec{p} cannot simply be reduced to the classical picture of $m_0\vec{x}$. Also, one must consider the measurement of time and distance for different observers. An observer at rest with respect to the oscillating body will, for instance, measure a different time period than an observer stationary with respect to the frame, relative to which the body is executing the oscillation (henceforth referred to as the 'stationary observer'). However, the degree of difference in the measurements remains to be seen. Throughout the article:

$$\gamma = \frac{1}{\sqrt{1 - \frac{x^2}{c^2}}} \quad (3)$$

PRELIMINARIES

We assume a lab frame (ct, x) relative to the origin of which the body with coordinates (x', y', z') with rest mass m_0 executes a simple harmonic motion within the bound $x = \pm A$. For simplicity, I have taken the motion along only one dimension.

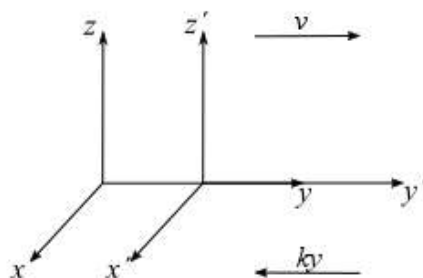


Fig-1: Motion of the oscillating frame

If $p = m_0 \gamma \dot{x}$ denotes the momentum as measured by the stationary observer, the corresponding equation of motion remains as:

$$\frac{dp}{dt} + kx = 0 \quad (4)$$

It is important to note that the restoring force considered in this treatment exists only in the lab frame, which gives the proportionality constant the property of being constant throughout the motion. Such a treatment holds for two cases-

1. For a sufficiently large ω .
2. For a sufficiently large amplitude.

For both cases, conceptions of length and time do not change for the stationary observer, however they do change for the oscillating body by virtue of the Lorentz transformations. To account for the manifestation of these changes for the oscillating body, one could argue that the relativistic mass of the oscillating body increases through the motion, thus providing the variation (decrease) in the effective angular frequency, and therefore the time period of oscillation.

$$\bar{\omega} = \sqrt{\frac{k}{m_{rel}}} \quad (5)$$

$$m_{rel} = m_0 \gamma \quad (6)$$

One could discount the use of the term relativistic mass entirely by suggesting that the change occurs purely due to the relativistic effects on the measurement of momentum and energy, which have a more intimate relationship with the aforementioned changes in the geometric properties of Spacetime in the frame of the oscillating body[2].

There is also the case in which the restoring force is entirely dependent on the parameters of the frame of the oscillating body, in which case the equation of motion is:

$$\frac{dp}{d\tau} + ks = 0 \quad (7)$$

where τ is the proper time and s is the proper length for the oscillating body. A system of the nature of a spring-mass system could possibly be governed by this equation, since the restoring force relies on the lengths measured in the moving frame. However, this case will not be treated in the article.

At the end of the following section, a relation between the proper time period and the time period measured by the stationary observer for this motion is obtained. It's interesting to note the causality involving such a relation, since the time period measured by the stationary observer is also seen to be impacted by the motion of the oscillating body.

The Time Period

For (4), on evaluating \dot{p} , where $p = m_0 \dot{x} \gamma$ is the momentum of the oscillating body measured by the stationary observer, we find that [3]:

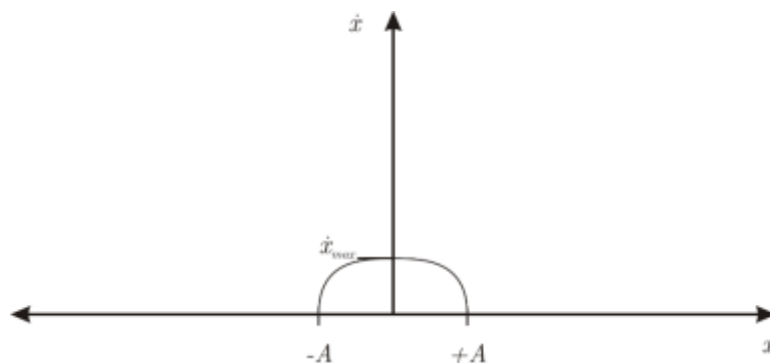


Fig-2: Phase curve of the motion

$$\frac{dp}{dt} = \gamma^3 m_0 \ddot{x} \quad (8)$$

Therefore, (4) expands to:

$$\gamma^3 m_0 \ddot{x} = -kx \quad (9)$$

Taking $\omega^2 = \sqrt{k/m_0}$:

$$\gamma^3 m_0 \dot{x} dx = -\omega^2 x dx \quad (10)$$

The solution to which has been shown to be periodic [4]. (10) evaluates to:

$$\frac{c^3}{\sqrt{c^2 - x^2}} = C - \frac{\omega^2 x^2}{2} \quad (11)$$

Taking conditions $x = \pm A$ and $\dot{x} = 0$:

$$C = c^2 + \frac{\omega^2 A^2}{2} \quad (12)$$

On rearrangement:

$$\dot{x} = c \left(1 - \frac{c^4}{\left(c^2 - \frac{\omega^2 x^2}{2} \right)^2} \right)^{1/2} \quad (13)$$

If T is taken to be the time period of oscillation and symmetry is accounted for the motion:

$$T = \frac{4}{c} \int_0^A \left(1 - \frac{c^4}{\left(c^2 - \frac{\omega^2 x^2}{2} \right)^2} \right)^{-1/2} dx \quad (14)$$

This can be further evaluated to:

$$T = \frac{4}{c} \int_0^A \left(1 - \left(1 + \frac{\omega^2}{2c^2} (A^2 - x^2) \right)^{-2} \right)^{-1/2} dx \quad (15)$$

On analysing (11) with C as found in (12), it is found that the Lorentz factory expressed as:

$$\gamma(\dot{x}) = \frac{1}{\sqrt{1 - \frac{x^2}{c^2}}} = 1 + \frac{\omega^2}{2c^2} (A^2 - x^2) = \gamma(x) \quad (16)$$

This simplifies the integral in (15) to:

$$T = \frac{4}{c} \int_0^A \frac{\gamma(x)}{\sqrt{\gamma(x)^2 - 1}} dx \quad (17)$$

Thus, if τ ($d\tau = dt/\gamma$) is taken to be the proper time period for the oscillating body:

$$\tau = \frac{4}{c} \int_0^A \frac{dx}{\sqrt{\gamma(x)^2 - 1}} \quad (18)$$

(16) makes it clear that $x(t)$ can be expressed in its usual sinusoidal form only for slower velocities.

THE CLASSICAL LIMIT

Here, the expression obtained above is shown to reduce to the classical expression, viz. $T = 2\pi/\omega$. It is to be noted that at all times

$$\omega x < \omega A < c \quad (19)$$

and particularly for this approximation:

$$\omega A \ll c \quad (20)$$

Using the binomial approximation in (12) the expression for the time period of the oscillation becomes:

$$T \approx \frac{4}{c} \int_0^A \left(\frac{\omega^2}{c^2} (A^2 - x^2) \right)^{-\frac{1}{2}} dx = \frac{4}{\omega} \int_0^A (A^2 - x^2)^{-\frac{1}{2}} dx = \frac{2\pi}{\omega} \quad (21)$$

The expression for the time period of the harmonic oscillator for relativistic speeds thus seems to, at least, agrees with classical approximations.

CONCLUSION

The expression for the time period here is left as an integral. Whether a closed form solution for this integral can be obtained or not is uncertain. If it comes to be known that a closed form indeed doesn't exist, the usefulness of the expression is limited only to techniques of numerical integration. In other words, it might be necessary to involve a computer for each calculation.

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