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New Technique for Solving Fractional Physical Equations

A. S. Abedl-Rady[1], S. Z. Rida[1], A. A. M. Arafa[2], H. R. Abedl-Rahim[1]
[1]Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt
[2]Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt

***Corresponding Author:**
H. R. Abedl-Rahim
Email: hamdy.ragab2013@yahoo.com

Abstract: In this paper, we introduce a new technique for solving fractional physical equations in the form of a rapid convergence series arrives at the exact solutions called variational iteration natural transform method(VINTM).It is a coupling of variational iteration method and the natural transform method. The results reveal that the method is very effective, simple and can be applied to other physical differential equations. The fractional derivatives are described in the Caputo sense.
Keywords: Natural transform, Variational iteration natural transform method (VINTM).), Fractional physical differential equations.

INTRODUCTION

The natural transform, initially was defined by Waqar *et al,*. [1] as the N - transform, which studied their properties and applications. Later, Belgacem *et al*., [2, 3] defined its inverse and studied some additional fundamental properties of this integral transform and named it the natural transform. Applications of natural transform in the solution of differential and integral equations and for the distribution and Bohemians spaces can be found in [3-10]. Now, we mention the following basic definitions of natural transform and its properties are introduced as follows:

Definition 1.1 [11]
Over the set of functions
$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,\left|\ f(t)\ \right|<Me^{|t|/\tau_j},\ if\ t\in(-1)^j\times[0,\infty)\right\}$$

The natural transform of $f(t)$ is $N[f(t)]=R(s;u)=\int\limits_0^\infty f(ut)e^{-st}dt, u>0, s>0$ (1)

where $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables u and s are the natural transform variables.

Theorem 1.2. We derives the relationship between Natural and Laplace, Sumudu transform in successive theorems [11] as follows:

1- If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A, $G(u)$ is Sumudu transform then,

$$N[f(t)]=R(s;u)=\frac{1}{u}\int\limits_0^\infty f(t)e^{-\frac{st}{u}}dt=\frac{1}{u}F(\frac{s}{u}), \tag{2}$$

2. If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A then, $G(u)$ is Sumudu transform of function $f(t)$ in A, then:

$$N[f(t)]=R(s;u)=\frac{1}{s}\int\limits_0^\infty f(\frac{ut}{s})e^{-t}dt=\frac{1}{s}G(\frac{u}{s}) \tag{3}$$

3- If $f^n(t)$ is the *n*th derivative of function $f(t)$ then, its natural transform is given by:

$$N[f^{n}(t)] = R_n(s,u) = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0), n \geq 1 \tag{4}$$

4. If $F(s,u)$, $G(s,u)$ are the natural transform of respective functions $f(t)$, $g(t)$ both defined in set A then,

$$N[f * g] = uF(s,u)G(s,u) \tag{5}$$

where $f * g$ is convolution of two functions f and g .

5. If $N[f(t)]$ is the natural transform of the function $f(t)$, then the natural transform of fractional derivative of order α is defined as:

$$N[f^{(\alpha)}(t)] = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0) \tag{6}$$

6. Let the function $f(t)$ belongs to set A be multiplied with weight function $e^{\pm t}$ then,

$$N[e^{\pm t} f(t)] = \frac{s}{s \mp u} R[\frac{s}{s \mp u}] \tag{7}$$

7. Let the function $f(at)$ belongs to set A, where a is non-zero constant then,

$$N[f(at)] = \frac{1}{a} R[\frac{s}{a}, u] \tag{8}$$

8. If $w^n(t)$ is given by $w^n(t) = \int_0^t ....\int_0^t f(t)(dt)^n dt$, then, the natural transform of $w^n(t)$ is given by:

$$N[w^n(t)] = \frac{u^n}{s^n} R(s,u) \tag{9}$$

9. The natural transform of T-periodic function $f(t) \in A$ such that $f(t+nT) = f(t), n = 0,1,2,...$ is given by:

$$N[f(t)] = R(s,u) = [1 - e^{\frac{-sT}{u}}]^{-1} \frac{1}{u} \int_0^T e^{-\frac{st}{u}} f(t)dt \tag{10}$$

10. The function $f(t)$ in set A is multiplied with shift function t^n , then,

$$N[t^n f(t)] = \frac{u^n}{s^n} \frac{d}{du^n}^n u^n R(s,u) \tag{11}$$

2 Analysis of the proposed method(VINTM).

In the case of an algebraic equation $f(x) = 0$, the Lagrange multipliers can be evaluated by an iteration formula for finding the solution of the algebraic equation $f(x) = 0$ that can be constructed as[12]:

$$x_{n+1} = x_n + \lambda f(x_n). \tag{12}$$

The optimality condition for the extreme $\frac{\partial x_{n+1}}{\partial x_n} = 0$ leads to

$$\lambda = -\frac{1}{f'(x_n)}, \tag{13}$$

Where ∂ is the classical variational operator. From (12) and (13), for a given initial value X_0 , we can find the approximate solution X_{n+1} by the iterative scheme for (12) as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f(x_0) \neq 0, n = 0,1,2,..... \tag{14}$$

This algorithm is well known as the Newton-Raphson method and has quadratic convergence. To illustrate the basic idea of variational iteration natural transform method, we consider the following fractional differential equation:

$$_0^c D_t^\alpha U + R[U(t)] + N[U(t)] = K(t), 0 < \alpha \tag{15}$$

Where R is a linear operator, N is a nonlinear operator and $K(t)$ is a given continuous function. Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the natural transform to eq. (15). Then the linear part is transformed into an algebraic equation as follows:

$$\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-\frac{s^{\alpha}U(0)}{u^{\alpha}}..........-\frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}}+N[R[U(t)]+N[U(t)]-K(t)]=0, \tag{16}$$

Where $N[f(t)]=R(s;u)=\int_{0}^{\infty}f(ut)e^{-st}dt, u>0, s>0$

The iteration formula of (15) can be used to suggest the main iterative scheme involving the Lagrange multiplier as:

$$R_{n+1}=R_n+\lambda(\frac{s^{\alpha}}{u^{\alpha}}R_n(s,u)-\frac{s^{\alpha}U(0)}{u^{\alpha}}..........-\frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}}+N[R[U(t)]+N[U(t)]-K(t)]). \tag{17}$$

Considering $N[R[U(t)]+N[U(t)]$ as restricted terms, one can derive a Lagrange multiplier as:

$$\partial R_{n+1}=\partial R_n+\partial(\lambda\frac{s^{\alpha}}{u^{\alpha}}R_n),$$

$$\partial R_{n+1}=\partial R_n+\frac{s^{\alpha}}{u^{\alpha}}(\lambda' R_n+\lambda\partial R_n)$$

This yields the stationary conditions of eq. (17) as follows;

$$\partial R_n(1+\frac{s^{\alpha}\lambda}{u^{\alpha}})=0,$$

$$\frac{s^{\alpha}}{u^{\alpha}}(\lambda' R_n)=0.$$

with eq. (17) and the inverse-natural transform N^{-1}, the iteration formula (16) can be explicitly given as**:**

$$U_{n+1}=U_n-N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}(\frac{s^{\alpha}}{u^{\alpha}}R_n-\frac{s^{\alpha}U(0)}{u^{\alpha}}..........-\frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}}+N[R[U(t)]+N[U(t)]-K(t)])] \tag{18}$$

$$U_{n+1}=N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}(\frac{s^{\alpha}}{u^{\alpha}}U(0)..........-\frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}})-\frac{u^{\alpha}}{s^{\alpha}}(N[R[U(t)]+N[U(t)]-K(t))], \tag{19}$$

$$U_0(t)=N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}(\frac{s^{\alpha}}{u^{\alpha}}U(0)..........-\frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}})]$$

Consequently the exact solution may be produced by using

$$U(x,t)=\lim_{n\to\infty}U_n(x,t) \tag{20}$$

APPLICATIONS

Application 3.1

Consider the following one–dimensional linear inhomogeneous fractional linear Schrodinger equation:

$$D_t^{\alpha}U=iU_{xx}, t>0, 0<\alpha\le 1, x\in R \tag{21}$$

Subject to initial condition:

$$U(x,0)=e^{ix}$$

where α is parameter describing the order of the fractional derivative. The function $U(x,t)$ is the unknown function, t is the time and x is the spatial coordinate. The derivative is understood in the Caputo sense. The general response expression contains parameter describing the order of the fractional derivative that can be varied to obtain various responses.

By applying the natural transform on both sides of eq.(21),then

$$\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-\frac{s^{\alpha-1}}{u^{\alpha}}U(x,0)-N[iU_{xx}]=0 \tag{22}$$

The iteration formula of eq. (21) can be constructed as:

$$R_{n+1} = R_n + \lambda(s,u)[\frac{s^\alpha}{u^\alpha} R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} U(x,0) - N[iU_{nxx}] \tag{23}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, $\tilde{U}_0 = 0$ and $N[iU_{nxx}]]$ is a restricted variation, that is , $\partial \tilde{U}_n = 0$
i.e.

$$\partial R_{n+1} = \partial R_n + \partial(\lambda \frac{s^\alpha}{u^\alpha} R_n) \quad , \qquad \partial R_{n+1} = \partial R_n + \frac{s^\alpha}{u^\alpha}(\lambda' R_n + \lambda \partial R_n) \tag{24}$$

This yields the stationary conditions, which gives $\lambda = -\frac{u^\alpha}{s^\alpha}$.

Substituting this value of Lagrangian multiplier in eq. (23) we get the following iteration formula:

$$R_{n+1} = R_n - \frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha} R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} U(x,0) - N[iU_{nxx}]] \tag{25}$$

Applying inverse natural transform on both sides of eq. (25) we get:

$$U_0 = e^{ix}, \tag{26}$$

$$U_1 = e^{ix} - ie^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{27}$$

$$U_2 = e^{ix} - ie^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)} - e^{ix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{28}$$

$$U_3 = e^{ix} - ie^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)} - e^{ix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ie^{ix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, ... \tag{29}$$

Finally, approximate analytical solution $U(x,t)$ is given by:

$$U(x,t) = e^{ix}[1 - \frac{it^\alpha}{\Gamma(\alpha+1)} + \frac{(it^\alpha)^2}{\Gamma(2\alpha+1)} - \frac{(it^\alpha)^3}{\Gamma(3\alpha+1)} + ...]$$

hence,

$$U(x,t) = e^{ix} E_\alpha(-it^\alpha) \tag{30}$$

where $\sum_{k=0}^{\infty} \frac{(-it^\alpha)^k}{\Gamma(\alpha k+1)} = E_\alpha(-it^\alpha)$ is the famous Mittag–Leffler function, then:

$$U(x,t) = e^{i(x-t^\alpha)} \tag{31}$$

For the special case α = 1, we obtain[See figures (1,2)]

$$U(x,t) = e^{i(x-t)} \tag{32}$$

which is the exact solution of eq(21) obtained by [13] .

Application 3.2

Consider the one–dimensional linear inhomogeneous fractional Burger's equation:

$$D_t^\alpha U + U_x - U_{xx} = 2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, t > 0, 0 < \alpha \le 1, x \in R \tag{33}$$

Subject to initial condition:

$U(x,0) = x^2$

As the previous application, by applying VINTM, we obtain:

$$U_0 = x^2, \tag{34}$$

$$U_1 = x^2 + t^2 , \tag{35}$$

$$U_2 = x^2 + t^2,... \tag{36}$$

Finally, approximate analytical solution $U(x,t)$ is given by

$$U(x,t) = x^2 + t^2 \tag{37}$$

which is the exact solution of eq (33) obtained by LTADM [14] .

Application 3.3

We next consider the linear inhomogeneous fractional KdV equation:

$$D_t^{\alpha} U(x,t) + U_x(x,t) + U_{xxx}(x,t) = 2t\cos(x), t \succ 0, 0 \prec \alpha \le 1, x \in R \tag{38}$$

Subject to initial condition

$U(x,0) = 0$

By applying VINTM, we obtain

$$U_0 = 0, \tag{39}$$

$$U_1 = 2\cos x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{40}$$

$$U_2 = 2\cos x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{41}$$

$$U_3 = 2\cos x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},... \tag{42}$$

Finally, approximate analytical solution $U(x,t)$ is:

$$U(x,t) = 2\cos x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{43}$$

For the special case $\alpha = 1$, we obtain

$$U(x,t) = t^2 \cos(x) \tag{44}$$

which is the exact solution received by HAM [15] and VIM [16].

Application 3.4

Consider the following linear Fokker-Plank equation:

$$\frac{\partial^{\alpha} U}{\partial t^{\alpha}} = -\frac{\partial}{\partial x} A(x,t)U + \frac{\partial^2}{\partial x^2} B(x,t)U, \tag{45}$$

$$A(x,t) = e^t \coth x \cosh x + e^t \sinh x - \coth x, \ B(x,t) = e^t \cosh x \tag{46}$$

Subject to initial condition

$U(x,0) = \sinh x., x \in R$

Similar to the previous applications, by applying VINTM,. we obtain:

$$U_0 = \sinh x, \tag{47}$$

$$U_1 = \sinh x \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \quad , \tag{48}$$

$$U_2 = \sinh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \ldots \tag{49}$$

Finally, we approximate the analytical solution $U(x,t)$ by:

$$U(x,t) = \sinh x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sinh x + \sinh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots. \tag{50}$$

hence,

$$U(x,t) = \sinh x E_{\alpha,1}(t^{\alpha}) \tag{51}$$

Where $\sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(\alpha k+1)} = E_{\alpha,1}(t^{\alpha})$ is the famous Mittag–Leffler function.

For the special case $\alpha = 1$, we obtain [See figure 3]

$$U(x,t) = e^t \sinh x \tag{52}$$

which is the exact solution and is same as obtained by ADM [17], VIM[18] and HPM[19].

CONCLUSION

It is obvious that the new technique (VINTM), has been successes to find exact solution of linear Schrödinger equation, inhomogeneous Burger’s equation , KdV equation , and Fokker-Plank equation. The results state that proposed technique is very powerful and efficient in finding the analytical solutions for a large class of physical differential equations of fractional order.

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