DOI: 10.21276/sjpms.2016.3.4.2

Scholars Journal of Physics, Mathematics and Statistics **ISSN 2393-8056 (Print)**
Sch. J. Phys. Math. Stat. 2016; 3(4):127-138 **ISSN 2393-8064 (Online)**
©Scholars Academic and Scientific Publishers (SAS Publishers)
(An International Publisher for Academic and Scientific Resources)

On Separation Properties and $(R_0)_R$ Spaces

Reyadh Delfi Ali
Department of Mathematics, College of Education, University of Al-Gadisiya, Iraq

***Corresponding Author:**
Reyadh Delfi Ali
Email: reyadh_delphi@yahoo.co.uk

Abstract: The author introduces some open sets in bitopological spaces and studies some of their basic properties. Certain separation properties and $(R_0)_R$ obtained from standard separation by replacing open sets by R–open sets in their definitions are studied.
Keywords: bitopological, R–open

1-INTRODUCTION

The concept of bitopological spaces was initiated by Kelly [1] . Let (X,Γ_1,Γ_2) be a bitopological spaces. For a set $A\subset X$ by A^{-i} and A^{0_i}, we denoted the Γ_i –closure and Γ_i –interior of A for $i=1,2$. In this work we study some sort of R –separation properties of topological spaces by using the notion of R –open sets instead of open sets.

2- R –OPEN SETS

Definition(2.1):

A subset A in bitopological space (X,Γ_1,Γ_2) is be termed R –open if there exists an T_1–open O such that $O\subset A\subset O^{-_2 0_1}$,the family of all R –open set in bitopological space was not necessary topology on X . The family of all R –open set in bitopological space (X,Γ_1,Γ_2) is denoted by $R.O.(X)$. It clear that every T_1–open is R –open but the converse is not true. The complement of R –open set will be called a R –closed set.
The following theorems gives some properties of R –open sets .

Theorem(2.2):

Let (X,Γ_1,Γ_2) be topological space. Let $A\subset X$, A is R –open if $A\subset A^{0_1-_2 0_1}$.

Proof:

Let A be R –open set. Then $O\subset A\subset O^{-_2 0_1}$ for some Γ_1 –open set O . But $O^{-_2}\subset A^{0_1-_2}$ and thus $O^{-_2 0_1}\subset A^{0_1-_2 0_1}$ .Hence $A\subset A^{0_1-_2 0_1}$.

Lemma(2.3):

Let (X,Γ_1,Γ_2) be a bitopological space and $O\in\Gamma_1$, then $O^{-_2 0_1-_2 0_1}=O^{-_2 0_1}$.

Proof :

It is know that $O^{-_2 0_1}\subset O^{-_2}$ .then $O^{-_2 0_1-_2}\subset O^{-_2-_2}=O^{-_2}$ .So $O^{-_2 0_1-_2 0_1}\subset O^{-_2 0_1}$. On the other hand, $O\subset O^{-_2}$ then $O^{0_1}\subset O^{-_2 0_1}$ .But $O=O^{0_1}$,for $O\in\Gamma_1$ .So $O\subset O^{-_2 0_1}$ .Hence $O^{-_2}\subset O^{-_2 0_1-_2}$. This implies that $O^{-_2 0_1}\subset O^{-_2 0_1-_2 0_1}$ .Therefore $O^{-_2 0_1-_2 0_1}=O^{-_2 0_1}$.

Theorem(2.4):

Let (X,Γ_1,Γ_2) be a bitopological space and $B\subset X$. Then $B\in R.O.(X)$ if there exist $A\in R.O.(X)$ such that $A\subset B\subset A^{-_2 0_1}$.

Proof: Obvious.

Theorem(2.5):

Let (X,Γ_1,Γ_2) be a bitopological space. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of R –open sets in a bitopological space X . Then UA_α is R –open.

Proof :

For each $\alpha\in\Lambda$,we have Γ_1 –open O_α such that $O_\alpha\subset A_\alpha\subset O_\alpha{}^{-_2 0_1}$ .Then $\bigcup_{\alpha\in\Lambda}O_\alpha\subset\bigcup_{\alpha\in\Lambda}A_\alpha\subset\bigcup_{\alpha\in\Lambda}O_\alpha{}^{-_2 0_1}\subset\overset{-_2 0_1}{\bigcup_{\alpha\in\Lambda}}O_\alpha$ .Hence ,let $O=\bigcup_{\alpha\in\Lambda}O_\alpha$ .then $O\subset\bigcup A_\alpha\subset O^{-_2 0_1}$.

Theorem(2.6):

Let (X,Γ_1,Γ_2) be bitopological space. Then $\Gamma_1=\operatorname{int}_1 R.O.(X)$ where $\operatorname{int}_1 R.O.(X)=\{O:O=A^{o_1}\}$ for some $A\in R.O.(X)$.

Proof: Obvious.

Theorem(2.7):

Let $A\subset Y\subset X$ where X is a bitopological space and Y is a subspace of bitopological space (X,Γ_1,Γ_2). Let $A\in R.O.(X)$ then $A\in R.O.(Y)$.

Proof:

$O\subset A\subset O^{-_2 0_1}$, where O is T_1 –open in X . now $O\subset Y$ and thus $O=O\bigcap Y\subset A\bigcap Y\subset Y\bigcap O^{-_2 0_1}\subset O^{-_{2Y}0_{1Y}}$ or $O\subset A\subset O^{-_{2Y}0_{1Y}}$ since $O=O\bigcap Y$, O is Γ_1/Y –open in Y and the theorem is proved.

We introduce the following definitions of the R –neighbourhood, R -derived, R -closure and R –interior of a set which is similar to that of standard neighbourhood ,derived closure and interior .

Definition(2.8):

A Set $N_x\subset X$ is said to be R –neighbourhood of a point $x\in X$ if there exists a $A\in R.O.(X)$ such that $x\in A\subset N_x$.

Theorem(2.9):

$A\in R.O.(X)$ iff A is R –neighbourhood of each $x\in A$.

Proof: Obvious.

Definition(2.10):

A point $x\in X$ is said to be R –limit point of A iff for each $U\in R.O.(X)$, $x\in U$ and $U-\{x\}\bigcap A\neq\phi$. The set of all R –limit points of A is said to be R –derived set of A and is denoted by $R-der(A)$.

The following theorems give some properties of R –limit points.

Theorem(2.11):

A is R –closed if f it contains the set of its R –limit points.

Proof: Obvious.

Theorem(2.12):

If A, B by subsets of bitopological space (X, Γ_1, Γ_2) .Then:

1) If $A \subset B$, then $R - der(A) \subset R - der(B)$.

2) $R - der(A) \bigcup R - der(B) \subset R - der(A \bigcup B)$.

3) $R - der(A \bigcap B) \subset R - der(A) \bigcap R - der(B)$.

4) $R - der(R - der(A)) / A \subset R - der(A)$.

5) $R - der(A \bigcup R - der(A)) \subset A \bigcup R - der(A)$.

Proof :

We prove parts (5) and the others follow directly from definition.

5) Let $x \in R - der(A \bigcup R - der(A))$. If $x \in A$, the result is obvious. So let $x \in R - der(A)/A$. Then, if $U \in R.O.(X)$ containing x , $U - \{x\} \bigcap (A \bigcup R - der(A)) \neq \phi$, then $U - \{x\} \bigcap A \neq \phi$ or $U - \{x\} \bigcap R - der(A) \neq \phi$. If $U - \{x\} \bigcap A \neq \phi$, then $x \in R - der(A)$. If $U - \{x\} \bigcap R - der(A) \neq \phi$ then $U - \{x\} \bigcap A \neq \phi$. Therefore $x \in R - der(A)$.

Thus in any case $R - der(A \bigcup R - der(A)) \subset A \bigcup R - der(A)$.

Definition(2.13):

Let A be a subset of a bitopological space (X, Γ_1, Γ_2) , $A \bigcup R - der(A)$ is defined to be the R –closure of A and is denoted by $R - cl(A)$.

Theorem(2.14):

Let (X, Γ_1, Γ_2) be a bitopological space. Let A and B be two subsets of X . Then:

1) If $A \subset B$, then $R - cl(A) \subset R - cl(B)$.

2) $R - cl(A) \bigcup R - cl(B) \subset R - cl(A \bigcup B)$.

3) $R - cl(A \bigcap B) \subset R - cl(A) \bigcap R - cl(B)$.

4) $R - cl(R - cl(A)) = R - cl(A)$.

5) A is a R –closed iff $R - cl(A) = A$, $R - cl(A)$ is R -closed .

6) $R - cl(A) = \bigcap \{ F, F$ is R –closed and $A \subset F \}$, $R - cl(A)$ is the smallest R –closed set containing A .

Proof:

We prove parts (4) and the others follow directly from definition.

4) $R - cl(R - cl(A)) = R - cl(A \bigcup R - der(A))$.

$= (A \bigcup R - der(A)) \bigcup R - der(A \bigcup R - der(A)) = A \bigcup R - der(A) = R - cl(A)$.

Definition(2.15):

A point $x \in X$ is said to be a R –interior point of A if f there exist $U \in R.O.(X)$ containing x , such that $U \subset A$. The set of all R –interior of A is said to be the R –interior of A and is denoted by $R - \text{int}(A)$.

The following theorem gives some properties of R –interior sets.

Theorem(2.16):

Let (X,Γ_1,Γ_2) be a bitopological space. Let A and B be two subsets of X . Then:

1) $R-\text{int}$ is R -open.
2) $R-\text{int}(A)$)is the largest R -open set contained in A .
3) A is R -open if f $A=R-\text{int}(A)$.
4) $R-\text{int}(R-\text{int}(A))=R-\text{int}(A)$.
5) If $A\subset B$ then $R-\text{int}(A)\subset R-\text{int}(B)$.
6) $R-\text{int}(A)\bigcup R-\text{int}(A)\subset R-\text{int}(A\bigcup B)$.
7) $R-\text{int}(A\bigcap B)\subset R-\text{int}(A)\bigcap R-\text{int}(B)$.

Proof:
We prove parts (1),(2) and the others follow directly from definition .
1) Let $x\in R-\text{int}(A)$ .then $U\subset A$ for some $U\in R.O.(X)$ containing x .Also $y\in U$,then $y\in R-\text{int}(A)$, therefore $x\in U\subset R-\text{int}(A)$.Hence $R-\text{int}(A)$ is R –neighbourhood of x .
There by theorem 2.9, $R-\text{int}(A)$ is R -open.
2) Let $V\in R.C.(X)$, $V\subset A$. Then $y\in V$ .implies that $y\in A$,so that $y\in R-\text{int}(A)$ .Therefore $V\subset R-\text{int}(A)$.

3- $R-T_i$ $spaces$, $i=0,1,2$:

In this section we introduce $(R-T_i)$spaces, $i=0,1,2$ and study some of their properties.

Definition(3.1):

Let (X,Γ_1,Γ_2) be a bitopological space .

a) $(X,\Gamma_1)$is called $R-T_0$ iff for each x , $y\in X$ such that $x\neq y$, there exists a R -open set in X containing exactly one of them.
b) $(X,\Gamma_1)$is called $R-T_1$ iff for each x , $y\in X$ such that $x\neq y$, there exists a R -open set V_1 containing x but not y and a R -open set V_2 containing y but not x .
c) $(X,\Gamma_1)$is called $R-T_2$ or R -Hausdoffiff for each x , $y\in X$ such that $x\neq y$, therefore exists disjoint R -open sets V_1 and V_2, $x\in V_1$ and $y\in V_2$.

The following theorems gives properties of $(R-T_i)$spaces, $i=0,1,2$.

Theorem(3.2):

Let (X,Γ_1,Γ_2) be a bitopological space.

1) If (X,Γ_1) is T_0 then $(X,\Gamma_1)$is $R-T_0$.
2)If (X,Γ_1) is T_1 then $(X,\Gamma_1)$is $R-T_1$.
3)If (X,Γ_1) is T_2 then $(X,\Gamma_1)$is $R-T_2$.

Proof:

We prove parts (3) and the others similar prove part (3). Suppose that (X,Γ_1) is T_2 and $x, y \in X$ such that $x \neq y$, then there exists disjoint Γ_1–open sets V_1 and V_2, $x \in V_1$ and $y \in V_2$. Now since every Γ_1–open is R-open. Then (X,Γ_1) is $R-T_2$.

Theorem(3.3):

Let (X,Γ_1,Γ_2) be a bitopological space.

1) If (X,Γ_1) is $R-T_1$ then (X,Γ_1) is $R-T_0$.

2) If (X,Γ_1) is $R-T_2$ then (X,Γ_1) is $R-T_1$.

Proof: Obvious.

Theorem(3.4):

Let (X,Γ_1,Γ_2) be a bitopological space .Then (X,Γ_1) is $R-T_1$ iff for each $x \in X$, $R-cl\{x\}=\{x\}$.

Proof :

Suppose that (X,Γ_1) is $R-T_1$, and suppose $y \in \{x\}^C$. Then $x \neq y$ and by $R-T_1$, then exist $V \in R.O.(X)$ such that $y \in V$ but $x \notin V$. Hence $y \in V \subset \{x\}^C$. Therefore by theorem 2.16 part (3) $\{x\}^C$ R-open. Thus $\{x\}$ R–closed by theorem 2.14 part (5), $R-cl\{x\}=\{x\}$ .The converse is clear.

Corollary(3.5):

Let (X,Γ_1,Γ_2) be a bitopological space. Then (X,Γ_1) is $R-T_1$ iff for each $x \in X$, $R-der\{x\}=\phi$.

Proof: Obvious.

Theorem(3.6):

Let (X,Γ_1,Γ_2) be a bitopological space. Then (X,Γ_1) is $R-T_2$ iff for each $x, y \in X$ such that $x \neq y$ there exists a R-open set V such that $x \in V$ and $y \notin R-cl(V)$.

Proof:

Let (X,Γ_1) is $R-T_2$ and $x, y \in X$ such that $x \neq y$. Then there exist disjoint R-open sets V and U such that $x \in V$ and $y \in U$. If $y \in R-cl(V)$ this contradicts that V and U disjoint. Hence $y \notin R-cl(V)$ .the converse is clear .

Theorem(3.7) [2]:

Let (X,Γ) be a topological space and $A \in \Gamma$ .Then $A \cap \overline{B} \subset \overline{A \cap B}$ for every subset B of X .

Theorem(3.8):

Let (X,Γ_1,Γ_2) be a bitopological space. Let $V \in R.O.(X)$ and $Y \in \Gamma_2$ .then $V \cap Y \in R.O.(Y)$.

Proof :

$O \subset V \subset O^{-_2 0_1}$, where O is Γ_1–open in X .Thus $O \cap Y \subset V \cap Y \subset O^{-_2 0_1} \cap Y$.

Now $V \cap Y \subset O^{-_2} \cap Y$ and by theorem 3.7 $V \cap Y \subset \overline{O \cap Y}^{\,2}$ .Hence $V \cap Y \subset \overline{O \cap Y}^{\,2Y}$. (where $(-_{2Y})$ is the Γ_2–closure operator Y). It follow that $O \cap Y \subset V \cap Y \subset \overline{O \cap Y}^{\,2Y\, 0_{1Y}}$ (where (0_{1Y}) is the Γ_1–interior operator in Y).Therefore $V \cap Y \in R.O.(Y)$.

Theorem(3.9):

Let (X,Γ_1,Γ_2) be a bitopological space. Then

1) Every Γ_2–open subspace of $R-T_0$ space is $R-T_0$.

2)Every Γ_2–open subspace of $R-T_1$ space is $R-T_1$.

3)Every Γ_2–open subspace of $R-T_2$ space is $R-T_2$.

Proof:

We prove part (3) and the others similar prove (3).

3)Let (X,Γ_1) be $R-T_2$. Let x and y by two points of the Γ_2 –open subspace Y of X . If V_1 and V_2 are disjoint R–open sets in X of x and y , respectively, then by theorem 3.8 $V_1\bigcap Y$ and $V_2\bigcap Y$ are disjoint R–open sets of x and in Y .

4- $(R_0)_R$ –SPACES

In this section we introduce $(R_0)_R$ –space and study some of their properties.

Definition(4.1):

Let (X,Γ_1,Γ_2) be a bitopological space. (X,Γ_1) Is called $(R_0)_R$ if for each $V\in R.O.(X)$ and each $x\in V$, $R-cl\{x\}\subset V$.

Since a space (X,Γ_1) is $R-T_1$ iff the singletons are R–closed (theorem 3.4), it is clear that every $R-T_1$ space is $(R_0)_R$.

Definition(4.2) [3]:

A Topological space (X,Γ) is called (R_0) iff for each $U\in T$ and each $x\in U$, $\overline{\{x\}}\subset U$.

Theorem(4.3):

Let (X,Γ_1,Γ_2) be a bitopological space if (X,Γ_1) is R_0, then (X,Γ_1) is $(R_0)_R$.

Proof: Obvious.

Definition(4.4) [5]:

A Topological space (X,Γ) is called $(R_0)_S$ iff for each semi open U and each $x\in U$, $scl\{x\}\subset U$, where $scl\{x\}$ denotes the semi closure of $\{x\}$.

Definition(4.5):

Let (X,Γ_1,Γ_2) be a bitopologicalspace, the R of X is defined to be the set $R-Ker\{x\}=\{y:x\in R-cl\{y\}\}$.

Theorem(4.6):

Let (X,Γ_1,Γ_2) be a bitopological space, then (X,Γ_1) is $R-T_1$ iff it is $R-T_0$ and $(R_0)_R$.

Proof:

Suppose that (X,Γ_1) is $R-T_0$ and $(R_0)_R$. Let x , $y\in X$ such that $x\neq y$. Since (X,Γ_1) is $R-T_0$ it follow that there exist R–open set V such that $x\in V$ and $y\notin V$ and since (X,Γ_1) is $(R_0)_R$ it follow that $R-cl\{x\}\subset V$. Hence $y\in(R-cl\{x\})^c$ .Therefore (X,Γ_1) is $R-T_1$. The converse is clear.

Theorem(4.7):

Let (X,Γ_1,Γ_2) be a bitopologicalspace, then (X,Γ_1) is $(R_0)_R$ iff for every R–closed set F and $x \notin F$, there exists a R–open set V such that $F \subset V$, $x \notin V$.

Proof:

Obvious.

Definition(4.8) [4]:

Let (X,Γ) be a bitopological space. A Subset A of X is called semi open iff for some O, $O \subset A \subset \overline{O}$, where $\overline{O}$ denotes the closure of O.

Definition(4.9) [5]:

Let (X,Γ) be a bitopologicalspace, the semi of X is defined to be the $sKer\{x\} = \{y : x \in scl\{y\}\}$ where $scl\{y\}$ denotes the semi closure of $\{y\}$.

Theorem(4.10):

Let (X,Γ_1,Γ_2) be a bitopologicalspace. Then for x, $y \in X$, $R-Ker\{x\} \neq R-Ker\{y\}$ iff $R-cl\{x\} \neq R-cl\{y\}$.

Proof: Obvious.

Theorem(4.11):

Let (X,Γ_1,Γ_2) be a bitopological space, the following are equivalent:

a) (X,Γ_1) is $(R_0)_R$.

b) For every $x \in X$, $R-cl\{x\} \subset R-Ker\{x\}$.

c) If F is R–closed in X, then $F = \bigcap \{V : V$ is R–open, $F \subset V\}$.

d) If V is R–open in X, then $V = U\{F : F$ is R–closed, $F \subset V\}$.

e) For any nonempty set A and R–open set V in X such that $A \bigcap V \neq \phi$, there exists a R–closed set F for which $F \subset V$ and $A \bigcap F \neq \phi$.

f) For any R–closed set F in X and $x \notin F$. $R-cl\{x\} \bigcap F = \phi$.

Proof:

(a)$\rightarrow$ (b): Let $y \in R-cl\{x\}$.let V be any R–open set $x \in V$. Now by (a), $y \in V$. This gives that $x \in R-cl(y)$. Therefor $y \in R-Ker(x)$.

(b)$\rightarrow$(c): Suppose x does not belong to the R–closed F. And F^c is R–open and $x \in F^c$. Let $y \in R-cl\{x\}$. Then by (b) $x \in R-cl\{y\}$. Therefore, every R–open set which contains x, contains y .Hence, $R-cl\{x\} \subset F^c$. Now $(R-cl\{x\})^c$ R–openset containing F to which x does not belong. Consequently. x does not belong to the intersection of all the R–open sets which contain F .Thus (c) hold

(c)$\rightarrow$(d): Let V is R–open sets in X, then V^c is R–closedset. Now by (c),

$V^c = \bigcap\{u : u$ is R–open, $V^c \subset u\}$.Therefore $V = U\{u^c : u^c$ is R–closed, $u^c \subset V\}$.

(d)$\rightarrow$ (e): Let V be R–open and A is non-empty such that $A \bigcap V \neq \phi$. Let $x \in A \bigcap V$ .by (d) there exists a R–closed F such that $x \in F \subset V$.clearly, $A \bigcap F \neq \phi$.

(e) $\rightarrow$ (f): Let F be a R –closed set and $x \notin F$. Then F^c is R –open and $\{x\} \cap F^c = \phi$. By (e), there exists a R –closed set H such that $H \cap \{x\} \neq \phi$ and $H \subset F^c$. Therefore $R-cl\{x\} \subset F^c$ .consequently, $F \cap R-cl\{x\} = \phi$.

(f) $\rightarrow$ (a): Let F be R –closed set in X and $x \notin F$, $R-cl\{x\} \cap F = \phi$. The $(R-cl\{x\})^c$ is R –open set such that $F \subset (R-cl\{x\})^c$, $x \notin (R-cl\{x\})^c$,therefore by theorem 4.4 (X,Γ_1) is $(R_0)_R$.

Theorem(4.12):

Let (X,Γ_1,Γ_2) be bitopological space if for every point x of a $(R_0)_R$ space (X,Γ_1), $R-cl\{x\} \cap R-\ker\{x\} = \{x\}$,then $R-cl\{x\} = \{x\}$.

Proof: Obvious.

Theorem(4.13):

Let (X,Γ_1,Γ_2) be bitopological space if (X,Γ_1) is an $(R_0)_R$ space and x , $y \in X$, then $R-cl\{x\} = R-cl\{y\}$ or $R-cl\{x\} \cap R-cl\{y\} = \phi$.

Proof:

Suppose $R-cl\{x\} \cap R-cl\{y\} \neq \phi$ .Let $a \in R-cl\{x\} \cap R-cl\{y\}$ .Then $R-cl\{a\} \subset R-cl\{x\} \cap R-cl\{y\}$.

Now by theorem (2.7) part (b)

$a \in R-cl\{x\} \rightarrow a \in R-\ker\{x\}$

$\rightarrow x \in R-cl\{a\}$

$\rightarrow R-cl\{x\} \subset R-cl\{a\}$

$\rightarrow R-cl\{x\} \subset R-cl\{y\}$

Similarly

$a \in R-cl\{y\} \rightarrow R-cl\{y\} \subset R-cl\{x\}$

Consequently, $R-cl\{x\} = R-cl\{y\}$.

Theorem(4.14):

Let (X,Γ_1,Γ_2) be bitopological space and Y is Γ_2 –open. Then $R-cl_Y A \subset R-cl_X A$ (where $R-cl_Y A$ is the R –closure operator in Y)

Theorem(4.15):

Let (X,Γ_1,Γ_2) be bitopological space .then every Γ_1 –open and Γ_2 –open subspace of $(R_0)_R$ space is $(R_0)_R$.

Proof:

Let Y Γ_1 –open and Γ_2 –open subspace of $(R_0)_R$ space (X,Γ_1) and $A \in R.O.(Y)$ and $x \in A$. Now to prove that there is $B \in R.O.(x) \ni A = B \cap Y$.

Since $A \in R.O.(Y)$,then there is an Γ_1/Y open set U in Y such that $U \subset A \subset \overline{U}^{2Y\,o_{1Y}}$ also there is $W \in \Gamma_1$ such that $W \cap Y = U$. Let $B = A \cup W$ .then $B \cap Y = A$. Now, we show that $W \subset B \subset \overline{W}^{2X\,o_{1X}}$.

Obviously $W \subset B$. Let $t \in B$ .Then $t \in W$ or $t \in A$ .If $t \in W$ then $t \in \overline{W}^{2X\,o_{1X}}$ .If $t \in A$, $t \in \overline{U}^{2Y\,o_{1Y}}$ and since Y Γ_1 –open set so $t \in \overline{W}^{2X\,o_{1X}}$ .Therefore $B \in R.O.(X)$. Again, to prove that $(Y,\Gamma_1/Y)$ $(R_0)_R$ space .

Since $A = B \cap Y$,so $x \in B$ and $x \in Y$. Hence $R - cl_x\{x\} \subset B$ and $R - cl_x\{x\} \subset Y$ .It follows that $R - cl_x\{x\} \subset A$. Then by theorem 2.14 $R - cl_Y\{x\} \subset A$ .Therefore $(Y, \Gamma_1/Y)$ is $(R_0)_R$.

5-EXAMPLES

In this section we shall that the converse of the above theorems not true.

Remark(5.1):

The family of all R –open sets in bitopological space was not necessary topology on X as shown by the following example .

Example(5.2):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}, \Gamma_2 = \{\phi, X, \{2\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(X) = \{\phi, X, \{1\}, \{2\}, \{1,3\}, \{2,3\}, \{1,2\}\}$.

Remark(5.3):

The reverse inclusion in the theorem 2.12 parts (2),(3) not true as shown by the following examples .

Example(5.4):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}, \Gamma_2 = \{\phi, X, \{1,2\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(X) = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ .take $A = \{2\}, B = \{1,3\}$ .Then $R - der(A) = \phi, R - der(B) = \phi$ .Also $R - der(A \cup B) = R - der(\{1,2,3\}) = \{3\} \not\subset R - der(A) \cup R - der(B) = \phi$.

Example(5.5):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}, \Gamma_2 = \{\phi, X, \{1,2\}, \{3\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(X) = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Take $A = \{1,3\}, B = \{2,3\}$. Then $R - der(A) = \{3\}, R - der(B) = \{3\}$ .Therefore $R - der(A) \cap R - der(B) = \{3\} \not\subset R - der(A \cap B) = \phi$.

Remark(5.6):

The reverse inclusion in theorem 2.14 parts (2),(3) are not true as shown by the following example .

Example(5.7):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2,3\}\}, \Gamma_2 = \{X, \phi, \{1,2\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(X) = \{X, \phi, \{1\}, \{2,3\}, \{1,3\}, \{1,2\}\}$. Take $A = \{1\}, B = \{2\}$ .Then $R - cl(A) = \{1\}, R - cl(B) = \{2\}$ .Therefore $R - cl(A \cup B) = R - cl(\{1,2\}) = X \not\subset R - cl(A) \cup R - cl(B) = \{1,2\}$. Take $A = \{2,3\}, B = \{1,2\}$ .The $R - cl(A) = \{2,3\}, R - cl(B) = \{1,2\}$ and $A \cap B = \{2\}, R - cl(A \cap B) = \{2\}$ .Therefore $R - cl(A) \cap R - cl(B) = \{2,3\} \not\subset R - cl(A \cap B) = \{2\}$.

Remark(5.8):

It is obvious that every Γ_1 –open is R –open but the converse is not true as shown by the following example.

Example(5.9):

Consider (X, Γ_1, Γ_2) defined in example 5.4 take $A = \{1,3\}$ is R –open set but not Γ_1 –open .

Remark(5.10):

The reverse inclusion in theorem 2.16 parts 6, 7 are not true as shown by the following example.

Example(5.11):

Consider (X, Γ_1, Γ_2) defined in example 5.4 take $A = \{3\} B = \{2\}$.

Then $R - \text{int}(A) = \phi, R - \text{int}(B) = \{2\}$, but $R - \text{int}(A \cup B) = \{2,3\}$ .Therefore $R - \text{int}(A \cup B) = \{2,3\} \not\subset R - \text{int}(A) \cup R - \text{int}(B) = \{2\}$. Consider (X, Γ_1, Γ_2) defined in example 5.7 ,take $A = \{1,3\} B = \{2,3\}$ .Then $R - \text{int}(A) = \{1,3\}, R - \text{int}(B) = \{2,3\}$,but $R - \text{int}(A \cap B) = \phi$ .Therefore $R - \text{int}(A) \cap R - \text{int}(B) = \{3\} \not\subset R - \text{int}(A \cap B) = \phi$.

Remark(5.12):

The converse of theorem 3.2 are not true as shown by the following examples .

Example(5.13):

Let $X=\{1,2,3\}$ and $\Gamma_1=\{X,\phi,\{1\}\},\Gamma_2=\{X,\phi,\{1,3\}\}$. Then it can be verified that (X,Γ_1,Γ_2) is bitopological space and $R.O.(x)=\{X,\phi,\{1\},\{1,2\},\{1,3\}\}$.Therefore (X,Γ_1) is a $R-T_0$ space, but it is not T_0.

Example(5.14):

Let $X=\{1,2,3\}$ and $\Gamma_1=\{X,\phi,\{1\},\{2\},\{1,2\},\{2,3\}\},\Gamma_2=\{X,\phi,\{1,3\}\}$. Then it can be verified that (X,Γ_1,Γ_2) is bitopological space and $R.O.(x)=\{X,\phi,\{1\},\{2\},\{1,2\},\{2,3\},\{1,3\}\}$. Therefore (X,Γ_1) is a $R-T_1$ space, but it is not T_1 and (X,Γ_1) is $R-T_2$ space, but it is not T_2 .

Remark(5.15):

The converses theorem 3.3 is not true as shown by the following examples.

Example(5.16):

Consider (X,Γ_1,Γ_2) defined in examples 5.13 (X,Γ_1) is a $R-T_0$ space, but it is not $R-T_1$.

Example(5.17):

Let X be infinite set and $\Gamma_1=$ confinite topology on X , $\Gamma_2=$ discrete topology on X . Then it can be verified that (X,Γ_1,Γ_2) bitopological space and $R.O.(x)=\Gamma_1$. Therefore (X,Γ_1) is a $R-T_1$ space but is not $R-T_2$.

Remark(5.18):

The property of $R-T_0$, $R-T_1$ and $R-T_2$ are not hereditary property as shown by the following .

Example(5.19):

Let $X=\{1,2,3,4\}$ and $\Gamma_1=\{X,\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{2.4\},\{1,2,4\},\{2,3,4\}\},\Gamma_2=\{X,\phi,\{1,3\},\{1,2,4\},\{1\}\}$. Then it can be verified that (X,Γ_1,Γ_2) is bitopological space and $R.o.(x)=\{X,\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{2,4\},\{1,2,4\},\{2,3,4\},\{1,3,4\},\{1,4\}\}$. Therefore (X,Γ_1) is $R-T_1$ and $R-T_2$. Now let $Y=\{2,3,4\}$ is Γ_1 –open subspace of (X,Γ_1) . $\Gamma_1/Y=\{Y,\phi,\{2\},\{3\},\{2,3\},\{2,4\}\}$, $\Gamma_2/Y=\{Y,\phi,\{3\},\{2,4\}\}$.Then it can be verified (Y,Γ_1,Γ_2)bitopological space and $R.o.(Y)=\{Y,\phi,\{2\},\{3\},\{2,3\},\{2,4\}\}$. Therefore $(Y,\Gamma_1/Y)$ is not $R-T_1$ and also not $R-T_2$.

Example(5.20):

Let $X=\{1,2,3,4\}$ and $\Gamma_1=\{X,\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\},\Gamma_2=\{X,\phi,\{1,2,4\}\}$. Then it can be verified that (X,Γ_1,Γ_2) is bitopological space and $R.o.(x)=\{X,\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{1,4\},\{1,3,4\},\{1,2,4\},\{2,4\}[2,3,4]\}$. Therefore (X,Γ_1) is $R-T_1$ and $R-T_2$. Now, let $Y=\{2,3,4\}$ is subspace of (X,Γ_1), then $R.o.(Y)=\{Y,\phi,\{2\},\{3\},\{2,3\},\{2,4\}\}$. Therefore $(Y,\Gamma_1/Y)$ is not $R-T_1$ and also not $R-T_2$.

Example(5.21):

Let $X=\{1,2,3\}$ and $\Gamma_1=\{X,\phi,\{1\},\{2,3\}\},\Gamma_2=\{X,\phi,\{1,3\}\}$. Then it can be verified that (X,Γ_1,Γ_2) is bitopological space and $R.o.(x)=\{X,\phi,\{1\},\{1,2\},\{1,3\},\{2,3\}\}$. Therefore (X,Γ_1) is $R-T_0$. Now, let $Y=\{2,3\}$ is subspace of (X,Γ_1) .then $\Gamma_1/Y=\{Y,\phi\},\Gamma_2/Y=\{Y,\phi,\{3\}\}$ and it can be verified that $(Y,\Gamma_1/Y,\Gamma_2/Y)$ is bitopological space and $R.o.(Y)=\{Y,\phi\}$. Therefore $(Y,\Gamma_1/Y)$ is not $R-T_0$.

Remark(5.22):

The axiom of $R-T_0$ and $(R_0)_R$ are independent as shown by the following examples.

Example(5.23):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2,3\}\}, \Gamma_2 = \{X, \phi, \{2,3\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.o.(x) = (X, \phi, \{1\}, \{2,3\})$.Therefore (X, Γ_1) is a $(R_0)_R$, but it is not $R - T_0$.

Example(5.24):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}, \Gamma_2 = \{X, \phi, \{3\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.o.(x) = (X, \phi, \{1\}, \{2\}, \{1,2\})$ .Therefore $(X, \Gamma_1)$is $R - T_0$,but it is not $(R_0)_R$.

Remark(5.25):

It is obvious that every space $R - T_1$ is $(R_0)_R$, but the converse is not true as shown by the following example.

Example(5.26):

Consider (X, Γ_1, Γ_2) defined in example 5.23 (X, Γ_1) is a $(R_0)_R$,but it is not $R - T_1$.

Remark(5.27):

The axiom of $(R_0)_s$ and $(R_0)_R$ are independent as shown by the following examples.

Example(5.28)

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}, \Gamma_2 = \{X, \phi, \{3\}\}$.then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(x) = \{X, \phi, \{1\}, \{1,2\}\}$.

Therefore $(X, \Gamma_1)$is $(R_0)_S$,but it is not $(R_0)_R$.

Example(5.29):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{1,2\}, \{2,3\}, \{2\}\}, \Gamma_2 = \{X, \phi, \{1,3\}\}$. Then $R.O.(X) = \{X, \phi, \{1\}, \{1,2\}, \{2,3\}, \{2\}, \{1,3\}\}$. Therefore $(X, \Gamma_1)$is $(R_0)_R$,but it is not $(R_0)_s$.

Remark(5.30):

The axiom of $s\ker\{x\}$ and $R - \ker\{x\}$ are independent as shown by the following examples .

Example(5.31):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2,3\}\}, \Gamma_2 = \{X, \phi, \{2,3\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(X) = \{X, \phi, \{1\}, \{2,3\}, \{1,2\}, \{1,3\}\}$. Therefore $s\ker\{2\} = \{2,3\} \neq R - \ker\{2\} = \{2\}$.

Example(5.32):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}, \Gamma_2 = \{X, \phi, \{2,3\}, \{3\}\}$. Then it can be verified that (X, Γ_1, Γ_2) is bitopological space and $R.O.(X) = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$. Therefore $R - \ker\{3\} = \{1,2,3\} = X \neq s\ker\{3\} = \{3\}$.

Remark(5.33):

The converse of theorem 4.3 are not true as shown by the following example.

Example(5.34):

Consider (X, Γ_1, Γ_2) defined in example 5.29 (X, Γ_1) is $(R_0)_R$, but it is not (R_0).

Remark(5.35):

The property of $(R_0)_R$ is not hereditary property as shown by the following examples.

Example(5.36):

Let $X = \{1,2,3\}$ and $\Gamma_1 = \{X, \phi, \{2,3\}\}, \{3\}, \{1\}, \{1,3\}\}, \Gamma_2 = \{X, \phi, \{1,3\}\}$.

Then $R.O.(X) = \{X, \phi, \{2,3\}, \{3\}, \{1\}, \{1,3\}, \{1,2\}\}$. Therefore $(X, \Gamma_1)$is $(R_0)_R$. Now, let $Y = \{2,3\}$ is subspace of (X, Γ_1), then $R.O.(Y) = \{Y, \phi, \{3\}\}$ .therefore $(Y, \Gamma_1 / Y)$ is not $(R_0)_R$.

Example(5.37):

Let $X=\{1,2,3,4\}$ and $\Gamma_1=\{X,\phi,\{1,2\},\{3,4\},\{1,2,4\},\{4\}\},\Gamma_2=\{X,\phi,\{1,3\},\{1,2,3\},\{1,3,4\}\}$ .Then $R.O.(X)=\{X,\phi,\{1,2\},\{3,4\},\{1,2,4\},\{4\},\{1,2,3\},\{1,2,4\},\{2,3,4\},\{1,3,4\}\}$. Therefore (X,Γ_1) is $(R_0)_R$. Now let $Y=\{2,3,4\}$ is subspace of (X,Γ_1) .

Then $\Gamma_1/Y=\{\phi,Y,\{2\},\{3,4\},\{2,4\},\{4\}\},\Gamma_2/Y=\{\phi,Y,\{3\},\{2,3\},\{3,4\}\}$ and it can be verified that $(Y,\Gamma_1/Y,\Gamma_2/Y)$ is bitopological space and $R.O.(Y)=\{\phi,Y,\{2\},\{3,4\},\{2,4\},\{4\}\}$. Therefore $(Y,\Gamma_1/Y)$ is not $(R_0)_R$.

REFERENCES

1. Kelly JC. Bitopological spaces proc. London Math. 1963;13:71-89.
2. Bcurbaki. General Topology Addision Wesley puplishing co. 1966;part 1.
3. Davis AS. Indexed systems of neighborhoods for general topological spaces. The American Mathematical Monthly. 1961 Nov 1;68(9):886-93.
4. Levine N. Semi-open sets and semi-continuity in topological spaces. The American Mathematical Monthly. 1963 Jan 1;70(1):36-41.
5. Maheshwar SN. On $(R0)_S$ -spaces partugaliaeMathematics. 1975;34:213-217 .