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Infinite-Dimensional Conservation Law and Liouville Integrability of Free Vibration Equation of a Beam

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Abstract: According to the infinite-dimensional Liouville theorem, we give the infinite-dimensional conservation law and the Liouville integrability of a forth order free vibration equation of a beam by two cases including discrete spectrum and continuous spectrum. This equation can be considered as the infinite-dimensional Neumann models without any constraints.
Keywords: Hamilton-Jacobi theory, Liouville integrability, beam vibration equation, infinite-dimensional Neumann model

INTRODUCTION

It is well-known that the finite-dimensional Liouville theorem [5] means that if there exist n independent first integrals in evolution, and the liouville integrability is just based on the Liouville theorem [1]. For a given infinite dimensional Hamilton system, a necessary condition to make such system integrable is that it has an infinity number of first integrals. However, Calogero [2] pointed out that due to the ambiguities in the counting of infinities, this condition is not sufficient. A natural problem is how many constants of motion are sufficient to ensure that such system is solvable. In [3], Liu proved an infinite-dimensional liouville theorem based on the infinite-dimensional Hamilton-Jacobi theory, and gave a definition of the infinite-dimensional Liouville integrability. In some degree, Liu's theorems and definitions clarify some relations between the first integrals and solvability of infinite-dimensional Hamitonian systems. As example, Liu [3] discussed the second order wave equation and Neumann model. For other respects of integrable systems, we can see the Refs [4-7].

In the present paper, our aim is to study a model of the forth order vibration equation of a beam, and obtain its infinite-dimensional Liouville integrability. We discuss the problem by two cases: one is to consider the discrete spectrum, another is continuous spectrum. We construct the complete set of first integrals and prove that these Hamitonian systems are the Liouville integrable.

INFINITE-DIMENSIONAL LIOUVILLE THEOREM AND LIOUVILLE INTEGRABILITY

Consider the case of countably infinite variables. $P=(p_1,\cdots,p_n,\cdots)$ and $Q=(q_1,\cdots,q_n,\cdots)$ are a pair of canonical variables. Here, $H=H(P,Q,t)$ is the Hamilton function. The Hamilton canonical equations are given by

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}, \tag{1}$$

$$\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}, \tag{2}$$

for $i=1,2,\cdots$. S denotes the action function which takes its value on the classical path. We have $p_i=\frac{\partial S}{\partial q_i}$ for $i=1,2,\cdots$ and denote them by $P=\frac{\partial S}{\partial Q}$ for simplicity. We write the Hamilton-Jacobi equation as follows

$$\frac{\partial S}{\partial t}=-H(Q,\frac{\partial S}{\partial Q},t). \tag{3}$$

If there exists a general integral $S = S(Q,\alpha)$ for the H-J equation, where $\alpha = (\alpha_1, \alpha_2, \cdots)$, we can solve the Hamilton canonical equation. A crucial step is to solve out $Q = Q(t,\alpha,\beta)$ from the following system of equations

$$\frac{\partial S}{\partial \alpha_i} = \beta_i, \tag{4}$$

for $i = 1,2,\cdots$, where $\beta = (\beta_1, \beta_2, \cdots)$. In the finite dimensional case, this condition can be represented as $\det \frac{\partial^2 S}{\partial q_i \alpha_j} \neq 0$. In the infinite dimensional case, we use the invertible property of the operator $\frac{\partial^2 S}{\partial q_i \alpha_j}$ instead of $\det \frac{\partial^2 S}{\partial q_i \alpha_j} \neq 0$ (see,[3]). Liu[3] proved the following results.

Theorem 1[3]. If the operator $\frac{\partial^2 S}{\partial q_i \alpha_j}$ is invertible,

$$Q = Q(t,\alpha,\beta), \tag{5}$$

$$P = P(t,\alpha,\beta), \tag{6}$$

are the solutions of the Hamilton canonical equations (1) and (2).

Theorem 2[3]. Suppose that the Hamilton system has an infinite number of first integrals (or motion constants)

$$f_i(P,Q,t) = \alpha_i, i = 1,2,\cdots. \tag{7}$$

If these first integrals satisfy the following conditions, the Hamilton system is integrable.

1^0. $[f_i, f_j] = 0$, where $[f_i, f_j] = \sum_{k=1}^{+\infty} (\frac{\partial f_i}{\partial q_k}\frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\frac{\partial f_j}{\partial q_k})$ is the Poisson bracket.

2^0. The operator $(\frac{\partial f_i}{\partial p_j})$ is invertible, where $(\frac{\partial f_i}{\partial p_j})$ denotes the infinite-dimensional matrix with the general element $\frac{\partial f_i}{\partial p_j}$.

Based on the above theorem, Liu[3] gave the following definitions.

Definition 1. An infinite number of motion constants (or first integrals) (13) is called a complete set of motion constants if the condition 2^0 is satisfied.

Definition 2. If a Hamilton system has a complete set of motion constants, the system is called to possess the Liouville integrability or to be Liouville integrable.

THE LIOUVILLE INTEGRABILITY OF THE FREE VIBRATION EQUATION OF A BEAM: DISCRETE SPECTRUM

We consider free vibrations of a beam. When the influence of the dynamical axial force $\frac{ES}{2l} w_{xx} \int_0^l w_x^2 dx$ is neglected, the governing equation of free vibration is given by

$$ELw_{xxxx} + \rho S w_{tt} = 0, \tag{8}$$

where w is the lateral displacement, E is the Young's modulus, ρ is the density of the beam, S is the area of the cross section, and I is the moment of inertia of the cross section. Furthermore, by re-scaling of t, x and w, we get a forth order equation (see, for example, [7,8])

$$u_{tt} + u_{xxxx} = 0, \tag{9}$$

with the corresponding initial and boundary conditions

$$u(0,t) = u(2\pi,t) = 0, \tag{10}$$

$$u(x,0) = \psi(t), u_t(0,x) = \varphi(t). \tag{11}$$

This is an infinite-dimensional problem, and the corresponding Lagrangian function and Hamilton function are

$$L=\frac{1}{2}\int_0^{2\pi}((u_t)^2-(u_{xx})^2)dx, \tag{12}$$

and

$$H=\frac{1}{2}\int_0^{2\pi}((u_t)^2+(u_{xx})^2)dx. \tag{13}$$

Let $q=u$ and $p=u_t$ be a pair of canonical variables. Then Hamiltonian function is given as

$$H(p,q)=\frac{1}{2}\int_0^{2\pi}(p^2+(q_{xx})^2)dx. \tag{14}$$

Therefore the Hamilton-Jacobi equation is given by

$$\frac{\partial S}{\partial t}=-\frac{1}{2}\int_0^{2\pi}\{(\frac{\delta S}{\delta q})^2+(q_{xx})^2\}dx \tag{15}$$

In order to use the method of the separation of variables, we take the Fourier transformation of u with respect to x,

$$u(x,t)=\sum_{n=1}^{+\infty}a_n(t)\sin(nx), \tag{16}$$

and then

$$u_t(x,t)=\sum_{n=1}^{+\infty}a'_n(t)\sin(nx). \tag{17}$$

Hence the Hamiltonian function becomes

$$H=\frac{1}{2}\sum_{n=1}^{+\infty}\{a'^2_n(t)+n^4a_n^2(t)\}. \tag{18}$$

Taking the action as

$$S(a_1(t),a_2(t),\cdots)=S_0(t)+\sum_{n=1}^{+\infty}S_n(a_n), \tag{19}$$

and substituting it into Hamilton-Jacobi equation yield

$$(\frac{dS_n}{da_n})^2+n^4a_n^2=E_n, n=1,2,\cdots, \tag{20}$$

where $E_n s$ are constants and satisfy the following condition

$$\sum_{n=1}^{+\infty}E_n=2E. \tag{21}$$

Solving Eq.(20), we get

$$S_n=\int\sqrt{E_n-n^4a_n^2}\,da_n. \tag{22}$$

According to the calculus, we can solve out the solutions of a_n, for $n=1,2,\cdots$. Hence we can use the Hamilton-Jacobi theory to solve the beam free vibration problem.

Next we obtain the L-integrability of the free vibration of a beam. We first give an infinite number of first integrals

$$f_n(u,u_t)=\frac{1}{2}n^4(\int_0^{2\pi}u(x,t)\sin(nx)dx)^2+\frac{1}{2}(\int_0^{2\pi}u_t(x,t)\sin(nx)dx)^2, \tag{23}$$

for $n=1,2,\cdots$. In fact, we have

$$\frac{d}{dt}f_n(u,u_t)=n^4\int_0^{2\pi}u(x,t)\sin(nx)dx\int_0^{2\pi}u_t(x,t)\sin(nx)dx \tag{24}$$

$$+\int_0^{2\pi}u_t(x,t)\sin(nx)dx\int_0^{2\pi}u_{tt}(x,t)\sin(nx)dx=0, \tag{25}$$

where we use $u_{tt}+u_{xxxx}=0$ and integration by part in last step. Rewriting the first integrals in terms of variables $a_n s$, we have

$$f_n=\frac{1}{2}(a'^2_n(t)+n^4a_n^2(t)), n=1,2,\cdots. \tag{26}$$

Therefore, every f_n is just the energy of the n th mode. T

Now we prove that these first integrals constitute a complete set. Indeed, in this case, the canonical variables are $q_n=a_n$ and $p_n=a'_n$. From the set of first integrals represented by

$$f_n=\frac{1}{2}(p_n^2(t)+n^4q_n^2(t)) \tag{27}$$

in terms of q_n and p_n, we can solve out the $p_n s$. It follows that this is a complete set. On the other hand, we have

$$\frac{\partial f_n}{\partial q_m}=\delta mn, \tag{28}$$

where δmn is the Dirac sign in infinite dimension, that is, the operator (matrix) $(\frac{\partial f_n}{\partial q_m})$ is invertible. It is easy to prove $[f_n, f_m]=0$. According to theorem 2, the beam free vibration problem is the infinite-dimensional liouville integrable. If we remove some first integrals in the set, for example, f_1, the set will be not complete, since we can't solve out p_1.

THE LIOUVILLE INTEGRABILITY OF THE FREE VIBRATION OF A BEAM: CONTINUOUS SPECTRUM

We consider the Cauchy problem for an infinite vibrating beam

$$u_{tt}+u_{xxxx}=0, \tag{29}$$

$$u(-\infty,t)=u(+\infty,t)=0, \tag{30}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x). \tag{31}$$

We take the Fourier transformation of $u(x,t)$ with respect to the variable x,

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a(y,t)\sin(xy)dy. \tag{32}$$

It is easy to prove that

$$f(y,t)=\frac{1}{2}\{a_t{}^2(y,t)+y^4a^2(y,t)\}, \tag{33}$$

is a first integral for every y, that is, $\frac{d}{dt}f(y,t)=0$.

Another form is

$$f(y,t)=\frac{1}{2}\{(\frac{1}{2\pi}\int_{-\infty}^{+\infty}u_t(x,t)\sin(xy)dx)^2+y^4(\frac{1}{2\pi}\int_{-\infty}^{+\infty}u(x,t)\sin(xy)dx)^2\}. \tag{34}$$

This first integral is just the energy of the y-th mode. They constitute a set of the first integrals with uncountably infinite elements.

DISCUSSION

By the infinite-dimensional Liouville theorem, we prove that the free vibration problems of a beam are infinite-dimensional Liouville integrable by two cases including discrete spectrum and continuous spectrum. For the infinite-dimensional Neumann models, as pointed out by Liu [3], if we consider the constraints on the solution u, we will deal with the infinite genus Riemann surfaces. Our models can be considered as the trivial infinite-dimensional Neumann models without any constraints.

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