DOI: 10.21276/sjpms.2017.4.1.1

Scholars Journal of Physics, Mathematics and Statistics **ISSN 2393-8056 (Print)**
Sch. J. Phys. Math. Stat. 2017; 4(1):1-9 **ISSN 2393-8064 (Online)**
©Scholars Academic and Scientific Publishers (SAS Publishers)
(An International Publisher for Academic and Scientific Resources)

The Analysis of an M/G/1 Retrial Queue with Two Vacation Policies (SWVI+MV)

Tao Li[1], Liyuan Zhang[2]
[1]School of Science, Shandong University of Technology, Zibo, China
[2]School of Business, Shandong University of Technology, Zibo, China

***Corresponding Author:**
Tao Li
Email: liltaot@126.com

Abstract: In this paper, we consider an M/G/1 retrial queue with two vacation policies which comprise single working vacation, vacation interruption and multiple vacations, denoted by SWVI+MV. Only the customer at the head of the orbit queue is allowed access to the server. When the orbit becomes empty at the end of each regular service period, the server goes for a working vacation during which the server continues to serve the customers with a slower rate. At a service completion instant in the working vacation period, if there are customers in the system at that moment, the server will come back to the normal busy period which means that vacation interruption happens. At the end of each working vacation, the server starts a new busy period if there are customers in the system. Otherwise, the server begins an ordinary vacation during which the server stops the service completely. If the system is empty at the instant of an ordinary vacation completion, the server takes another new ordinary vacation. Using the matrix-analytic method, we obtain the necessary and sufficient condition for the system to be stable. By applying the supplementary variable technique, we obtain the steady state joint distribution of the server and the number of customers in the orbit. Various interesting performance measures are also derived. Finally, some numerical examples are presented.
Keywords: Retrial, Working vacation, Vacation, Supplementary variable technique

INTRODUCTION

Retrial queueing systems are described by the feature that the arriving customers who find the server busy join the retrial orbit to try again for their requests. In the past years, retrial queueing systems with server vacations have attracted considerable attentions and successfully applied in manufacturing and production systems, service systems and communication systems. For more details we refer the readers to the surveys of Artalejo [1], Yang *et al.* [2], Lakshmi and Ramanath [3], Padmavathi *et al.* [4] and Jain and Bhagat [5].

On the basis of ordinary vacation, Servi and Finn [6] first introduced the concept of working vacation, where the server provides service at a lower speed during the vacation period rather than stopping service completely. Recently, retrial queueing system with working vacation has become an important aspect. Do [7] first studied an M/M/1 retrial queue with working vacations which is motivated by the performance analysis of a Media Access Control function in wireless networks. Using the matrix-analytic method, Liu and Song [8] introduced non-persistent customers into the discrete time Geo/Geo/1 retrial queue with working vacations. Using the method of a supplementary variable, Arivudainambi *et al.* [9] and Jailaxmi et al. [10] both generalized the model of [7] to an M/G/1 retrial queue with working vacation and constant retrial policy. In order to utilize the server effectively, Li and Tian [11] introduced working vacation interruption policy. During the working vacation period, if at least one customer is present in the system at a service completion epoch, the server will interrupt the vacation and resume regular service. The retrial queueing systems with working vacations and vacation interruption have also been investigated extensively. Gao *et al.* [12] discussed an M/G/1 retrial queue with general retrial times and working vacation interruption, the discrete-time Geo^X/G/1 queue was analyzed by Gao and Wang [13]. Rajadurai *et al.* [14] introduced negative customers into an M/G/1 retrial queue with unreliable server, working vacation and vacation interruption.

In light of the classical vacation and working vacation, Ye and Liu [15] first considered a queue with single working vacation and vacations which is characterized by the following features: When the system becomes empty in regular service period, the server takes a working vacation during which the possible arriving customers are served with a lower rate. After that, if there are customers left in the system, the system will resume to the regular service period, otherwise, the server will enter an ordinary vacation during which the server stops the service completely. At the end of each ordinary vacation, the server takes another new ordinary vacation if the system is empty. Recently, Ye and Liu [16] extended the M/M/1 queue to the GI/M/1 queue. In this paper, we introduce this new vacation policy into an M/G/1 retrial queue with general retrial times. Moreover, vacation interruption policy is also considered, i.e., if there are customers in the orbit at a service completion instant in the working vacation period, the server will stop the vacation and come back to the normal working level. To the authors' best of knowledge, no special work focused on this model has appeared in open literatures.

This paper is organized as follows. A brief description of this model is given in Section 2. The stability condition is obtained by the matrix-analytic method in Section 3. In Section 4, we deal with the steady state joint distribution of the server and the number of customers in the orbit. Various performance measures of this model are also discussed. Section 5 presents some numerical examples and Section 6 concludes the paper.

SYSTEM MODEL

In this paper, we consider an M/G/1 retrial queue with single working vacation, vacation interruption and multiple ordinary vacations. The customers arrive according to a Poisson process with rate λ, and there is no waiting space in front of the server. If the customer finds the server busy when he arrives, he will join the orbit and wait for his service again later. If the customer finds the server idle, on the other hand, the arriving customer will commence his service immediately, and the normal service time S_b has a distribution function $G_b(x) = 1 - exp\{-\int_0^x \mu(t)dt\}$. We assume that only the customer at the head of the orbit queue is allowed to the server, and the retrial time R has a distribution function $R(x) = 1 - exp\{-\int_0^x \alpha(t)dt\}$. The server takes a working vacation when the system becomes empty, and the lower service time S_w has a distribution function $G_w(x) = 1 - exp\{-\int_0^x \eta(t)dt\}$. We assume the working vacation time W follows an exponential distribution with parameter θ. At a service completion instant in the working vacation period, if there are customers in the system at that moment, the server will stop the vacation and come back to the normal working level. The working vacation will continue if the system is empty. At the end of each working vacation, the server starts a new busy period if there are customers in the system. Otherwise, the server begins an ordinary vacation during which the server completely stops working, and the ordinary vacation time V has a distribution function $V(x) = 1 - exp\{-\int_0^x \beta(t)dt\}$. If there are customers in the system at the instant of an ordinary vacation completion, the server will resume to a regular serving level with normal service rate. If the system is empty, on the other hand, the server takes another new ordinary vacation.

We assume that all the random variables defined above are independent. Throughout the rest of the paper, for a distribution function $F(x)$, we define $\overline{F}(x) = 1 - F(x)$, $\tilde{F}(s) = \int_0^\infty e^{-sx}\,dF(x)$ and $\overline{F}^*(s) = \int_0^\infty e^{-sx}\,\overline{F}(x)dx$. Clearly, we have $\overline{F}^*(s) = \frac{1-\tilde{F}(s)}{s}$.

Let $N(t)$ represent the number of customers in the retrial group at time t, and $I(t)$ denote the server state: if $I(t) = 0$, the server is in a working vacation period at time t and the server is free; if $I(t) = 1$, the server is in a working vacation period at time t and the server is busy; if $I(t) = 2$, the server is in an ordinary vacation period at time t and the server is free; if $I(t) = 3$, the server is during a normal service period at time t and the server is free; if $I(t) = 4$, the server is during a normal service period at time t and the server is busy.

At time $t \geq 0$, we define the random variable $\xi(t)$ as follows: if $I(t)$=1, $\xi(t)$ denotes the elapsed lower service time; if $I(t)$=2, $\xi(t)$ represents the elapsed ordinary vacation time; if $I(t)$=3, $\xi(t)$ stands for the elapsed retrial time; if $I(t)$=4, $\xi(t)$ denotes the elapsed normal service time. Therefore, the system can be described by Markov process $X(t) = \{I(t), N(t), \xi(t)\}$ with state space

$$\Omega = \{(0,0)\} \cup \{(i,0,x), i = 1,2,4, x \geq 0\} \cup \{(i,n,x), i = 1,2,3,4, n \geq 1, x \geq 0\}.$$

Let $\{t_n; = 1,2,\cdots\}$ be the sequence of epoches at which a service completion occurs or an ordinary vacation time ends. Then the sequence of random variables $Y_n = \{I(t_n^+), N(t_n^+)\}$ forms an embedded Markov chain with state space $\{(0,0)\} \cup \{(2,0)\} \cup \{(3,k), k \geq 1\}$.

STABLE CONDITION

To develop the transition matrix of $\{Y_n; n \geq 1\}$, we introduce a few definitions:

(1) Define

$$a_k = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dG_b(x), \quad k \geq 0,$$

which explains the probability that k customers arrive during S_b, and we have

$$A(z)=\sum_{k=0}^{\infty}a_k z^k=\int_0^{\infty}e^{-\lambda(1-z)x}\,dG_b(x)=\tilde{G}_b(\lambda-\lambda z),\quad A(1)=1,$$
$$A'(1)=\lambda\int_0^{\infty}x\,dG_b(x),\quad A''(1)=\lambda^2\int_0^{\infty}x^2\,dG_b(x).$$

(2) Define

$$b_k=\int_0^{\infty}\frac{(\lambda x)^k}{k!}e^{-\lambda x}e^{-\theta x}dG_w(x),\quad k\geq 0,$$

which represents the probability that $W\geq S_w$ and k customers arrive during S_w, and we get

$$B(z)=\sum_{k=0}^{\infty}b_k z^k=\int_0^{\infty}e^{-(\theta+\lambda-\lambda z)x}\,dG_w(x)=\tilde{G}_w(\theta+\lambda-\lambda z),\quad B(1)=\tilde{G}_w(\theta),$$
$$B'(1)=\lambda\int_0^{\infty}x\,e^{-\theta x}dG_w(x),\quad B''(1)=\lambda^2\int_0^{\infty}x^2\,e^{-\theta x}dG_w(x).$$

(3) Define

$$c_k=\int_0^{\infty}\frac{(\lambda x)^k}{k!}e^{-\lambda x}\theta e^{-\theta x}\overline{G}_w(x)dx,\quad k\geq 0,$$

which explains the probability that $W<S_w$ and k customers arrive during W, and we obtain

$$C(z)=\sum_{k=0}^{\infty}c_k z^k=\theta\overline{G}_w^{*}(\theta+\lambda-\lambda z)=\frac{\theta}{\theta+\lambda-\lambda z}[1-B(z)],$$
$$C(1)=1-\tilde{G}_w(\theta), C'(1)=\frac{\lambda}{\theta}\left(1-\tilde{G}_w(\theta)\right)-B'(1),$$
$$C''(1)=2(\frac{\lambda}{\theta})^2(1-\tilde{G}_w(\theta))-2(\frac{\lambda}{\theta})B'(1)-B''(1).$$

(4) Define

$$d_k=\int_0^{\infty}\frac{(\lambda x)^k}{k!}e^{-\lambda x}dV(x),\quad k\geq 0,$$

which represents the probability that k customers arrive during V, and we derive

$$D(z)=\sum_{k=0}^{\infty}d_k z^k=\int_0^{\infty}e^{-(\lambda-\lambda z)x}\,dV(x)=\tilde{V}(\lambda-\lambda z),\quad D(1)=1,$$
$$D'(1)=\lambda\int_0^{\infty}x\,dV(x),\quad D''(1)=\lambda^2\int_0^{\infty}x^2\,dV(x).$$

(5) Define

$$u_k=\sum_{j=0}^{k}c_j\,a_{k-j},\quad k\geq 0,$$

which explains the probability that $W<S_w$ and k customers arrive during W plus S_b, and we have

$$U(z)=\sum_{k=0}^{\infty}u_k z^k=C(z)A(z),\quad U(1)=C(1)A(1)=1-\tilde{G}_w(\theta),$$
$$U'(1)=C'(1)A(1)+C(1)A'(1), U''(1)=C''(1)A(1)+2C'(1)A'(1)+C(1)A''(1).$$

Using the lexicographical sequence for the states, the transition probability matrix of $\{Y_n;n\geq 1\}$ can be written as the block-Jacobi matrix

$$P=\begin{pmatrix} W_0 & W_1 & W_2 & W_3 & \cdots \\ B_0 & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix},$$

where

$$W_0=\begin{pmatrix}\frac{\lambda}{\lambda+\theta}(b_0+u_0) & \frac{\theta}{\lambda+\theta} \\ 0 & d_0\end{pmatrix},\quad W_k=\begin{pmatrix}\frac{\lambda}{\lambda+\theta}(b_k+u_k) \\ d_k\end{pmatrix},\quad k\geq 1,$$
$$B_0=(\tilde{R}(\lambda)a_0\quad 0),\quad A_0=\tilde{R}(\lambda)a_0,\quad A_k=\tilde{R}(\lambda)a_k+\lambda\overline{R}^{*}(\lambda)a_{k-1},\quad k\geq 1.$$

Theorem 1. *The embedded Markov chain $\{Y_n; n \geq 1\}$ is ergodic if and only if $A'(1) < \tilde{R}(\lambda)$.*

Proof. It is not difficult to see that $\{Y_n; n \geq 1\}$ is an irreducible and aperiodic Markov chain, so we just need to prove that $\{Y_n; n \geq 1\}$ is positive recurrent if and only if $A'(1) < \tilde{R}(\lambda)$. Since $A = \sum_{k=0}^{\infty} A_k$=1, from the chapter 2 of Neuts [17], the embedded Markov chain $\{Y_n; n \geq 1\}$ is positive recurrent if and only if $\sum_{k=0}^{\infty} k\, A_k = \lambda \overline{R}^*(\lambda) + A'(1) < 1$, i.e., $A'(1) < \tilde{R}(\lambda)$. □

Since the arrival process is Poisson, using PASTA property, it can be showed from Burke's theorem (see [18], pp.187-188) that the steady state probabilities of the Markov process $X(t)$ exist if and only if the stable condition $A'(1) < \tilde{R}(\lambda)$ holds. Now we define the limiting probability and limiting probability densities:

$$P_{0,0} = \lim_{t\to\infty} P\,(I(t) = 0, N(t) = 0),$$
$$P_{1,n}(x)dx = \lim_{t\to\infty} P\,(I(t) = 1, N(t) = n, x \leq \xi(t) < x + dx), \quad n \geq 0,$$
$$P_{2,n}(x)dx = \lim_{t\to\infty} P\,(I(t) = 2, N(t) = n, x \leq \xi(t) < x + dx), \quad n \geq 0,$$
$$P_{3,n}(x)dx = \lim_{t\to\infty} P\,(I(t) = 3, N(t) = n, x \leq \xi(t) < x + dx), \quad n \geq 1,$$
$$P_{4,n}(x)dx = \lim_{t\to\infty} P\,(I(t) = 4, N(t) = n, x \leq \xi(t) < x + dx), \quad n \geq 0.$$

STEADY STATE ANALYSIS AND PERFORMANCE MEASURES

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system

$$(\lambda + \theta)P_{0,0} = \int_0^\infty P_{1,0}\,(x)\eta(x)dx + \int_0^\infty P_{4,0}\,(x)\mu(x)dx, \tag{1}$$
$$\frac{d}{dx}P_{1,n}(x) = -\big(\lambda + \theta + \eta(x)\big)P_{1,n}(x) + \big(1 - \delta_{n,0}\big)\lambda P_{1,n-1}(x), n \geq 0, \tag{2}$$
$$\frac{d}{dx}P_{2,n}(x) = -(\lambda + \beta(x))P_{2,n}(x) + (1 - \delta_{n,0})\lambda P_{2,n-1}(x), \quad n \geq 0, \tag{3}$$
$$\frac{d}{dx}P_{3,n}(x) = -(\lambda + \alpha(x))P_{3,n}(x), \quad n \geq 1, \tag{4}$$
$$\frac{d}{dx}P_{4,n}(x) = -(\lambda + \mu(x))P_{4,n}(x) + (1 - \delta_{n,0})\lambda P_{4,n-1}(x), \quad n \geq 0, \tag{5}$$

where $\delta_{n,0}$ is the Kronecker's symbol. The boundary conditions are

$$P_{1,0}(0) = \lambda P_{0,0}, \quad P_{1,n}(0) = 0, \quad n \geq 1, \tag{6}$$
$$P_{2,0}(0) = \theta P_{0,0} + \int_0^\infty P_{2,0}\,(x)\beta(x)dx, \quad P_{2,n}(0) = 0, \quad n \geq 1, \tag{7}$$
$$P_{3,n}(0)=\int_0^\infty P_{1,n}\,(x)\eta(x)dx+\int_0^\infty P_{2,n}\,(x)\beta(x)dx+\int_0^\infty P_{4,n}\,(x)\mu(x)dx, \quad n\geq 1, \tag{8}$$
$$P_{4,n}(0)=\theta\int_0^\infty P_{1,n}(x)dx+\big(1-\delta_{n,0}\big)\lambda\int_0^\infty P_{3,n}(x)dx+\int_0^\infty P_{3,n+1}(x)\alpha(x)dx, \quad n\geq 0, \tag{9}$$

and the normalization condition is

$$P_{0,0}+\sum_{n=0}^{\infty}\Big(\int_0^\infty P_{1,n}\,(x)dx+\int_0^\infty P_{2,n}\,(x)dx+\int_0^\infty P_{4,n}\,(x)dx\Big)+\sum_{n=1}^{\infty}\int_0^\infty P_{3,n}\,(x)dx=1. \tag{10}$$

By introducing the generating functions $P_i(x,z) = \sum_{n=b}^{\infty} P_{i,n}\,(x)z^n, i = 1,3,4, b = 0; i = 2, b = 1$, from (1)-(5), we have

$$(\lambda + \theta)P_{0,0} = P_{1,0}(0)b_0 + P_{4,0}(0)a_0, \tag{11}$$
$$P_1(x,z) = P_1(0,z)e^{-(\theta+\lambda(1-z))x}\overline{G}_w(x), \tag{12}$$
$$P_2(x,z) = P_2(0,z)e^{-\lambda(1-z)x}\overline{V}(x), \tag{13}$$
$$P_3(x,z) = P_3(0,z)e^{-\lambda x}\overline{R}(x), \tag{14}$$
$$P_4(x,z) = P_4(0,z)e^{-\lambda(1-z)x}\overline{G}_b(x). \tag{15}$$

From (6)-(7), after some computation, we can get

$$P_1(0,z) = P_{1,0}(0) = \lambda P_{0,0}, \tag{16}$$
$$P_2(0,z) = P_{2,0}(0) = \frac{\theta}{1 - d_0}P_{0,0}. \tag{17}$$

Using (16)-(17), from (8)-(9), we can obtain

$$P_3(0,z)=\lambda B(z)P_{0,0}+A(z)P_4(0,z)+\frac{\theta}{1-d_0}D(z)P_{0,0}-\Big(\lambda+\frac{\theta}{1-d_0}\Big)P_{0,0}, \tag{18}$$

$$zP_4(0,z) = \lambda z C(z) P_{0,0} + (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z) P_3(0,z). \tag{19}$$

Inserting (19) into (18) yields

$$P_3(0,z) = \frac{\lambda(B(z) + U(z) - 1) + \frac{\theta}{1-d_0}(D(z)-1)}{z - (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z) A(z)} z P_{0,0}. \tag{20}$$

And $P_4(0,z)$ is given by

$$P_4(0,z) = \frac{\lambda z C(z) + (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z)[\lambda(B(z)-1) + \frac{\theta}{1-d_0}(D(z)-1)]}{z - (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z) A(z)} P_{0,0}. \tag{21}$$

Remark 1. *If $A'(1) < \tilde{R}(\lambda)$, the equation $z - (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z) A(z) = 0$ has no root in the range $0 < z < 1$ and has the minimal nonnegative root $z = 1$. (see Lemma 3.1 [12])*

Define the marginal generating functions $\Phi_i(z) = \int_0^\infty P_i(x,z)dx, i = 1,2,3,4$. Substituting (16), (17), (20) and (21) into (12)-(15), after some calculations we have the following theorem.

Theorem 2.

$$\Phi_1(z) = \frac{\lambda}{\theta} C(z) P_{0,0},$$

$$\Phi_2(z) = \frac{\theta}{1-d_0} \frac{1-D(z)}{\lambda(1-z)} P_{0,0},$$

$$\Phi_3(z) = \frac{\lambda \overline{R}^*(\lambda)(B(z) + U(z) - 1) + \frac{\theta}{1-d_0} \overline{R}^*(\lambda)(D(z)-1)}{z - (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z) A(z)} z P_{0,0},$$

$$\Phi_4(z) = \frac{\lambda z C(z) + (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z)[\lambda(B(z)-1) + \frac{\theta}{1-d_0}(D(z)-1)]}{z - (\tilde{R}(\lambda) + \lambda \overline{R}^*(\lambda) z) A(z)} \frac{1-A(z)}{\lambda(1-z)} P_{0,0}.$$

From Theorem 2, we can obtain some system performance measures.

The probability that the server is in a working vacation period and is busy is given by

$$\Phi_1(1) = \frac{\lambda}{\theta} C(1) P_{0,0}.$$

The probability that the server is in an ordinary vacation period is given by

$$\Phi_2(1) = \frac{\theta}{1-d_0} \frac{D'(1)}{\lambda} P_{0,0}.$$

The probability that the server is during a normal service period and is free is given by

$$\Phi_3(1) = \frac{\lambda \overline{R}^*(\lambda)(B'(1) + U'(1)) + \frac{\theta}{1-d_0} \overline{R}^*(\lambda) D'(1)}{\tilde{R}(\lambda) - A'(1)} P_{0,0}.$$

The probability that the server is during a normal service period and is busy is given by

$$\Phi_4(1) = \frac{\lambda \tilde{R}(\lambda) C(1) + \lambda(B'(1) + C'(1)) + \frac{\theta}{1-d_0} D'(1)}{\tilde{R}(\lambda) - A'(1)} \frac{A'(1)}{\lambda} P_{0,0}.$$

Moreover, $P_{0,0}$ can be determined by the normalization condition

$$P_{0,0} + \Phi_1(1) + \Phi_2(1) + \Phi_3(1) + \Phi_4(1) = 1,$$

which leads to

$$P_{0,0} = \frac{\tilde{R}(\lambda) - A'(1)}{\tilde{R}(\lambda) - A'(1) + (\frac{\lambda}{\theta} + A'(1)) C(1) + \frac{\theta}{1-d_0} \frac{D'(1)}{\lambda}}.$$

The probability that the server is busy is given by

$$P_b = \Phi_1(1) + \Phi_4(1) = \frac{\frac{\lambda}{\theta} C(1)(\tilde{R}(\lambda) - A'(1)) + \tilde{R}(\lambda) C(1) A'(1) + (B'(1) + C'(1)) A'(1) + \frac{\theta}{1-d_0} \frac{D'(1)}{\lambda} A'(1)}{\tilde{R}(\lambda) - A'(1)} P_{0,0}.$$

The probability that the server is free is given by

$$P_f = P_{0,0} + \Phi_2(1) + \Phi_3(1) = \frac{(1+\frac{\theta}{1-d_0}\frac{D'(1)}{\lambda})(\tilde{R}(\lambda) - A'(1)) + \lambda\overline{R}^*(\lambda)(B'(1)+U'(1)) + \frac{\theta}{1-d_0}\overline{R}^*(\lambda)D'(1)}{\tilde{R}(\lambda) - A'(1)} P_{0,0}.$$

Let $E[L_j]$ denote the average number of customers in the orbit when the server's state is $i, i = 1,2,3,4$. From Theorem 2, after some calculations we can get

$$E[L_1] = \frac{\lambda}{\theta} C'(1) P_{0,0}, \quad E[L_2] = \frac{\theta}{1-d_0} \frac{D''(1)}{2\lambda} P_{0,0},$$

$$E[L_3] = \frac{\lambda\overline{R}^*(\lambda)(B''(1)+U''(1)) + \frac{\theta}{1-d_0}\overline{R}^*(\lambda)D''(1)}{2(\tilde{R}(\lambda)-A'(1))} P_{0,0} + (1 + \frac{2\lambda\overline{R}^*(\lambda)A'(1)+A''(1)}{2(\tilde{R}(\lambda)-A'(1))})\Phi_3(1),$$

$$E[L_4] = \frac{2\lambda C'(1) + \lambda\left(B''(1)+C''(1)\right) + 2\lambda\overline{R}^*(\lambda)\left(\lambda B'(1) + \frac{\theta}{1-d_0}D'(1)\right) + \frac{\theta}{1-d_0}D''(1)}{2(\tilde{R}(\lambda)-A'(1))} \frac{A'(1)}{\lambda} P_{0,0} + (\frac{A''(1)}{2A'(1)} + \frac{2\lambda\overline{R}^*(\lambda)A'(1)+A''(1)}{2(\tilde{R}(\lambda)-A'(1))})\Phi_4(1).$$

Clearly, the probability generating function of the number of customers in the orbit is given by

$$\Phi(z) = P_{0,0} + \Phi_1(z) + \Phi_2(z) + \Phi_3(z) + \Phi_4(z).$$

The probability generating function of the number of customers in the system is given by

$$\tilde{\Phi}(z) = P_{0,0} + z\Phi_1(z) + \Phi_2(z) + \Phi_3(z) + z\Phi_4(z).$$

Therefore, the mean orbit length ($E[L]$) is given by

$$E[L] = E[L_1] + E[L_2] + E[L_3] + E[L_4].$$

And the mean system length ($E[\tilde{L}]$) is derived as

$$E[\tilde{L}] = E[L] + \Phi_1(1) + \Phi_4(1) = E[L] + P_b.$$

Let $E[W]$ ($E[\widetilde{W}]$) be the expected waiting (sojourn) time of a customer in the orbit (system), using Little's formula,

$$E[W] = \frac{E[L]}{\lambda}, \quad E[\widetilde{W}] = \frac{E[\tilde{L}]}{\lambda}.$$

NUMERICAL RESULTS

In this section, taking the M/E_2/1 queue as an especial case, we present some numerical examples to illustrate the effect of the varying parameters on the mean orbit length $E[L]$, where the normal service time (the lower service time) follows the Erlang distribution of order 2 with $\tilde{G}_b(s) = (\frac{\mu}{s+\mu})^2$ ($\tilde{G}_w(s) = (\frac{\eta}{s+\eta})^2$). In order to make a comparison, two different retrial time distributions are also considered, it is assumed that the retrial time follows the exponential distribution with $\tilde{R}(s) = \frac{\alpha}{s+\alpha}$ or Erlang distribution of order 2 with $\tilde{R}(s) = (\frac{\alpha}{s+\alpha})^2$. We also assume that the ordinary vacation time follows the exponential distribution with $\tilde{V}(s) = \frac{\beta}{s+\beta}$. Under the stable condition $A'(1) < \tilde{R}(\lambda)$, the various parameters of this model are arbitrarily chosen as λ=1.2, μ=5, η=0.8, α=4, θ=0.5 and β=1, unless they are considered as variables in the respective figures.

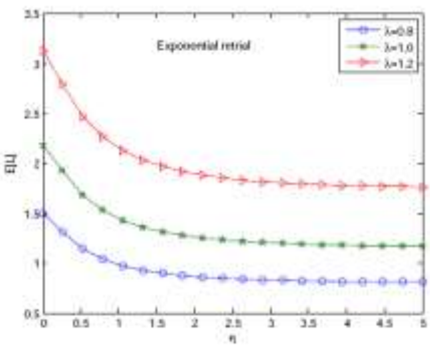

Fig-1: The effect of η on $E[L]$ for different values of λ

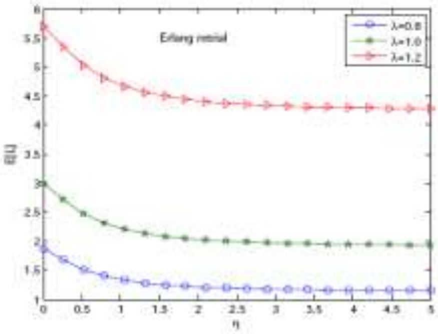

Fig- 2: The effect of η on $E[L]$ for different values of λ

From Figs.1-2, it is obvious that $E[L]$ decreases evidently as the values of η increase. Since we assume working vacation interruption policy, when η is large, we can see that η has little effect on $E[L]$ as η increases. As expected, the arrival rate λ has a noticeable effect on $E[L]$.

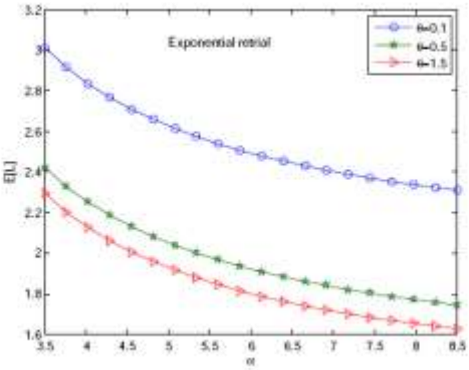

Fig-3: The effect of α on $E[L]$ for different values of θ

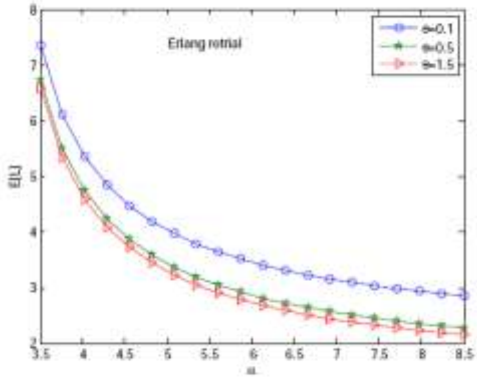

Fig-4: The effect of α on $E[L]$ for different values of θ

Figs.3-4 illustrate that $E[L]$ decreases dramatically with α increasing, this is because that as the value of α increases, the mean retrial time decreases. And the smaller the mean retrial time is, the bigger the probability that the server is busy is, which decreases the value of $E[L]$. Since the service rate η is less than μ, it can be observed that increasing θ decreases the value of $E[L]$.

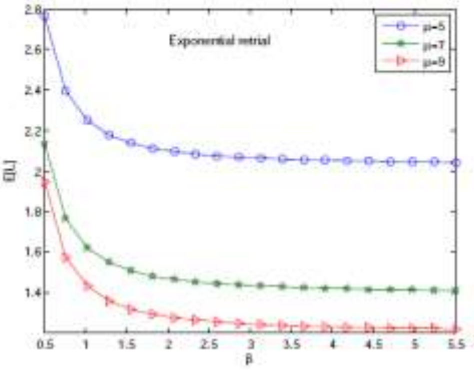

Fig-5: The effect of β on $E[L]$ for different values of μ

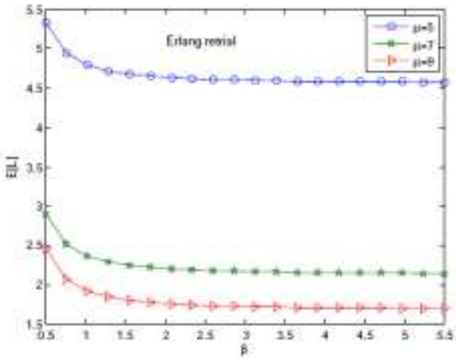

Fig-6: The effect of β on $E[L]$ for different values of μ

Figs.5-6 indicate that $E[L]$ decreases with increasing values of β, and the effect of β on $E[L]$ is more obvious when β is smaller, the reason is that the expected ordinary vacation time is $1/\beta$. We can also find that as μ increases, $E[L]$ decreases, which agrees with the intuitive expectation.

Furthermore, under the same condition, the mean retrial time with exponential distribution is shorter than that with Erlang distribution. Thus, from Figs.1-6, we can see that $E[L]$ with exponential retrial time is smaller than that with Erlang retrial time.

CONCLUSION

In this work, we investigate an M/G/1 retrial queue with single working vacation, vacation interruption and ordinary vacations. Using embedded Markov chain and matrix-analytic method, we get the condition of stability. Supplementary variable technique is employed to obtain the expressions for the probability generating functions of the server state and the number of customers in the orbit. Various important performance measures are also derived. Finally, the effect of various parameters on the mean orbit length are examined numerically. For future research, using the same method, one can discuss a similar model but with batch arrival customers.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (No.11301306).

REFERENCES

1. Artalejo JR. Accessible bibliography on retrial queues: Progress in 2000-2009. Mathematical and Computer Modelling. 2010;51:1071-1081.
2. Yang S, Wu J, Liu Z. An $M^{[X]}$/G/1 retrial G-queue with single vacation subject to the server breakdown and repair. Acta Mathematicae Applicatae Sinica. 2013;29(3):579-596.
3. Lakshmi K, Ramanath K. An M/G/1 retrial queue with a single vacation scheme and general retrial times. American Journal of Operational Research. 2013;3:7-16.
4. Padmavathi I, Sivakumar B, Arivarignan G. A retrial inventory system with single and modified multiple vacation for server. Annals of Operations Research. 2015;233(1):335-364.
5. Jain M, Bhagat A. $M^{[X]}$/G/1 retrial vacation queue for multi-optional services, phase repair and reneging. Quality

Technology Quantitative Management. 2016;13(3):263-288.

6. Servi L, Finn S. M/M/1 queue with working vacations (M/M/1/WV). Performance Evaluation. 2002;50:41-52.
7. Do T. M/M/1 retrial queue with working vacations. Acta Informatica. 2010;47:67-75.
8. Liu Z, Song Y. Geo/Geo/1 retrial queue with non-persistent customers and working vacations. Journal of Applied Mathematics and Computing. 2013;42:103-115.
9. Arivudainambi D, Godhandaraman P, Rajadurai P. Performance analysis of a single server retrial queue with working vacation. OPSEARCH. 2014;51(3):434-462.
10. Jailaxmi V, Arumuganathan R, Kumar MS. Performance analysis of single server non-Markovian retrial queue with working vacation and constant retrial policy. RAIRO Operations Research. 2014;48:381-398.
11. Li J, Tian N. The M/M/1 queue with working vacations and vacation interruption. Journal of Systems Science and Systems Engineering. 2007;16:121-127.
12. Gao S, Wang J, Li W. An M/G/1 retrial queue with general retrial times, working vacations and vacation interruption. Asia-Pacific Journal of Operational Research. 2014;31(2):1-25.
13. Gao S, Wang J. Discrete-Time Geo^X/G/1 retrial queue with general retrial times, working vacations and vacation interruption. Quality Technology Quantitative Management. 2013;10:493-510.
14. Rajadurai P, Chandrasekaran VM, Saravanarajan MC. Analysis of an unreliable retrial G-queue with working vacations and vacation interruption under Bernoulli schedule. Ain Shams Engineering Journal. on line.
15. Ye Q, Liu L. The analysis of the M/M/1 queue with two vacation policies (M/M/1/SWV+MV). International Journal of Computer Mathematics. on line.
16. Ye Q, Liu L. Performance analysis of the GI/M/1 queue with single working vacation and vacations. Methodology & Computing in Applied Probability. on line.
17. Neuts MF. Structured Stochastic Matrices of M/G/1 Type and Their Applications. Marcel Dekker, New York, 1989.
18. Cooper RB. Introduction to Queueing Theory. North-Holland, New York, 1981.