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Another Proof of the Brezis-Lieb Lemma

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Abstract: The Breis-Lieb Lemma was first came up with by the famous French mathematician Haim Brezis and American mathematician Elliott Lieb, it is an improvement of Fatou's Lemma, which has numerous applications mainly in calculus of variations when it faced the problem whether an infimum or supremum can be achieved. In this paper we use the Clarkson's inequality combined with the Fatou's Lemma to prove the Brezis-Lieb lemma.
Keywords: Brezis-Lieb lemma, Fatou's lemma, Clarkson's inequality

INTRODUCTION

Brezis-Lieb Lemma

Brezis-Lieb Lemma was first came up with by Haim Brezis and Elliott Lieb [1], they give the L^p case and the general case, which are the improvements of Fatou's Lemma.

Theorem 1.1. (the L^p case) Let (Ω,Σ,μ) be a measure space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex valued measurable functions, suppose $\|f_n\|_p \le C < +\infty$ for all n and for some $0 < p < \infty$, and $f_n \to f$ $a.e.$ then

$$\lim_{n\to\infty}\left\{\|f_n\|_p^p - \|f_n - f\|_p^p\right\} = \|f\|_p^p \tag{1}$$

More generally,

Theorem 1.2. (the general case) Let $j: \boldsymbol{C} \to \boldsymbol{C}$ be a continuous function, where $\boldsymbol{C}$ is complex domain, with $j(0)=0$, for every sufficiently small $\varepsilon > 0$, there exist two continuous, nonnegative function φ_ε and ψ_ε such that

$$\left|j(a+b) - j(a)\right| \le \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b) \tag{2}$$

for all $a, b \in \boldsymbol{C}$.

Let $f_n = f + g_n$ be a sequence of measurable functions from Ω to $\boldsymbol{C}$ such that:

(i) $g_n \to 0$ $a.e.$

(ii) $j(f) \in L^1$.

(iii) $\int \varphi_\varepsilon(g_n(x))\,d\mu(x) \le C < +\infty$, *for some constant* C, *independent of* ε *and* n.

$(iiii)$ $\int \psi_\varepsilon(f(x))\,d\mu < +\infty$, *for all* $\varepsilon > 0$.

then, as $n \to \infty$,

$$\int\left|j(f+g_n) - j(g_n) - j(f)\right| d\mu \to 0. \tag{3}$$

Preliminary knowledge

In order to give a complete proof of an important corollary of Brezis-Lieb lemma, we should know some preliminary knowledge used in this paper.

For $1 < p < +\infty$, L^p is reflexive, separable, and the dual of L^p is $L^{p'}$, where

$$\frac{1}{p}+\frac{1}{p'}=1 \tag{4}$$

Lemma 1.3 (Clarkson's inequality[2])
(Clarkson's first inequality) Let $2\le p<\infty$, we claim that

$$\left\|\frac{f+g}{2}\right\|_p^p+\left\|\frac{f-g}{2}\right\|_p^p\le\frac{1}{2}\left(\|f\|_p^p+\|g\|_p^p\right),\quad \forall f,g\in L^p. \tag{5}$$

(Clarkson's second inequality) Let $1<p\le 2$,we claim that

$$\left\|\frac{f+g}{2}\right\|_p^{p'}+\left\|\frac{f-g}{2}\right\|_p^{p'}\le\left(\frac{1}{2}\|f\|_p^p+\frac{1}{2}\|g\|_p^p\right)^{1/(p-1)},\quad \forall f,g\in L^p. \tag{6}$$

Lemma 1.4 (Fatou's lemma[2])
Let f_n be a sequence of functions in L^1, which satisfies

(i) *for all* n, $f_n\ge 0$ *a.e.*

(ii) $sup_n\int f_n<+\infty.$

For almost all $x\in\Omega$, we set $f(x)=liminf_{n\to\infty}f_n(x)\le+\infty$. Then $f\in L^1$ and

$$\int f\le\liminf_{n\to\infty}\int f_n \tag{7}$$

USING CLARKSON'S INEQUALITY TO PROVE ONE IMPORTANT COROLLARY OF BREZIS-LIEB LEMMA

In Brezis-Lieb Lemma, we replace the condition $\{f_n\}$ being a bounded sequence in L^p by $\|f_n\|_p\to\|f\|_p$, then we can give a direct proof by using Clarkson's inequalities:

Proposition 2.1. ([2]) Let $1<p<+\infty$, $f_n\in L^p(\Omega)$, such that

(i) $f_n(x)\to f(x)$ *a.e.*

(ii) $\|f_n\|_p\to\|f\|_p$

Then we have

$$\|f_n-f\|_p\to 0 \tag{8}$$

Proof. (1)When $2\le p<+\infty$, by Clarkson's first inequality, then we have

$$\left\|\frac{f_n+f}{2}\right\|_p^p+\left\|\frac{f_n-f}{2}\right\|_p^p\le\frac{1}{2}\left(\|f_n\|_p^p+\|f\|_p^p\right) \tag{9}$$

By moving the first term in the left hand side to the right hand side, we have

$$\left\|\frac{f_n-f}{2}\right\|_p^p\le\frac{1}{2}\left(\|f_n\|_p^p+\|f\|_p^p\right)-\left\|\frac{f_n+f}{2}\right\|_p^p \tag{10}$$

By taking the upper limit on both sides of the above inequality, then

$$\overline{\lim_{n\to\infty}}\left\|\frac{f_n-f}{2}\right\|_p^p\le\|f\|_p^p-\underline{\lim_{n\to\infty}}\left\|\frac{f_n+f}{2}\right\|_p^p \tag{11}$$

In the other hand, by Fatou's Lemma,

$$\underline{\lim}_{n\to\infty}\int\left|\frac{f_n+f}{2}\right|^p \geq \int \underline{\lim}_{n\to\infty}\left|\frac{f_n+f}{2}\right|^p \tag{12}$$

Since $f_n \xrightarrow{a.e.} f$, $\frac{f_n+f}{2} \xrightarrow{a.e.} f$, thus $\left|\frac{f_n+f}{2}\right|^p \to |f|^p$ $a.e.$,then

$$\int \underline{\lim}_{n\to\infty}\left|\frac{f_n+f}{2}\right|^p = \int |f|^p = \|f\|_p^p \tag{13}$$

Returning to（10）, we have $\overline{\lim}_{n\to\infty}\left\|\frac{f_n-f}{2}\right\|_p^p \leq 0$, applying squeeze rule,

$$\lim_{n\to\infty}\left\|\frac{f_n-f}{2}\right\|_p^p = 0, i.e. \|f_n - f\|_p \to 0 \tag{14}$$

(2)When $1 < p \leq 2$, by Clarkson's second inequality, then we have

$$\left\|\frac{f_n+f}{2}\right\|_p^{p'} + \left\|\frac{f_n-f}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|f_n\|_p^p + \frac{1}{2}\|f\|_p^p\right)^{1/(p-1)} \tag{15}$$

By moving the first term in the left hand side to the right hand side,

$$\left\|\frac{f_n-f}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|f_n\|_p^p + \frac{1}{2}\|f\|_p^p\right)^{1/(p-1)} - \left\|\frac{f_n+f}{2}\right\|_p^{p'} \tag{16}$$

By taking the upper limit on both sides of the above inequality, then we have

$$\overline{\lim}_{n\to\infty}\left\|\frac{f_n-f}{2}\right\|_p^{p'} \leq \left(\|f\|_p^p\right)^{p'/p} - \underline{\lim}_{n\to\infty}\left\|\frac{f_n+f}{2}\right\|_p^{p'} \tag{17}$$

By Fatou’s Lemma, we have

$$\underline{\lim}_{n\to\infty}\left(\int\left|\frac{f_n+f}{2}\right|^p\right)^{\frac{p'}{p}} \geq \left(\int \underline{\lim}_{n\to\infty}\left|\frac{f_n+f}{2}\right|^p\right)^{\frac{p'}{p}} \tag{18}$$

Since $f_n \xrightarrow{a.e.} f$, $\frac{f_n+f}{2} \xrightarrow{a.e.} f$, thus $\left|\frac{f_n+f}{2}\right|^p \to |f|^p$ $a.e.$, thus we have

$$\int \underline{\lim}_{n\to\infty}\left|\frac{f_n+f}{2}\right|^p = \int |f|^p = \|f\|_p^p \tag{19}$$

Returning to （17）, we have

$$\overline{\lim}_{n\to\infty}\left\|\frac{f_n-f}{2}\right\|_p^{p'} \leq 0 \tag{20}$$

It follows from squeeze rule that

$$\|f_n - f\|_p \to 0 \tag{21}$$

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