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Development and Implementation of Four-Step Predictor-Corrector Method with an Improvement Strategy for Fourth-Order Ordinary Differential Equations with Applications

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Abstract **Original Research Article**

This article focuses on the development, theoretical analysis, and implementation of a four-level predictor-corrector method with an improved strategy for solving fourth-order ordinary differential equations using initial conditions. When developing the method, Chebyshev polynomials of the first kind were adopted as the basis functions for solving the IVP. Chebyshev polynomials of the first kind were interpolated at some selected grid and off-grid points, and the fourth derivative of the approximate solution was collocated at all grid and off-grid points. The methods was derived and implemented in such a way that the correctors are of the same order with the predictors in other to overcome setbacks associated with existing predictor-corrector, thus enhanced a better accuracy and stability. Theoretical analysis of the methods were validated to ensure that the methods are usable and efficient. The methods was applied to three numerical test problems in order to establish the accuracy of the methods. The results show better accuracy over some existing approach in the literature.

Keywords: Enhanced Accuracy, Convergence of the Methods, Chebyshev-Fitted Polynomial, Four-Step Predictor-Corrector, Multistep Collocation Schemes, Off-Grid Points.

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1 INTRODUCTION

Numerous physical problems in sciences and engineering are usually modelled mathematically as a fourth-order ordinary differential equations (ODEs) with initial conditions. Unfortunately, some of such models does not have an analytical and cannot be solved by separation of variables. Thus, numerical models are developed to solve and give better understanding of physical phenomena to such problems. In this paper, we consider general fourth-order ordinary differential equations with initial conditions of the form:

$$y^4 = f(x, y, y', y'', y''')\ y^r(x_0) = y_0^r, r = 0,1,2,3 \quad (1)$$

Generally, this type of differential equations are often encountered by scientists, physicist and mathematician due to its applications in various physical problems in sciences and engineering [1-2]. Some of these physical problems describe certain phenomena related to theory of elastic stability, control theory and beam theory.

These models of physical problems are functions of time, Therefore, the study of solutions is of great interest to researchers [3-4].

The traditional way to solve the equation (1) is to reduce it to a first-order system of ordinary differential equations. [5-7], discuss various approaches to reduce to first-order systems and their weaknesses. A direct method of solving (1) using an implicit linear multistep method has been proven to be more efficient in terms of speed and accuracy than the reduction method to a system of first-order ordinary differential equations [8 − 10]. Predictor-corrector methods for solving high-order ordinary differential equations for initial value problems have been proposed by many authors in [11-15]. Implicit linear multi step method requires determination of starting values for its implementation to solving problems (1). The methods that produce initial value are called predictors while the implicit linear multi step method itself is the corrector. The combination of the two when used to obtain the numerical solution of a particular problem is widely referred to as predictor-corrector method [13, 14].

In the development of a predictor [10-12], adopted explicit linear multi step method by avoiding collocation at x_{n+k} for k -step method. This is repeated until all the desired results are achieved. Taylor's series was applied, as the starting values in the implementation of the scheme. The major disadvantage of this method stated in literature has been the use of predictors of lower order to implement the scheme reported in [16-23]. Furthermore, in Ref [20], the authors developed a zero-stability method that directly solves fourth-order ordinary differential equations. The methods used were collocation of differential systems and interpolation of approximate solutions to problems using power series as basis functions. The method was consistent and symmetric with optimal order. The predictor of lower order affected the accuracy of the method.

For researchers in this field, the need to improve predictive correction methods to obtain better results has become crucial. It is therefore a continuous linear multistep method that computes discrete values ??at multiple points simultaneously, called a block method. The scientists that have worked in this area includes the authors in [24-27]. The problem of predictors of lower order was addressed by developing the predictors which are of the same order with the corrector for a lower order ordinary differential equations as reported in [21-23], and the results was competitive.

Thus, this article proposes a Four-step predictor-corrector methods for solving directly fourth-order ordinary differential equations. In the development of the methods, we emphasized that the order of the corrector is also the same as the order of the predictor. The purpose of this, is to guarantee a more accuracy and enhanced accuracy while retaining its good stability properties. It should be noted that zero-stability condition barrier was circumvented by the introduction of hybrid formula which incorporates a function evaluation at an off-step point. Such formulae simultaneously proposed by [18,23] as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^n \left(\sum_{j=0}^{k} \beta_j f_{n+j}(x) + \sum_{j=0}^{k} \beta_v f_{n+v} \right) \tag{2}$$

where v are the hybrid points and n is the order of the differential equations.

2 METHODOLOGY

Consider Chebyshev polynomial of the first kind of the form,

$$y(x) = \sum_{i=0}^{C+I-1} a_i T_i(x) \tag{3}$$

where

C is called the number of collocation points

I represents the number of interpolation points

$a_i's$ are parameters to be determined, x is continuous and differentiable

$T_i(x)$ are Chebyshev polynomial's parameters given as,

$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x,$
$T_4(x) = 8x^4 - 8x^2 + 1, T_{n+1}(x) = 2xT_n - T_{n-1}(x), n \geq 1$

Taking the fourth-derivative of (3) gives,

$$y^{iv}(x) = \sum_{i=0}^{C+I-1} a_i T_i^{iv}(x) \tag{4}$$

Equations (3) and (4) are respectively the interpolation and collocation equations which would be evaluated at selected grid and off-grid points. The interpolation equation is given in expanded form below,

$$\begin{aligned} y(x) = &a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) + a_4(8x^4 - 8x^2 + 1) \\ &+ a_5(16x^5 - 20x^3 + 5x) + a_6(32x^6 - 48x^4 + 18x^2 - 1) \\ &+ a_7(64x^7 - 112x^5 + 56x^3 - 7x) + a_8(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1) \\ &+ a_9(256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x) \\ &+ a_{10}(512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1) \\ &+ a_{11}(1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x) \\ &+ a_{12}(2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1) \end{aligned} \tag{5}$$

while the collocation equations is given in expanded form as follow,

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$$y^{iv}(x) = f_{n+c} = 192a_4 + 1920a_5x + a_6(11520x^2 - 1152) + a_7(53760x^3 - 13440x) + a_8(215040x^4 - 92160x^2 + 3840) + a_9(774144x^5 - 483840x^3 + 51840x) + a_{10}(2580480x^6 - 2150400x^4 + 403200x^2 - 9600) + a_{11}(8110080x^7 - 8515584x^5 + 2365440x^3 - 147840x) + a_{12}(24330240x^8 - 30965760x^6 + 11612160x^4 - 1290240x^2 + 20160) \quad (6)$$

Interpolating (5) at points $x = x_{n+i}, i = 0,1,p,2,3,$ and collocating equation (6) at the points $x = x_{n+i}, i = 0,r,s,1,2,3,u,v,4$, where $1 \leq p \leq 2, 0 \leq r,s \leq 1$ and
$2 \leq u,v, \leq 4$.

Specifically, the values of r,s,p,u,v are taken to be $\frac{1}{3},\frac{2}{3},\frac{3}{2},\frac{10}{3},\frac{11}{3}$ respectively. Gaussian elimination approach was employed to solve the resulting equations arising from the interpolation and collocation of equations (5) and (6) to determine the values of $a_j's, j = 0(1)11$ taking$x_n = 0$. The values of $a_j, j = 0,1,\dots,13$ are substituting into (5) to gives a linear hybrid multi-step method with continuous coefficients in the form:

$$y_{n+4} = \alpha_0(t)y_n + \alpha_1(t)y_{n+1} + \alpha_{\frac{3}{2}}(t)y_{n+\frac{3}{2}} + \alpha_2(t)y_{n+2} + \alpha_3(t)y_{n+3} + \beta_3(t)f_{n+3} + \beta_{\frac{10}{3}}(t)f_{n+\frac{10}{3}} + \beta_{\frac{11}{3}}(t)f_{n+\frac{11}{3}} + \beta_4(t)f_{n+4}\Big) \quad (7)$$

Using the transformation in [8],

$$t = \frac{x - x_{n+k-1}}{h}$$
$$\frac{dt}{dx} = \frac{1}{h}$$

The coefficients of y_{n+j} and f_{n+j} are obtained as:

$$\alpha_0(t) = -\frac{10373120}{6634761}t^7 - \frac{1969040737}{862518930}t - \frac{13824}{225897815}t^{13} + \frac{177152}{1579705}t^{10} - \frac{5632}{315941}t^{11} + 1 + \frac{261815179}{156390795}t^2 + \frac{19345408}{14217345}t^6 + \frac{27648}{17376755}t^{12} - \frac{61485647}{187668954}t^3 \quad (8)$$

$$\alpha_1(t) = \frac{124416}{225897815}t^{13} + \frac{1622016}{315941}t^5 + \frac{40633541}{20852106}t^3 + \frac{1012893513}{143753155}t + \frac{50688}{315941}t^{11} + \frac{31119360}{2211587}t^7 - \frac{297732543}{34753510}t^2 - \frac{19345408}{1579705}t^6 - \frac{248832}{17376755}t^{12} - \frac{1594368}{1579705}t^{10} - \frac{14873088}{1579705}t^8 + \frac{3688192}{947823}t^9 \quad (9)$$

$$\alpha_{\frac{3}{2}}(t) = -\frac{165969920}{6634761}t^7 - \frac{59011072}{8530407}t^9 - \frac{221184}{225897815}t^{13} - \frac{3102048256}{431259465}t - \frac{241659904}{93834477}t^3 + \frac{2834432}{1579705}t^{10} - \frac{2883584}{315941}t^5 + \frac{309526528}{14217345}t^6 + \frac{79323136}{4739115}t^8 + \frac{1686790144}{156390795}t^2 - \frac{90112}{315941}t^{11} + \frac{442368}{17376755}t^{12} \quad (10)$$

$$\alpha_2(t) = \frac{124416}{225897815}t^{13} + \frac{1622016}{315941}t^5 + \frac{19781435}{20852106}t^3 - \frac{1594368}{1579705}t^{10} + \frac{50688}{315941}t^{11} + \frac{31119360}{2211587}t^7 + \frac{3688192}{947823}t^9 - \frac{70670874}{17376755}t^2 - \frac{19345408}{1579705}t^6 - \frac{248832}{17376755}t^{12} - \frac{14873088}{1579705}t^8 + \frac{732008631}{287506310}t \quad (11)$$

$$\alpha_3(t) = -\frac{13824}{225897815}t^{13} - \frac{180224}{315941}t^5 - \frac{3688192}{8530407}t^9 + \frac{1070671}{187668954}t^3 + \frac{177152}{1579705}t^{10}$$
$$-\frac{5632}{315941}t^{11} - \frac{10373120}{6634761}t^7 + \frac{4957696}{4739115}t^8 + \frac{54457973}{312781590}t^2 + \frac{19345408}{14217345}t^6$$
$$+\frac{27648}{17376755}t^{12} - \frac{16708287}{143753155}t \quad (12)$$

$$\beta_0(t) = \frac{29689343}{1712652972800}h^4t^{12} + \frac{2608450241}{38923931200}h^4t^6 - \frac{331785144503}{1402662784723200}h^4t$$
$$-\frac{11452843121}{175157690400}h^4t^5 - \frac{6869439}{11132244323200}h^4t^{13} - \frac{2257365675193}{138724890796800}h^4t^3 \quad (13)$$
$$-\frac{3576665443}{504454148352}h^4t^9 + \frac{20475115427}{934174348800}h^4t^8 + \frac{719327837}{467087174400}h^4t^{10} - \frac{227222930747}{4904415331200}h^4t^7$$
$$-\frac{1102696457}{5137958918400}h^4t^{11} + \frac{14860805023}{4403964787200}h^4t^2 + 1/24h^4t^4$$

$$\beta_{\frac{1}{3}}(t) = -\frac{10932747970689}{779257102624000}{}_{1050814593}h^4t - \frac{25831531971}{305830888000}{}_{61943859}h^4t^3$$
$$-\frac{1050814593}{8563264864000}h^4t^{12} + \frac{61943859}{13915305404000}h^4t^{13} - \frac{5566156443}{19461965600}h^4t^6 \quad (14)$$
$$+\frac{198518093703}{3425305945600}h^4t^2 + \frac{15613317}{88463480}h^4t^5 - \frac{3198135789}{311391449600}h^4t^{10} + \frac{25372801557}{17126529728000}h^4t^{11}$$

$$\beta_{\frac{2}{3}}(t) = -\frac{45175401}{4048088844800}h^4t^{13} - \frac{561579453}{155695724800}h^4t^{11} - \frac{3074047407}{311391449600}h^4t^3$$
$$+\frac{87990448797}{1245565798400}h^4t^2 + \frac{189232011}{622782899200}h^4t^{12} - \frac{1930563252711}{56673243827200}h^4t \quad (15)$$
$$+\frac{7050378303}{14154156800}h^4t^6 - \frac{1170815463}{11323325440}h^4t^9 - \frac{167453649}{707707840}h^4t^5$$

$$\beta_1(t) = \frac{27758829239}{44585593920}h^4t^7 - \frac{80918161}{155695724800}h^4t^{12} + \frac{1115000437}{186834869760}h^4t^{11} + \frac{21953789}{88463480}h^4t^5$$
$$+\frac{9917307}{506011105600}h^4t^{13} - \frac{46955985401}{630567685440}h^4t^3 - \frac{1008735953}{1769269600}h^4t^6$$
$$-\frac{16737764927}{42462470400}h^4t^8 - \frac{3288658319}{84924940800}h^4t^{10} - \frac{5886842337569}{127514798611200}h^4t \quad (16)$$

$$\beta_2(t) = \frac{33996567}{353853920}h^4t^5 - \frac{470308683}{1556957248000}h^4t^{12} - \frac{3581763209}{656841339000}h^4t^3 + \frac{2592149383}{778478624000}h^4t^{11}$$
$$+\frac{5432812878533}{425049328704000}h^4t + \frac{119235861}{10120222112000}h^4t^{13} - \frac{4914284791}{21231235200}h^4t^6$$
$$-\frac{33552172883}{1868348697600}h^4t^2 - \frac{156244530767}{849249408000}h^4t^8 - \frac{582064431}{28308313600}h^4t^{10} \quad (17)$$

$$\beta_3(t) = \frac{53569161631}{127514798611200}h^4t - \frac{3648376541}{2802523046400}h^4t^2 + \frac{15218039}{155695724800}h^4t^{12}$$
$$+\frac{1687923217}{630567685440}h^4t^3 - \frac{8511308941}{458594680320}h^4t^9 - \frac{13419109}{796171320}h^4t^5 \quad (18)$$
$$+\frac{80269463}{2022022400}h^4t^8 + \frac{464113861}{84924940800}h^4t^{10} - \frac{2375781539}{44585593920}h^4t^7$$

$$\beta_{\frac{10}{3}}(t) = -\frac{119899287}{44484492800}h^4t^3 + \frac{7795737}{7077078400}h^4t^{11} - \frac{4917755403}{113233254400}h^4t^8 - \frac{657906453}{14154156800}h^4t^6$$
$$+\frac{154163979}{177937971200}h^4t^2 - \frac{69452181}{622782899200}h^4t^{12} + \frac{19495647}{4048088844800}h^4t^{13}$$
$$-\frac{659335221}{5152113075200}h^4t + \frac{231107193}{11323325440}h^4t^9 + \frac{65018943}{3538539200}h^4t^5 \quad (19)$$

$$\beta_{\frac{11}{3}}(t) = \frac{2275881093}{2140816216000}h^4t^3 + \frac{789099687}{311391449600}h^4t^{10} + \frac{5472200511}{779257102624000}h^4t$$
$$-\frac{7995603933}{17126529728000}h^4t^{11} - \frac{946910169}{3425305945600}h^4t^2 + \frac{368226963}{19461965600}h^4t^6 \quad (20)$$
$$+\frac{411705207}{8563264864000}h^4t^{12} - \frac{58927257}{27830610808000}h^4t^{13} + \frac{13877608203}{778478624000}h^4t^8$$

$$\beta_4(t) = \frac{3825693}{11132244323200}h^4t^{13} - \frac{21799426639}{138724890796800}h^4t^3 + \frac{375013189}{5137958918400}h^4t^{11} - \frac{2618237}{342530594560}h^4t^{12} - \frac{332260879}{116771793600}h^4t^6 + \frac{1113362977}{30827753510400}h^4t^2 + \frac{396343}{353853920}h^4t^5 + \frac{2990859577}{1402662784723200}h^4t + \frac{17576183099}{4904415331200}h^4t^7 \quad (21)$$

The first derivatives of equation (7) is

$$y'_{n+4} = \alpha'_0(t)y_n + \alpha'_1(t)y_{n+1} + \alpha'_{\frac{3}{2}}(t)y_{n+\frac{3}{2}} + \alpha'_2(t)y_{n+2} + \alpha'_3(t)y_{n+3} + \beta'_3(t)f_{n+3} + \beta'_{\frac{10}{3}}(t)f_{n+\frac{10}{3}} + \beta'_{\frac{11}{3}}(t)f_{n+\frac{11}{3}} + \beta'_4(t)f_{n+4}\Big) \quad (22)$$

with the following coefficients

$$\alpha'_0(t) = -\frac{10373120}{947823}t^6 - \frac{1969040737}{862518930} - \frac{13824}{17376755}t^{12} + \frac{354304}{315941}t^9 - \frac{61952}{315941}t^{10} + \frac{523630358}{156390795}t + \frac{38690816}{4739115}t^5 + \frac{331776}{17376755}t^{11} - \frac{61485647}{62556318}t^2 + \frac{39661568}{4739115}t^7 - \frac{3688192}{947823}t^8 - \frac{901120}{315941}t^4 \quad (23)$$

(23)

$$\alpha'_1(t) = \frac{124416}{17376755}t^{12} + \frac{8110080}{315941}t^4 + \frac{40633541}{6950702}t^2 + \frac{1012893513}{143753155} + \frac{557568}{315941}t^{10} + \frac{31119360}{315941}t^6 - \frac{297732543}{17376755}t - \frac{116072448}{1579705}t^5 - \frac{2985984}{17376755}t^{11} - \frac{3188736}{315941}t^9 - \frac{118984704}{1579705}t^7 + \frac{11064576}{315941}t^8 \quad (24)$$

$$\alpha'_{\frac{3}{2}}(t) = \frac{165969920}{947823}t^6 - \frac{59011072}{947823}t^8 - \frac{221184}{17376755}t^{12} - \frac{3102048256}{431259465} - \frac{241659904}{31278159}t^2 + \frac{5668864}{315941}t^9 - \frac{14417920}{315941}t^4 + \frac{619053056}{4739115}t^5 + \frac{634585088}{4739115}t^7 + \frac{3373580288}{156390795}t - \frac{991232}{315941}t^{10} + \frac{5308416}{17376755}t^{11} \quad (25)$$

(25)

$$\alpha'_2(t) = \frac{124416}{17376755}t^{12} + \frac{8110080}{315941}t^4 + \frac{19781435}{6950702}t^2 - \frac{3188736}{315941}t^9 + \frac{557568}{315941}t^{10} + \frac{31119360}{315941}t^6 + \frac{11064576}{315941}t^8 - \frac{141341748}{17376755}t - \frac{116072448}{1579705}t^5 - \frac{2985984}{17376755}t^{11} - \frac{118984704}{1579705}t^7 + \frac{732008631}{287506310} \quad (26)$$

(26)

$$\alpha'_3(t) = -\frac{13824}{17376755}t^{12} - \frac{901120}{315941}t^4 - \frac{3688192}{947823}t^8 + \frac{1070671}{62556318}t^2 + \frac{354304}{315941}t^9 - \frac{61952}{315941}t^{10} - \frac{10373120}{947823}t^6 + \frac{39661568}{4739115}t^7 + \frac{54457973}{156390795}t + \frac{38690816}{4739115}t^5 + \frac{331776}{17376755}t^{11} - \frac{16708287}{143753155} \quad (27)$$

(27)

$$\beta'_0(t) = \frac{89068029}{428163243200}h^4t^{11} + \frac{7825350723}{19461965600}h^4t^5 - \frac{331785144503}{1402662784723200}h^4 - \frac{11452843121}{35031538080}h^4t^4 - \frac{6869439}{856326486400}h^4t^{12} - \frac{2257365675193}{46241630265600}h^4t^2 - \frac{3576665443}{56050460928}h^4t^8 + \frac{20475115427}{116771793600}h^4t^7 + \frac{719327837}{46708717440}h^4t^9 - \frac{227222930747}{700630761600}h^4t^6 - \frac{1102696457}{467087174400}h^4t^{10} + \frac{14860805023}{2201982393600}h^4t + 1/6h^4t^3 \quad (28)$$

$$\beta'_{\frac{1}{3}}(t) = -\frac{10932747970689}{779257102624000}h^4 - \frac{77494595913}{305830888000}h^4t^2 + \frac{25372801557}{1556957248000}h^4t^{10} - \frac{3152443779}{2140816216000}h^4t^{11} + \frac{61943859}{1070408108000}h^4t^{12} - \frac{16698469329}{9730982800}h^4t^5 + \frac{198518093703}{1712652972800}h^4t + \frac{15613317}{17692696}h^4t^4 - \frac{3198135789}{31139144960}h^4t^9 + \frac{331427398761}{194619656000}h^4t^6$$

$$+\frac{126172156059}{311391449600}h^4t^8-\frac{100694676447}{97309828000}h^4t^7 \tag{29}$$
(29)

$$\beta'_{\frac{2}{3}}(t)=-\frac{45175401}{311391449600}h^4t^{12}-\frac{561579453}{14154156800}h^4t^{10}-\frac{9222142221}{311391449600}h^4t^2+\frac{87990448797}{622782899200}h^4t+\frac{567696033}{155695724800}h^4t^{11}-\frac{1930563252711}{56673243827200}h^4+\frac{21151134909}{7077078400}h^4t^5-\frac{10537339167}{11323325440}h^4t^8$$
$$-\frac{167453649}{141541568}h^4t^4+\frac{1382462847}{5661662720}h^4t^9-\frac{48331990197}{14154156800}h^4t^6+\frac{4560085539}{2022022400}h^4t^7 \tag{30}$$
(30)

$$\beta'_1(t)=\frac{27758829239}{6369370560}h^4t^6-\frac{242754483}{38923931200}h^4t^{11}+\frac{1115000437}{16984988160}h^4t^{10}+\frac{21953789}{17692696}h^4t^4$$
$$+\frac{9917307}{38923931200}h^4t^{12}-\frac{46955985401}{210189228480}h^4t^2-\frac{3026207859}{884634800}h^4t^5-\frac{16737764927}{5307808800}h^4t^7$$
$$-\frac{3288658319}{8492494080}h^4t^9-\frac{5886842337569}{127514798611200}h^4+\frac{71193202289}{50954964480}h^4t^8+\frac{258400671619}{1401261523200}h^4t \tag{31}$$
(31)

$$\beta'_2(t)=\frac{33996567}{70770784}h^4t^4-\frac{1410926049}{389239312000}h^4t^{11}-\frac{3581763209}{218947113000}h^4t^2+\frac{2592149383}{70770784000}h^4t^1$$
$$+\frac{5432812878533}{425049328704000}h^4+\frac{119235861}{778478624000}h^4t^{12}-\frac{4914284791}{3538539200}h^4t^5-\frac{33552172883}{934174348800}h^4t$$
$$-\frac{156244530767}{106156176000}h^4t^7-\frac{582064431}{2830831360}h^4t^9+\frac{6266599169}{3317380500}h^4t^6+\frac{59358168701}{84924940800}h^4t^8 \tag{32}$$
(32)

$$\beta'_3(t)=\frac{53569161631}{127514798611200}h^4-\frac{3648376541}{1401261523200}h^4t+\frac{45654117}{38923931200}h^4t^{11}+\frac{1687923217}{210189228480}h^4t^2$$
$$-\frac{8511308941}{50954964480}h^4t^8-\frac{13419109}{159234264}h^4t^4-\frac{1049859}{19461965600}h^4t^{12}+\frac{226267131}{884634800}h^4t^5$$
$$-\frac{183184301}{16984988160}h^4t^{10}+\frac{80269463}{252752800}h^4t^7+\frac{464113861}{8492494080}h^4t^9-\frac{2375781539}{6369370560}h^4t \tag{33}$$
(33)

$$\beta'_{\frac{10}{3}}(t)=-\frac{359697861}{44484492800}h^4t^2+\frac{85753107}{7077078400}h^4t^{10}-\frac{4917755403}{14154156800}h^4t^7-\frac{1973719359}{7077078400}h^4t^5 \tag{34}$$
$$+\frac{154163979}{88968985600}h^4t-\frac{208356543}{155695724800}h^4t^{11}+\frac{19495647}{311391449600}h^4t^{12}-\frac{659335221}{5152113075200}h^4$$

$$\beta'_{\frac{11}{3}}(t)=\frac{6827643279}{2140816216000}h^4t^2+\frac{789099687}{31139144960}h^4t^9+\frac{5472200511}{779257102624000}h^4$$
$$-\frac{7995603933}{1556957248000}h^4t^{10}-\frac{946910169}{1712652972800}h^4t+\frac{1104680889}{9730982800}h^4t^5 \tag{35}$$
$$+\frac{1235115621}{2140816216000}h^4t^{11}-\frac{58927257}{2140816216000}h^4t^{12}+\frac{13877608203}{97309828000}h^4t^7$$

$$\beta'_4(t)=\frac{3825693}{856326486400}h^4t^{12}-\frac{21799426639}{46241630265600}h^4t^2+\frac{375013189}{467087174400}h^4t^{10}$$
$$-\frac{7854711}{85632648640}h^4t^{11}-\frac{332260879}{19461965600}h^4t^5+\frac{1113362977}{15413876755200}h^4t+\frac{396343}{70770784}h^4t^4 \tag{36}$$
$$+\frac{2990859577}{1402662784723200}h^4+\frac{17576183099}{700630761600}h^4t^6+\frac{3252180649}{280252304640}h^4t^8-\frac{14432339}{667267392}h^4t^7$$

The second derivatives of equation (7) is,

$$y''_{n+4}=\alpha''_0(t)y_n+\alpha''_1(t)y_{n+1}+\alpha''_{\frac{3}{2}}(t)y_{n+\frac{3}{2}}+\alpha''_2(t)y_{n+2}+\alpha''_3(t)y_{n+3}$$
$$\left.+\beta''_3(t)f_{n+3}+\beta''_{\frac{10}{3}}(t)f_{n+\frac{10}{3}}+\beta''_{\frac{11}{3}}(t)f_{n+\frac{11}{3}}+\beta''_4(t)f_{n+4}\right) \tag{37}$$

with the following coefficients

$$\alpha_0{}''(t)=-\frac{20746240}{315941}t^5-\frac{165888}{17376755}t^{11}+\frac{3188736}{315941}t^8-\frac{619520}{315941}t^9+\frac{523630358}{156390795}+\frac{38690816}{947823}t^4$$
$$+\frac{331776}{1579705}t^{10}-\frac{61485647}{31278159}t+\frac{277630976}{4739115}t^6-\frac{29505536}{947823}t^7-\frac{3604480}{315941}t^3 \tag{38}$$
(38)

$$\alpha''_1(t)=\frac{1492992}{17376755}t^{11}+\frac{32440320}{315941}t^3+\frac{40633541}{3475351}t+\frac{5575680}{315941}t^9+\frac{186716160}{315941}t^5-\frac{297732543}{17376755}$$

$$-\frac{116072448}{315941}t^4-\frac{2985984}{1579705}t^{10}-\frac{28698624}{315941}t^8-\frac{832892928}{1579705}t^6+\frac{88516608}{315941}t^7 \quad (39)$$

$$\alpha''_{\frac{3}{2}}(t)=-\frac{331939840}{315941}t^5-\frac{472088576}{947823}t^7-\frac{2654208}{17376755}t^{11}-\frac{483319808}{31278159}t+\frac{51019776}{315941}t^8-\frac{57671680}{315941}t^3+\frac{619053056}{947823}t^4+\frac{4442095616}{4739115}t^6+\frac{3373580288}{156390795}-\frac{9912320}{315941}t^9+\frac{5308416}{1579705}t^{10} \quad (41)$$

$$\alpha''_2(t)=\frac{1492992}{17376755}t^{11}+\frac{32440320}{315941}t^3+\frac{19781435}{3475351}t-\frac{28698624}{315941}t^8+\frac{5575680}{315941}t^9+\frac{186716160}{315941}t^5+\frac{88516608}{315941}t^7-\frac{141341748}{17376755}-\frac{116072448}{315941}t^4-\frac{2985984}{1579705}t^{10}-\frac{832892928}{1579705}t^6 \quad (41)$$

$$\alpha''_3(t)=-\frac{165888}{17376755}t^{11}-\frac{3604480}{315941}t^3-\frac{29505536}{947823}t^7+\frac{1070671}{31278159}t+\frac{3188736}{315941}t^8-\frac{619520}{315941}t^9-\frac{20746240}{315941}t^5+\frac{277630976}{4739115}t^6+\frac{54457973}{156390795}+\frac{38690816}{947823}t^4+\frac{331776}{1579705}t^{10} \quad (42)$$

$$\beta''_0(t)=\frac{89068029}{38923931200}h^4t^{10}+\frac{7825350723}{3892393120}h^4t^4-\frac{11452843121}{8757884520}h^4t^3-\frac{20608317}{214081621600}h^4t^{11}-\frac{2257365675193}{23120815132800}h^4t-\frac{3576665443}{7006307616}h^4t^7+\frac{20475115427}{16681684800}h^4t^6+\frac{2157983511}{15569572480}h^4t^8-\frac{227222930747}{116771793600}h^4t^5-\frac{1102696457}{46708717440}h^4t^9+\frac{14860805023}{2201982393600}h^4+\frac{1}{2h^4t^2} \quad (43)$$

$$\beta_{\frac{1}{3}}{}''(t)=-\frac{77494595913}{152915444000}h^4t+\frac{25372801557}{155695724800}h^4t^9-\frac{3152443779}{194619656000}h^4t^{10}+\frac{185831577}{267602027000}h^4t^{11}-\frac{16698469329}{1946196560}h^4t^4+\frac{198518093703}{1712652972800}h^4+\frac{15613317}{4423174}h^4t^3-\frac{28783222101}{31139144960}h^4t^8 \quad (44)$$

$$\beta_{\frac{2}{3}}{}''(t)=-\frac{135526203}{77847862400}h^4t^{11}-\frac{561579453}{1415415680}h^4t^9-\frac{9222142221}{155695724800}h^4t+\frac{87990448797}{622782899200}h^4+\frac{567696033}{14154156800}h^4t^{10}+\frac{21151134909}{1415415680}h^4t^4-\frac{10537339167}{1415415680}h^4t^7-\frac{167453649}{35385392}h^4t^3+\frac{12442165623}{5661662720}h^4t^8-\frac{144995970591}{7077078400}h^4t^5+\frac{31920598773}{2022022400}h^4t^6 \quad (45)$$

$$\beta''_1(t)=\frac{27758829239}{1061561760}h^4t^5-\frac{242754483}{3538539200}h^4t^{10}+\frac{1115000437}{1698498816}h^4t^9+\frac{21953789}{4423174}h^4t^3+\frac{29751921}{9730982800}h^4t^{11}-\frac{46955985401}{105094614240}h^4t-\frac{3026207859}{176926960}h^4t^4-\frac{16737764927}{758258400}h^4t^6-\frac{9865974957}{2830831360}h^4t^8+\frac{71193202289}{6369370560}h^4t^7+\frac{258400671619}{1401261523200}h^4 \quad (46)$$

(46)

$$\beta''_2(t)=\frac{33996567}{17692696}h^4t^3-\frac{1410926049}{35385392000}h^4t^{10}-\frac{3581763209}{109473556500}h^4t+\frac{2592149383}{7077078400}h^4t^9+\frac{357707583}{194619656000}h^4t^{11}-\frac{4914284791}{707707840}h^4t^4-\frac{33552172883}{934174348800}h^4-\frac{156244530767}{15165168000}h^4t^6-\frac{5238579879}{2830831360}h^4t^8+\frac{6266599169}{552896750}h^4t^5+\frac{59358168701}{10615617600}h^4t^7 \quad (47)$$

$$\beta''_3(t)=-\frac{3648376541}{1401261523200}h^4+\frac{45654117}{3538539200}h^4t^{10}+\frac{1687923217}{105094614240}h^4t-\frac{8511308941}{6369370560}h^4t^7-\frac{13419109}{39808566}h^4t^3-\frac{3149577}{4865491400}h^4t^{11}+\frac{226267131}{176926960}h^4t^4-\frac{183184301}{1698498816}h^4t^9+\frac{561886241}{252752800}h^4t^6+\frac{1392341583}{2830831360}h^4t^8-\frac{2375781539}{1061561760}h^4t^5 \quad (48)$$

$$\beta''_{\frac{10}{3}}(t) = -\frac{359697861}{22242246400}h^4t + \frac{85753107}{707707840}h^4t^9 - \frac{4917755403}{2022022400}h^4t^6 - \frac{1973719359}{1415415680}h^4t^4 + \frac{154163979}{88968985600}h^4 - \frac{208356543}{14154156800}h^4t^{10} + \frac{58486941}{77847862400}h^4t^{11} + \frac{2079964737}{1415415680}h^4t^7 + \frac{65018943}{176926960}h^4t^3 - \frac{3089478753}{5661662720}h^4t^8 + \frac{17285139423}{7077078400}h^4t^5 \quad (49)$$

$$\beta''_{\frac{11}{3}}(t) = \frac{6827643279}{1070408108000}h^4t + \frac{7101897183}{31139144960}h^4t^8 - \frac{7995603933}{155695724800}h^4t^9 - \frac{946910169}{1712652972800}h^4 + \frac{1104680889}{1946196560}h^4t^4 + \frac{1235115621}{194619656000}h^4t^{10} - \frac{176781771}{535204054000}h^4t^{11} + \frac{13877608203}{13901404000}h^4t^6 - \frac{23654908791}{38923931200}h^4t^7 - \frac{7263351}{48654914}h^4t^3 - \frac{97062750927}{97309828000}h^4t^5 \quad (50)$$

$$\beta''_4(t) = \frac{11477079}{214081621600}h^4t^{11} - \frac{21799426639}{23120815132800}h^4t + \frac{375013189}{46708717440}h^4t^9 - \frac{7854711}{7784786240}h^4t^{10} - \frac{332260879}{3892393120}h^4t^4 + \frac{1113362977}{15413876755200}h^4 + \frac{396343}{17692696}h^4t^3 + \frac{17576183099}{116771793600}h^4t^5 \quad (51)$$

The third derivatives of equation (7) is,

$$y'''_{n+4} = \alpha'''_0(t)y_n + \alpha'''_1(t)y_{n+1} + \alpha'''_{\frac{3}{2}}(t)y_{n+\frac{3}{2}} + \alpha'''_2(t)y_{n+2} + \alpha'''_3(t)y_{n+3} + \beta'''_3(t)f_{n+3} + \beta''_{\frac{10}{3}}(t)f_{n+\frac{10}{3}} + \beta'''_{\frac{11}{3}}(t)f_{n+\frac{11}{3}} + \beta'''_4(t)f_{n+4}\Big) \quad (52)$$

with the following coefficients

$$\alpha''''_0(t) = -\frac{103731200}{315941}t^4 - \frac{165888}{1579705}t^{10} + \frac{25509888}{315941}t^7 - \frac{5575680}{315941}t^8 + \frac{154763264}{947823}t^3 + \frac{663552}{315941}t^9 - \frac{61485647}{31278159} + \frac{555261952}{1579705}t^5 - \frac{206538752}{947823}t^6 - \frac{10813440}{315941}t^2 \quad (53)$$

$$\alpha_1'''(t) = \frac{1492992}{1579705}t^{10} + \frac{97320960}{315941}t^2 + \frac{40633541}{3475351} + \frac{50181120}{315941}t^8 + \frac{933580800}{315941}t^4$$
$$- \frac{464289792}{315941}t^3 - \frac{5971968}{315941}t^9 - \frac{229588992}{315941}t^7 - \frac{4997357568}{1579705}t^5 + \frac{619616256}{315941}t^6 \quad (54)$$

$$\alpha_{\frac{3}{2}}'''(t) = -\frac{1659699200}{315941}t^4 - \frac{3304620032}{947823}t^6 - \frac{2654208}{1579705}t^{10} - \frac{483319808}{31278159} + \frac{408158208}{315941}t^7$$
$$- \frac{173015040}{315941}t^2 + \frac{2476212224}{947823}t^3 + \frac{8884191232}{1579705}t^5 - \frac{89210880}{315941}t^8 + \frac{10616832}{315941}t^9 \quad (55)$$

(55)

$$\alpha_2'''(t) = \frac{1492992}{1579705}t^{10} + \frac{97320960}{315941}t^2 + \frac{19781435}{3475351} - \frac{229588992}{315941}t^7 + \frac{50181120}{315941}t^8 + \frac{933580800}{315941}t^4$$
$$+ \frac{619616256}{315941}t^6 - \frac{464289792}{315941}t^3 - \frac{5971968}{315941}t^9 - \frac{4997357568}{1579705}t^5 \quad (56)$$

$$\alpha_3'''(t) = -\frac{165888}{1579705}t^{10} - \frac{10813440}{315941}t^2 - \frac{206538752}{947823}t^6 + \frac{1070671}{31278159} + \frac{25509888}{315941}t^7$$
$$- \frac{5575680}{315941}t^8 - \frac{103731200}{315941}t^4 + \frac{555261952}{1579705}t^5 + \frac{154763264}{947823}t^3 + \frac{663552}{315941}t^9 \quad (57)$$

$$\beta_0'''(t) = \frac{89068029}{3892393120}h^4t^9 + \frac{7825350723}{973098280}h^4t^3 - \frac{11452843121}{2919294840}h^4t^2 - \frac{20608317}{19461965600}h^4t^{10}$$
$$- \frac{2257365675193}{23120815132800}h^4 - \frac{3576665443}{1000901088}h^4t^6 + \frac{20475115427}{2780280800}h^4t^5 + \frac{2157983511}{1946196560}h^4t^7$$
$$- \frac{227222930747}{23354358720}h^4t^4 - \frac{3308089371}{15569572480}h^4t^8 + h^4t \quad (58)$$

$$\beta_{\frac{1}{3}}'''(t) = -\frac{77494595913}{152915444000}h^4 + \frac{228355214013}{155695724800}h^4t^8 - \frac{3152443779}{19461965600}h^4t^9 + \frac{185831577}{24327457000}h^4t^{10}$$
$$- \frac{16698469329}{486549140}h^4t^3 + \frac{46839951}{4423174}h^4t^2 - \frac{28783222101}{3892393120}h^4t^7 + \frac{994282196283}{19461965600}h^4t^4 \quad (59)$$

$$\beta_{\frac{2}{3}}'''(t) = -\frac{135526203}{7077078400}h^4t^{10} - \frac{5054215077}{1415415680}h^4t^8 - \frac{9222142221}{155695724800}h^4 + \frac{567696033}{1415415680}h^4t^9$$
$$+ \frac{21151134909}{353853920}h^4t^3 - \frac{10537339167}{202202240}h^4t^6 - \frac{502360947}{35385392}h^4t^2 + \frac{12442165623}{707707840}h^4t^7$$
$$- \frac{144995970591}{1415415680}h^4t^4 + \frac{95761796319}{1011011200}h^4t^5$$
$$+ \frac{29751921}{884634800}h^4t^{10} - \frac{46955985401}{105094614240}h^4 - \frac{3026207859}{44231740}h^4t^3 - \frac{16737764927}{126376400}h^4t^5 \quad (60)$$

$$\beta_2'''(t) = \frac{101989701}{17692696}h^4t^2 - \frac{1410926049}{3538539200}h^4t^9 - \frac{3581763209}{109473556500}h^4 + \frac{23329344447}{7077078400}h^4t^8 + \frac{357707583}{17692696000}h^4t^{10} - \frac{4914284791}{176926960}h^4t^3 - \frac{156244530767}{2527528000}h^4t^5 - \frac{5238579879}{353853920}h^4t^7 + \frac{6266599169}{110579350}h^4t^4 + \frac{59358168701}{1516516800}h^4t^6 \quad (61)$$

$$\beta_3'''(t) = \frac{45654117}{353853920}h^4t^9 + \frac{1687923217}{105094614240}h^4 - \frac{8511308941}{909910080}h^4t^6 - \frac{13419109}{13269522}h^4t^2 - \frac{3149577}{442317400}h^4t^{10} + \frac{226267131}{44231740}h^4t^3 - \frac{549552903}{566166272}h^4t^8 + \frac{1685658723}{126376400}h^4t^5 \quad (62)$$
$$+ \frac{1392341583}{353853920}h^4t^7 - \frac{2375781539}{212312352}h^4t^4$$
$$- \frac{208356543}{1415415680}h^4t^9 + \frac{58486941}{7077078400}h^4t^{10} + \frac{2079964737}{202202240}h^4t^6 + \frac{195056829}{176926960}h^4t^2 - \frac{3089478753}{707707840}h^4t^7 + \frac{17285139423}{1415415680}h^4t^4$$

$$\beta_{\frac{10}{3}}'''(t) = -\frac{359697861}{2224224640}h^4 + \frac{771777963}{70770780}h^4t^8 - \frac{14753266209}{101101200}h^4t^5 - \frac{1973719359}{35385392}h^4t^3 \quad (63)$$

$$\beta_{\frac{11}{3}}'''(t) = \frac{6827643279}{1070408108000}h^4 + \frac{7101897183}{3892393120}h^4t^7 - \frac{71960435397}{155695724800}h^4t^8 + \frac{1104680889}{486549140}h^4t^3 + \frac{1235115621}{19461965600}h^4t^9 - \frac{176781771}{48654914000}h^4t^{10} + \frac{41632824609}{6950702000}h^4t^5 - \frac{23654908791}{5560561600}h^4t^6 - \frac{21790053}{48654914}h^4t^2 - \frac{97062750927}{19461965600}h^4t^4$$
$$+ \frac{1235115621}{19461965600}h^4t^9 - \frac{176781771}{48654914000}h^4t^{10} + \frac{41632824609}{6950702000}h^4t^5 - \frac{23654908791}{5560561600}h^4t^6 - \frac{21790053}{48654914}h^4t^2 - \frac{97062750927}{19461965600}h^4t^4 \quad (64)$$

Evaluating (7) at $t = 1$, yield the discrete four-step method corrector below $y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n - \frac{1217}{221760}h^4f_{n+4} + \frac{5697}{123200}h^4f_{n+\frac{11}{3}} - \frac{1161}{8960}h^4f_{n+\frac{10}{3}}$

$$+ \frac{5857}{20160}h^4f_{n+3} + \frac{40091}{67200}h^4f_{n+2} + \frac{5857}{20160}h^4f_{n+1} - \frac{1161}{8960}h^4f_{n+\frac{2}{3}} + \frac{5697}{123200}h^4f_{n+\frac{1}{3}} \quad (65)$$

Evaluating (22) at $t = 1$, yield the first derivative discrete four-step corrector method below

$$y'_{n+4} = -\frac{1}{84159767083392000h}(1015969269589460h^4f_n - 54342520503584060h^4f_{n+1} - 105542916778000614h^4f_{n+2} - 58263192093056060h^4f_{n+3} + 995882709344660h^4f_{n+4} - 8577312482433348h^4f_{n+\frac{1}{3}} + 23903924962405455h^4f_{n+\frac{2}{3}} + 21047808772964655h^4f_{n+\frac{10}{3}} - 9758640260922948h^4f_{n+\frac{11}{3}} + 192127968488332800y_n - 929633929102771200y_{n+1} + 459002227773196800y_{n+2} - 326857261859251200y_{n+3} + 605360994700492800y_{n+\frac{3}{2}}\Big) \quad (66)$$

Evaluating (37) at $t = 1$, yield the second derivative discrete four-step corrector method below

$$y''_{n+4} = -\frac{1}{924832605312000h^2}(15654924378820h^4f_n - 849752578662460h^4f_{n+1} - 1550397492156678h^4f_{n+2} - 1022704950448060h^4f_{n+3} + 9480188047780h^4f_{n+4} - 132579119492676h^4f_{n+\frac{1}{3}} + 367322935870935h^4f_{n+\frac{2}{3}} + 238259653969095h^4f_{n+\frac{10}{3}} - 240290221583556h^4f_{n+\frac{11}{3}} + 3096540485068800y_n - 1769570570718 7200y_{n+1} - 3823216627507200y_{n+2} - 1527622541491200y_{n+3} + 19950004391116800y_{n+\frac{3}{2}}\Big) \quad (67)$$

Evaluating (52) at $t = 1$, yield the third derivative discrete four-step corrector method below

$$\begin{aligned} y'''_{n+4} = -\frac{1}{231208151328000h^3}(&2048511774010h^4f_n - 119831048002600h^4f_{n+1} \\ &-231482524751808h^4f_{n+2} - 226847646962200h^4f_{n+3} - 20307150711530h^4f_{n+4} \\ &-17733623224536h^4f_{n+\frac{1}{3}} + 49366995974505h^4f_{n+\frac{2}{3}} + 39411174041415h^4f_{n+\frac{10}{3}} \\ &-136380223193256h^4f_{n+\frac{11}{3}} + 454501902624000y_n - 2703268215648000y_{n+1} \end{aligned} \tag{68}$$

It should be noted that the predictors and their correctors for the method were derived using the same approximate solution (5). The procedure for the development of the predictor is the same with the corrector method but with different points of interpolation and collocation. The predictor and its first, second, and third derivatives developed are as follows

$$\begin{aligned} y_{n+4} = {} & \frac{3632732}{129303}y_{n+3} - \frac{3201722}{14367}y_{n+2} + \frac{49848320}{129303}y_{n+\frac{3}{2}} - \frac{3058052}{14367}y_{n+1} + \frac{2986217}{129303}y_n \\ & + \frac{5776899}{2191446400}h^4f_{n+\frac{11}{3}} + \frac{5068821}{472003840}h^4f_{n+\frac{10}{3}} + \frac{420006437}{5213496960}h^4f_{n+3} \\ & - \frac{237114259}{137923200}h^4f_{n+2} - \frac{24608012608}{5824453635}h^4f_{n+\frac{3}{2}} - \frac{365890319}{193092480}h^4f_{n+1} - \frac{1853889}{8581888}h^4f_{n+\frac{2}{3}} \\ & - \frac{8284299}{107273600}h^4f_{n+\frac{1}{3}} + \frac{5760061}{2048159520}h^4f_n \end{aligned} \tag{69}$$

$$\begin{aligned} y'_{n+4} = {} & \frac{1}{4477653572071680000h}(2608382763346360h^4f_n - 1820873777705521110h^4f_{n+1} \\ & -1674313331654237274h^4f_{n+2} + 1072665925306303900h^4f_{n+3} \\ & -735717170762325180h^4f_{n+\frac{1}{3}} - 210642990500910825h^4f_{n+\frac{2}{3}} \\ & -4079135577475727360h^4f_{n+\frac{3}{2}} + 234890648833949550h^4f_{n+\frac{10}{3}} \\ & +98192307214205820h^4f_{n+\frac{11}{3}} + 22240959825641356800y_n \\ & -204422409324240307200y_{n+1} - 211810537718158579200y_{n+2} \\ & +250021795284188928000y_{n+3} + 368989807688338636800y_{n+\frac{3}{2}}\Big) \end{aligned} \tag{70}$$

$$\begin{aligned} y''_{n+4} = {} & -\frac{1}{4920498430848000h^2}(7004675708680h^4f_n + 15564844122823710h^4f_{n+1} \\ & +13035106050973578h^4f_{n+2} - 3511469472844190h^4f_{n+3} + 429396755799726h^4f_{n+\frac{1}{3}} \\ & +2748838751563905h^4f_{n+\frac{2}{3}} + 38830850243706880h^4f_{n+\frac{3}{2}} \\ & -21970640312595h^4f_{n+\frac{10}{3}} - 877671059982894h^4f_{n+\frac{11}{3}} - 204976024962508800y_n \\ & +1898909707401907200y_{n+1} + 1972717183864627200y_{n+2} \\ & -2295785171167488000y_{n+3} - 3437072349187276800y_{n+\frac{3}{2}}\Big) \end{aligned} \tag{71}$$

$$\begin{aligned} y'''_{n+4} = {} & -\frac{1}{1230124607712000h^3}(174308514591640h^4f_n - 4366276337126632 5h^4f_{n+1} \\ & -4682295980866132 2h^4f_{n+2} - 534057120108297 5h^4f_{n+3} \\ & -252510539886504 9h^4f_{n+\frac{1}{3}} - 143927130182764 5h^4f_{n+\frac{2}{3}} \\ & -8317808931442688 0h^4f_{n+\frac{3}{2}} + 297211334602513 5h^4f_{n+\frac{10}{3}} \\ & -158408286869457 9h^4f_{n+\frac{11}{3}} + 476779750967520000y_n \\ & -4283637011061408000y_{n+1} - 4276256263415136000y_{n+2} \\ & +4743195017520960000y_{n+3} + 7608794021756928000y_{n+\frac{3}{2}}\Big) \end{aligned} \tag{72}$$

3 Basic Properties of the Methods

In this section, analysis of the derived schemes such as order and error constant of the predictors and correctors, consistency, zero-stability, convergence and the stability domain was carried out. Suppose the linear Operator defined on the method (7) be defined as,

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h^4 \beta_j y^{iv}(x_{n+jh}) \tag{73}$$

where $y(x)$ is any continuously differentiable test function on the interval $[a,b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh), y'''(x_n + jh), j = 0,1, \dots, k$ in (75)

Taylor series about x_n and collecting like terms in h and y gives;
$[y(x);h] = I_0 y(x) + I_1 h y'(x) + \cdots I_p h^p y^p(x) + I_{p+1} h^{p+1} y^{p+1}(x) + I_{p+2} h^{p+2} y^{p+2}(x) + \cdots$ Definition 1: [20] The term I_{p+4} is called an error constant, meaning that the local truncation error is given as $T_{n+k} = I_{p+4} h^{(p+4)} y^{p+4}(x) + 0(h^{p+5})$

Definition 2. The difference operator L associated with the discrete implicit 4 step method (75) are said to be of order p if $I_0 = I_1 = I_2 = \cdots = I_{p+3} = 0, I_{p+4} \neq 0$ see [23]
Definition 3 [21-28]. LMM is a computational method for determining the sequence y_n which takes the form of a linear relationship between y_{n+j} and $f_{n+j}, j = 0(1)k$. The general form of a linear k - step method for mth order general odes may be written as

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{74}$$

α_j, β_j are the coefficients of the method, $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y_{n+j}^{i}, y_{n+j}^{ii}, \dots, y_{n+j}^{m-1}\right)$, $j = 0(1)k, h$ is the steplength, m is the order of ode to be solved: $\alpha_k \neq 0. \alpha_0$ and β_0 are not both zero [26, 27]. Definition 4. A multi-step method is said to be P-satble, if its interval of periodicity is $(0, \infty)$ see [24]

3.1 Order and Error constant of the Method

Apply the linear operator L in (75) to determine the order and error constant of the derived method. Expanding the method (75) and its derivatives with the Taylor series and combining the coefficients of the same terms with h^n yields $I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = \frac{19}{51892402} = 3.661422343 * 10^{-7} = 0.0000003661$ Hence, the corrector (main method) (67) is of order 10 with error constant

$$I_{14} = -\frac{19}{51892402} = 3.661422343 * 10^{-7}$$

The error constant of the derivatives (68), (69) and (70) are as follows
$I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = \frac{25}{31025671} = 8.057843456 * 10^{-7}$
$I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = \frac{28}{24973633} = 0.000001121182488$
$I_0^{24973633} = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = \frac{39}{81403303} = 4.790960386 * 10^{-7}$
Hence the first, second and third derivatives of the method are of order 10 with error constants $I_{14} = \frac{25}{31025671} = 8.057843456 * 10^{-7}, I_{14} = \frac{28}{24973633} = 0.000001121182488$ and $I_{14} = \frac{\frac{39}{8102671}}{81403303} = 4.790960386 * 10^{-7}$ respectively.
Similarly, the predictor (main method) (71) is of order 10 with error constant $I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = -\frac{4}{103129829} = -3.878606254 * 10^{-8}$

$$I_{14} = -\frac{4}{103129829} = -3.878606254 * 10^{-8}$$

and
$I_0 = I_{11} = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = -\frac{11}{163358304} = -6.733664423 * 10^{-8}$
$I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = \frac{8}{21928057} = 3.648294059 * 10^{-7}$
$I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = I_8 = I_9 = I_{10} = I_{11} = I_{12} = I_{13} = 0, I_{14} = \frac{166}{2351545} = 0.000006959716867$
23851545 The error constant of the derivatives (72), (73) and (74) are as follows, $I_{14} = -\frac{11}{163358304} = -6.733664423 * 10^{-8}, I_{14} = \frac{8}{21928057} = 3.648294059 * 10^{-7}$ and
$I_{14} = \frac{166}{23851545} = 0.000006959716867$ respectively.
Hence the first, second and third derivatives of the predictor of the method are of order 10 .

3.2 Zero-stability of the Method

Suppose the first characteristics polynomial of (75) is,

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1$$

Solving $\rho(r), r = 0,1,1$ which satisfies $|R_J| \geq 1, j = 1, \dots, k$. The roots are inside the unit circle and the multiplicity is simple. Therefore, the method is zero stable.

3.3 Consistency of the Method

According to [23], the major condition is sufficient for method to be consistence is to have an order p equals or greater than one. Consequently, Our method is of order $p = 10$. So it's consistent.

3.4 Convergence of the Method

A numerical method is said to be convergence, If it is consistence and zerostable. Thus, the methods are convergence since it satisfied the conditions in Section 3.2 and Section 3.3.

3.5 Stability Domain of the Method

Here, we consider the stability polynomial

$$\prod (z, \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0 \tag{75}$$

To determine the absolute stability region in this work, a method was used that does not require computing polynomial roots or solving systems of inequalities. This method according to [24] is called the Boundary Locus Method (BLM).

Definition 5. For $\bar{h}$ the region R in the $\bar{h}-$ complex plane such that the zeros of $\Pi(r, \bar{h}) = 0$ lie within the unit circle being in the region is called the absolute stability region

Thus, we redefine (57) in terms of Euler's number, $e^i\theta$, as follows

$$\pi\left(e^{i\theta}, h\right) = \rho\left(e^{i\theta}\right) = 0 \tag{76}$$

so that, the locus of the boundary δR is given by

$$\bar{h}(\theta) = \frac{\rho\left(e^{i\theta}\right)}{\sigma(e^{i\theta})} \tag{77}$$

ρ is the first characteristic polynomial and σ is the second characteristic polynomial.

If $\Pi(z, \bar{h}) = 0, \bar{h} = \lambda h^2$

Thus,

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \tag{78}$$

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 \tag{79}$$

$$\sigma(r) = -\frac{1217}{221760}r^4 + \frac{5697}{123200}r^{\frac{11}{3}} - \frac{1161}{8960}r^{\frac{10}{3}} + \frac{5857}{20160}r^3 + \frac{40091}{67200}r^2 + \frac{5857}{20160}r - \frac{1161}{8960}r^{\frac{2}{3}} + \tag{80}$$

$$\frac{5697}{123200}\sqrt[3]{r} - \frac{1217}{221760} \tag{81}$$

Using Matlab to plot (57), we have the Figure 1 below.

$$\frac{5697}{123200}\sqrt[3]{r} - \frac{1217}{221760} \tag{82}$$

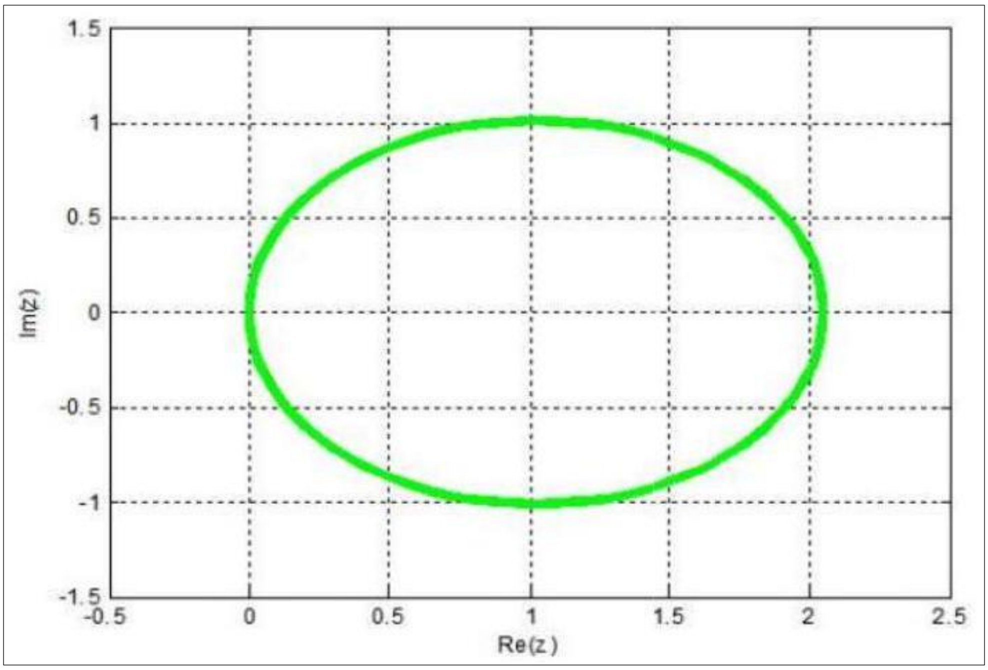

Figure 1: Stability domain of the proposed method

The inside of the curve represents the unstable region, while the outer segment of the curve represents the stable region [27]. The new method generated is absolutely-stable.

4 Numerical Examples

In this section, the predictors and correctors developed in this work are implemented using a code written in MATLAB for direct solution of problems ranging from special to nonlinear, and application problem namely Ship dynamics of fourth-order initial value problems of ordinary differential equations. Our results are compared with those obtained with existing methods. The following questions are taken as test questions.

4.1 Test problems

Problem 1

Consider the special fourth order below

$y^{iv} = x$

$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0, h = 0.1$

Exact solution: $y(x) = \frac{x^5}{120} + x$

Source: [26]

Problem 2

Consider the non-linear equation of fourth order below

$y^{iv} = (y')^2 - y(y)'' - 4(x)^2 + \exp(x)(1 - 4x + x^2),$

$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, h = \frac{1}{320}$

Exact solution: $y(x) = x^2 + \exp(x)$

Source: [27]

Problem 3

Consider an application problem from ship Dynamics below

$y^4 + 3y'' + y(2 + \varepsilon\cos(\Omega t)) = 0, t \geq 0$

Which is subjected to the following initial conditions

$y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0$ Where $\varepsilon = 0$ for the existence of the theoretical solution, $y(t) = 2\cos\ t - \cos\ t\sqrt{2}$

Source: [28]

4.2 Numerical Results

Table 1: Numerical results for a special fourth-order in problem 1 with $h = 0.1$

x	Exact solution	Computed solution	Error in the Proposed method	Error in $[26]$
0.10	0.100000083333333	0.100000083333333	$0.000000e + 00$	$1.66666667E - 10$
0.20	0.200002666666667	0.200002666666657	$9.242607e - 15$	$3.33333305E - 10$
0.30	0.30002025000000	0.300020249999992	$2.498002e - 15$	$5.99999994E - 10$
0.40	0.40008533333333	0.400085333333340	$6.494805e - 15$	$7.66666675E - 10$
0.50	0.500260416666667	0.500260416666706	$3.885781e - 14$	$9.33333300E - 10$
0.60	0.60064800000000	0.600648000000163	$1.629807e - 13$	$1.10000009E - 09$
0.70	0.70140058333333	0.701400583333709	$3.756995e - 13$	$1.27166666E - 09$
0.80	0.802730666666667	0.802730666667436	$7.696066e - 13$	$1.45333334E - 09$
0.90	0.90492075000000	0.904920750001487	$1.487144e - 12$	$1.64999991E - 09$
1.00	1.008333333333333	1.008333333335920	$2.586598e - 12$	$1.87666660E - 09$

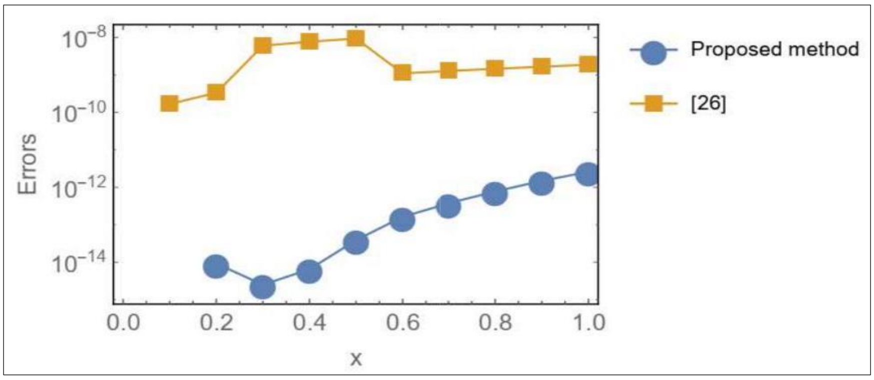

Figure 2: Comparison of the proposed method with [26] for test problem 1

Table 2: Numerical results for a nonlinear fourth-order in problem 2

x	Exact solution	Computed solution	Error in the Proposed method	Error in [27]
0.003125	1.00313965352774	1.00313965352774	$2.220446e - 15$	$1.14888400e - 12$
0.006250	1.00630863450376	1.0063086345032	$5.628831e - 13$	$1.88514680e - 11$
0.009375	1.00950697358907	1.0095069735885	$5.704326e - 13$	$9.77529800e - 11$
0.012500	1.012734701540635	1.01273470153874	$1.895373e - 12$	$3.15759129e - 10$
0.015625	1.015991849211686	1.01599184920765	$4.039213e - 12$	$7.86878775e - 10$
0.018750	1.019278447552026	1.01927844754401	$8.015588e - 12$	$1.66401853e - 09$
0.021875	1.02259452760833	1.0225945275963	$1.202927e - 11$	$3.14174019e - 09$
0.025000	1.02594012052443	1.02594012050551	$1.891598e - 11$	$5.45897125e - 09$
0.028125	1.029315257541654	1.029315257513647	$2.800626e - 11$	$8.90167956e - 09$
0.031250	1.032719969999102	1.03271996995882	$4.028355e - 11$	$1.38055022e - 08$

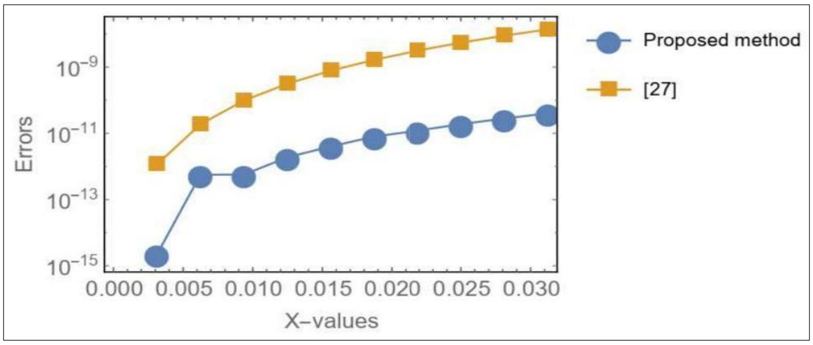

Figure 3: Comparison of the proposed method with [27] for test problem 2

Table 3: Numerical results for a fourth-order application problem in problem 3

X-value	Exact solution	Computed solution	Error in the Proposed method	Error in [28]
0.0031250	0.999999999992052690	0.999999999992053	$0.000000e+000$	$6.685763E-13$
0.006250	0.999999999872843830	0.999999999872713	$1.304512e-015$	$1.458489E-11$
0.009375	0.999999999356275480	0.999999999356270	$5.884182e-015$	$1.082968E-10$
0.012500	0.999999997965526630	0.999999997965507	$1.920686e-014$	$3.917803E-10$
0.015625	0.999999995033067470	0.999999995033026	$4.196643e-014$	$1.025145E-09$
0.018750	0.999999989700679490	0.999999989700584	$9.592327e-014$	$2.217319E-09$
0.021875	0.999999980919479500	0.999999980919320	$1.596501e-013$	$4.226068E-09$
0.025000	0.999999967449951230	0.999999967449622	$3.291811e-013$	$7.358019E-09$
0.028125	0.999999947861981100	0.999999947861397	$5.845324e-013$	$1.196868E-08$
0.031250	0.999999920534901050	0.999999920533798	$1.105338e-012$	$1.846249E-08$

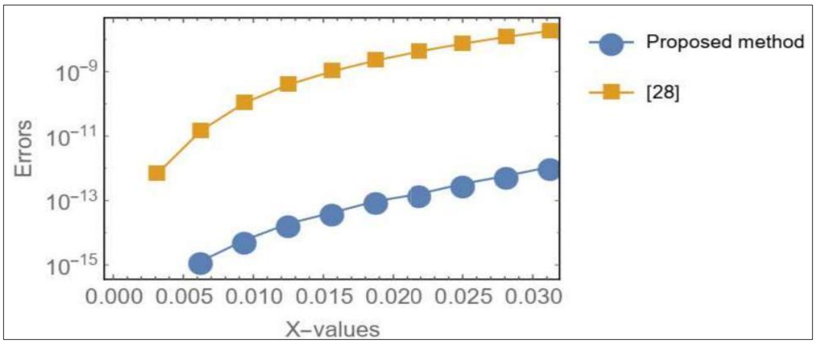

Figure 4: Comparison of the proposed method with [28] for test problem 3

4.3 DISCUSSION OF RESULTS

Tables 1 to 3 shows the computation of results for the Proposed method. The Four-step Predictor-corrector methods were used to solve test problem 1 and the results were compared with the existing methods of [26], who proposed four-step hybrid block. The results shown in Table 1 reveals that the proposed methods performs better in terms of accuracy. The proposed methods were also applied to solve test problem 2 and the results were compared with the existing methods of [27]. The results shown in Table 2 reveals that the developed methods performed favourably with [27], in terms of accuracy and convergency. Finally, The proposed methods were used to solve test problem 3; an application problem in engineering namely, Ship dynamics. The numerical results and comparison with [28], were made in Table 3. The results reveals that the proposed methods is very close to the exact solution and thus, give a better results in terms of accuracy.

5. CONCLUSION AND FUTURE RESEARCH

In this paper, the proposed Predictor-Corrector method for solving fourth-order ordinary differential equations have been successfully developed, analysed and implemented. The methods were found to be zero stable, consistent and converges. The region of absolute stability of the methods is absolutely stable as shown in Figure 1. The numerical examples where presented in Tables 1-3. Comparison of errors in the proposed method with other existing approach were displayed in curves in Figures 2-4. It is evident from the results and curves that the proposed method is not only an alternative methods to other approaches but the results shows that it performed favourably in terms of accuracy with existing methods. Our future research will be focus on the numerical solution of fourth-order boundary value problems using the approach proposed in this article.

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