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Properties of Solutions for a Degenerate Parabolic Equation with Nonlinear Sources

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Abstract: In this paper existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution that exists globally or blows up in finite time are obtained for the degenerate parabolic problem

$$x^n u_t - (u^m)_{xx} = au^p \int_0^l u^q(x,t)dx - ku^r(x,t) \quad \text{in} \quad (0,l)\times(0,T),$$

where $0 < T \le +\infty, a, k, l > 0, p, q \ge 0, p+q \ge r > m > 1$.
Keywords: Blow-up; Nonlocal source; Degenerate parabolic equation

INTRODUCTION

In this paper, we consider the following degenerate nonlinear reaction-diffusion equation with nonlocal source

$$\begin{cases} x^n u_t - (u^m)_{xx} = au^p \int_0^l u^q(x,t)dx - ku^r(x,t), & (x,t)\in(0,l)\times(0,T), \\ u(0,t) = u(l,t) = 0, & t\in(0,T), \\ u(x,0) = u_0(x), & x\in[0,l] \end{cases} \qquad (1.1)$$

where $T \le +\infty$. m, n, p, q, l, a and k are constants with $a, k, l > 0$, $p, q \ge 0$, $p+q \ge r > m > 1$. Let $D = (0, l)$ and $\Omega_r = D \times (0, r]$. Let $\overline{D}$ and $\overline{\Omega}_r$ be their respective closures and $u_0(x)$ satisfies compatibility conditions. Since $n > 0$, the coefficient of u_t may tend to 0 as x tends to 0, we can regard the equation as degenerate.

This type of equations arise in the study of the flow of a fluid through a porous medium or in study of population dynamics. It has also been suggested that nonlocal growth terms present more realistic models of population dynamics under many situations. Recently, much effort has been devoted to the study of blow-up properties for nonlinear parabolic equations with nonlocal sources, see [1-7] and references therein.

When $n = p = 0, m = a = 1$, the problem (1.1) has been studied by Wang and Wang (see [8]). They proved that if $m > n > 1$ or $m = n > 1$ and $l > k$, then the solution $u(x,t)$ of (1.1) with large initial data blows up in finite time. When $n = p = 0, m = a = 1, q > r > 1$, Souplet (see [3]) obtained the asymptotic blow-up behavior of the solution.

Floater [9] and Chan [10] investigated the blow-up properties of the following problem:

$$\begin{cases} x^q u_t - u_{xx} = u^p, & (x,t)\in(0,a)\times(0,T), \\ u(0,t) = u(a,t) = 0, & t\in(0,T), \\ u(x,0) = u_0(x), & x\in[0,a] \end{cases} \qquad (1.2)$$

where $q > 0$ and $p > 1$.

The motivation for studying problem (1.2) comes from Ockendon's model (see [11]) for the flow in a channel of a fluid whose viscosity depends on temperature

$$xu_t - u_{xx} = e^u, \qquad (1.3)$$

where u represents the temperature of the fluid. Under certain conditions on the initial datum $u_0(x)$, Floater [9] proved that the solution $u(x,t)$ of (1.2) blows up at the boundary $x = 0$ for the case $1 < p \le q+1$. This contrasts with one of the results in [12], which showed that for the case $q=0$, the blow-up set of solution $u(x,t)$ of (1.2) is a proper compact subset of D.

Budd et al. [13] generalized the results in [9] to the following degenerate quasilinear parabolic equation:

$$x^q u_t - (u^m)_{xx} = u^p, \qquad (1.4)$$

with homogeneous Dirichlet conditions in the critical exponent $q=(p-1)/m$, where $q>0, m\geq 1$ and $p>1$.

They pointed out that the general classification of blowup solution for the degenerate equation (1.4) stays the same for the quasilinear equation (see [13,14])

$$u_t - (u^m)_{xx} = u^p. \qquad (1.5)$$

For the case $p>q+1$, in [10] Chan and Liu continued to study problem (1.2). Under certain conditions, they proved that $x=0$ is not a blowup point and the blowup set is a proper compact subset of D.

For nonlinear parabolic equations with nonlocal sources, in [5] Deng et al. studied the following problem

$$\begin{cases} u_t = (u^m)_{xx} + a\int_{-l}^{l} u^q dx, & (x,t)\in(0,l)\times(0,T), \\ u(0,t)=u(l,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x), & x\in[0,l] \end{cases} \qquad (1.6)$$

Where $a>0,\ q>m>1$. Under certain conditions, they obtained that the solution blows up in finite time and got the estimate of blow-up rate.

In [15], Liu et al. considered the following degenerate nonlinear reaction-diffusion equation with nonlocal source

$$\begin{cases} v_t - x^\alpha (v^m)_{xx} = \int_0^l v^{p_1} dx - kv^{q_1}, & (x,t)\in(0,l)\times(0,T), \\ v(0,t)=v(l,t)=0, & t\in(0,T), \\ v(x,0)=v_0(x), & x\in[0,l] \end{cases} \qquad (1.7)$$

they established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they also got some sufficient conditions for blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

In [16], C. Peng et al. considered the following problem

$$\begin{cases} x^q u_t - u_{xx} = \int_0^l u^m dx - ku^n, & (x,t)\in(0,l)\times(0,T), \\ u(0,t)=u(l,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x), & x\in[0,l] \end{cases} \qquad (1.8)$$

where $0<T\leq+\infty,\ l>0,\ m\geq n>1,\ k>0$ are constants. They obtained the sufficient conditions for the solution that exists globally or blows up in finite time.

Motivated by the results of the papers [9,10,15,16], we modify the method developed by Chan et al. [10] and Liu et al. [15] and consider a new degenerate parabolic (1.1). We will study the globally existence and blow-up of solutions of (1.1).

In this paper we first give a proof of the local existence for (1.1) by regularization procedure, since the degeneracy of (1.1) is introduced by $x^n|_{x=0}=0$. Then, under appropriate hypotheses, we prove that the solution blows up in finite time. The blow-up means that there exists a $T^*<\infty$ such that $||u(\cdot,t)||_\infty<\infty$ for $t\in(0,T^*)$ and $\lim_{t\to T^*}||u(\cdot,t)||_\infty=\infty$. If $T^*=+\infty$, then u is a global solution.

Before stating our main results, we make some assumptions for initial data $u_0(x)$ as follows:

(H_1)(Compatibility condition) $u_0(x)>0$ in $(0,l)$, $u_0(0)=u_0(l)=0$;

(H_2) $u_0(x)\in C^{2+\alpha}(0,l]\cap C[0,l]$ for some $0<\alpha<1,\ u_{0x}(l)<0$;

$(H_3)\, (u_0^m)_{xx} + au_0^p \int_0^l u_0^q dx - ku_0^r \geq 0,\ x \in (0,l);\ (u_0^m)_{xx} + au_0^p \int_0^l u_0^q dx - ku_0^r|_{x=\pm l} = 0.$

Remark 1.1. We can choose $u_0(x) = C\sin(\frac{\pi}{l}x)^{\frac{1}{m}}$ to satisfy the above conditions (H_1)-(H_3), where C is a sufficiently large positive constant. This work is organized as follows. In section 2, we state the local existence and uniqueness of the solution and prove that the solution is a classical one by adding some assumptions on $u_0(x)$. The results of global existence and finite time blow-up are shown in section 3.

LOCAL EXISTENCE AND UNIQUENESS

To investigate the local existence and uniqueness of the solution of problem (1.1), under the transformation of $u^m = v$, $t = \frac{1}{m}\tau$, then (1.1) becomes

$$\begin{cases} x^n v_\tau = v^{m_1}(v_{xx} + av^{p_1}\int_0^l v^{q_1}dx - kv^{r_1}), & (x,\tau) \in (0,l)\times(0,T'), \\ v(0,\tau) = v(l,\tau) = 0, & \tau \in (0,T'), \\ v(x,0) = v_0(x), & x \in [0,l] \end{cases} \tag{2.1}$$

where $0 < m_1 = \frac{m-1}{m} < 1$, $p_1 = \frac{p}{m} \geq 0, q_1 = \frac{q}{m} \geq 0, r_1 = \frac{r}{m}$, $p_1 + q_1 \geq r_1 > 1$, $v_0(x) = u_0^m(x)$.

Under this transformation, assumptions (H_1)-(H_4) become

(H_1') (Compatibility condition) $v_0(x) > 0$ in $(0,l)$, $v_0(0) = v_0(l) = 0$;

(H_2') $v_0(x) \in C^{2+\alpha}(0,l] \cap C[0,l]$ for some $0 < \alpha < 1$, $v_{0x}(l) < 0$;

(H_3') $v_{0xx} + av_0^{p_1}\int_0^l v_0^{q_1}dx - kv_0^{r_1} \geq 0,\ x \in (0,l);\ v_{0xx} + av_0^{p_1}\int_0^l v_0^{q_1}dx - kv_0^{r_1}|_{x=\pm l} = 0.$

Definition 2.1. A nonnegative functions $\overline{v}(x,t)$ is called an upper solution of (1.1), if $\overline{v}(x,t) \in C([0,l]\times[0,T'))$ and satisfies

$$\begin{cases} x^n \overline{v}_\tau \geq \overline{v}^{m_1}(v_{xx} + a\overline{v}^{p_1}\int_0^l \overline{v}^{q_1}dx - k\overline{v}^{r_1}), & (x,\tau) \in (0,l)\times(0,T'), \\ \overline{v}(0,\tau) \geq 0,\ \overline{v}(l,\tau) \geq 0, & \tau \in (0,T'), \\ \overline{v}(x,0) \geq \overline{v}_0(x), & x \in [0,l] \end{cases} \tag{2.2}$$

Similarly, $\underline{v}(x,t) \in C([0,l]\times[0,T'))$ is called a lower solution if it satisfies all the reversed inequalities in (2.2). In order to prove the existence of a unique positive solution to (1.1), we give the following lemma firstly.

Lemma 2.2. Suppose that $w(x,\tau) \in C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r)$ and satisfies

$$\begin{cases} x^n w_\tau - d(x,\tau)w_{xx} \geq c_1(x,\tau)w + c_3(x,\tau)\int_0^l c_2(x,\tau)w(x,\tau)dx, & (x,\tau) \in \Omega_r, \\ w(\pm l,\tau) \geq 0, & \tau \in (0,r], \\ w(x,0) \geq 0, & x \in [0,l] \end{cases}$$

where $c_1(x,\tau)$, $c_2(x,\tau)$, $c_3(x,\tau)$ are bounded functions and $c_2(x,\tau), c_3(x,\tau), d(x,\tau) \geq 0$ in Ω_r. Then $w(x,\tau) \geq 0$ on $\overline{\Omega}_r$.

The proof is similar to the proofs of Lemma 1 in [1] or Lemma 2.1 in [8], we omit it.

Remark 2.3. Lemma 1 of [1] lacked the condition $c_2(x,\tau) \geq 0$. The following counter example shows that the condition $c_2(x,\tau) \geq 0$ is necessary.

Let $c_1(x,\tau) = 0$, $c_2(x,\tau) = -180$, $d(x,\tau) = c_3(x,\tau) = 1$, $l = 1$, $n = 1$, $r = \frac{1}{4}$. Considering the function

$$w(x,\tau) = (2x-1)^2 - \tau \quad \text{in} \quad (0,1)\times(0,\frac{1}{4}].$$

Then we have

$$xw_\tau - w_{xx} \geq -9 \geq \int_0^1 c_2(x,\tau)w(x,\tau)dx = -180\int_0^1((2x-1)^2-\tau)dx = -180(\frac{1}{3}-\tau),$$
$$w(0,\tau) = w(1,\tau) = 1-\tau \geq 0 \quad \text{for} \quad \tau \in [0,\frac{1}{4}],$$
$$w(x,0) = (2x-1)^2 \geq 0 \quad \text{for} \quad x \in [-1,1].$$

But $w(\frac{1}{2},\tau) = -\tau < 0$. Because (2.1) is a degenerate equation, the standard parabolic theory can't be used to give the local existence of the solution, we consider the regularized problem.

$$\begin{cases} x^n v_{\varepsilon\tau} = (v_\varepsilon+\varepsilon)^{m_1}(v_{\varepsilon xx} + av_\varepsilon^{p_1}\int_0^l v_\varepsilon^{q_1}dx - kv_\varepsilon^{r_1}), & (x,\tau) \in (0,l)\times(0,T'), \\ v_\varepsilon(0,\tau) = v_\varepsilon(l,\tau) = 0, & \tau \in (0,T'), \\ v_\varepsilon(x,0) = v_0(x), & x \in [0,l] \end{cases} \tag{2.3}$$

By Lemmas 2.2, we have following comparison principle.

Lemma 2.4. Assume that $v_\varepsilon(x,\tau) \in C(\overline{\Omega}_{T'}) \cap C^{2,1}(\Omega_{T'})$ is a nonnegative solution of the (2.2) and a nonnegative function of

$$\begin{cases} x^n w_\tau \geq (\leq)(w+\varepsilon)^{m_1}(w_{xx} + aw^{p_1}\int_0^l w^{q_1}dx - kw^{r_1}), & (x,\tau) \in (0,l)\times(0,T'), \\ w(0,\tau) \geq (=)0, \quad w(l,\tau) \geq (=)0, & \tau \in (0,T'), \\ w(x,0) \geq (\leq)v_0(x), & x \in [0,l] \end{cases} \tag{2.4}$$

Then $w(x,\tau) \geq (\leq) v_\varepsilon(x,\tau)$ on $[0,l]\times[0,T')$.

Proof. We only consider the case " $\geq$ " (as for the other case " $\leq$ " the proof is similar). Let $\varphi(x,\tau) = w(x,\tau) - v_\varepsilon(x,\tau)$. Subtracting (2.3) from (2.4) and using the mean value theorem, we obtain

$$x^n\varphi_\tau = x^n w_\tau - x^n v_{\varepsilon\tau} \geq m_1(\eta_1+\varepsilon)^{m_1-1}(w_{xx} + aw^{p_1}\int_0^l w^{q_1}dx)\varphi + a(v_\varepsilon+\varepsilon)^{m_1}w^{p_1}q_1\int_0^l \eta_2^{q_1-1}\varphi dx$$
$$+[a(v_\varepsilon+\varepsilon)^{m_1}p_1\eta_3^{p_1-1}\int_0^l v_\varepsilon^{q_1}dx]\varphi + (v_\varepsilon+\varepsilon)^{m_1}\varphi_{xx} + (-k)[m_1(\eta_4+\varepsilon)^{m_1-1}\eta_4^{r_1} + r_1(\eta_4+\varepsilon)^{m_1}\eta_4^{r_1-1}]\varphi$$

with the initial-boundary conditions

$$\varphi(\pm l,\tau) \geq 0, \qquad \varphi(x,0) \geq 0,$$

where $\eta_i (i=1,\ldots,4)$ are some intermediate values between w, v_ε. Then Lemma 2.1 ensures that $\varphi(x,\tau) \geq 0$, that is, $w(x,\tau) \geq v_\varepsilon(x,\tau)$ on $[0,l]\times[0,T')$.□

By Lemma 2.4, we have the following result of monotonicity.

Lemma 2.5. Let $0 < \varepsilon_1 < \varepsilon_2 < 1$ and suppose that v_0 satisfies (H_1')-(H_4'), $v_{\varepsilon 1}(x,\tau)$ and $v_{\varepsilon 2}(x,\tau)$ are solution of (2.3). Then $v_{\varepsilon 1}(x,\tau) \leq v_{\varepsilon 2}(x,\tau)$.

Lemma 2.6. Suppose that v_0 satisfies (H_1')-(H_3'), $v_\varepsilon(x,\tau)$ is the solution of (2.3) on $\overline{\Omega}_{T'}$. Then $v_{\varepsilon\tau} \geq 0$ on $\overline{\Omega}_{T'}$.

Proof. Let $\phi = v_{\varepsilon\tau}$. Differentiating (2.3) with respect to τ gives

$$\phi_\tau = m_1x^{-n}(v_\varepsilon+\varepsilon)^{m_1-1}(v_{\varepsilon xx} + av_\varepsilon^{p_1}\int_0^l v_\varepsilon^{q_1}dx - kv_\varepsilon^{r_1})\phi$$
$$+x^{-n}(v_\varepsilon+\varepsilon)^{m_1}(\phi_{xx} + ap_1v_\varepsilon^{p_1-1}\phi\int_0^l v_\varepsilon^{q_1}dx + aq_1v_\varepsilon^{p_1}\int_0^l v_\varepsilon^{q_1-1}\phi dx - kr_1v_\varepsilon^{r_1-1}\phi).$$

By (H_1')-(H_3'), we have

$$\phi(x,0) = v_{\varepsilon\tau}(x,0) = x^{-n}(v_0+\varepsilon)^{m_1}(v_{0xx} + av_0^{p_1}\int_0^l v_0^{q_1}dx - kv_0^{r_1}) \geq 0.$$

In view of $\phi(0,\tau) = v_{\varepsilon\tau}(0,\tau) = 0, \phi(l,\tau) = v_{\varepsilon\tau}(l,\tau) = 0$. By Lemma 2.2, it follows that $\phi(x,\tau) \geq 0$ on $\overline{\Omega}_{T'}$.□

Lemma 2.7. Suppose that v_0 satisfies (H_1')-(H_3'). Then there exist T_0 and a priori bound M such that for all $\varepsilon \in (0,1)$, the solution of (2.3) satisfies

$$v_0(x) \leq v_\varepsilon(x,\tau) \leq M \quad \text{for} \quad (x,\tau) \in \overline{\Omega}_{T_0}.$$

Proof. By Lemma 2.5, we know that v_ε ismonotone with respect to ε. Suppose the solution of (2.3) is v_1 when $\varepsilon = 1$, and T_1 is the maximal existence time of v_1. For any T_0 in $(0, T_1)$, we can conclude that

$$v_\varepsilon(x,\tau) \leq v_1(x,\tau) \leq v_1(x,T_0) \leq \max_{[0,l]} v_1(x,T_0) = M.$$

Since $v_{\varepsilon\tau} \geq 0$, $v_\varepsilon(x,0) = v_0(x)$. It follows that $v_0(x) \leq v_\varepsilon(x,\tau) \quad \text{for} \quad (x,\tau) \in \overline{\Omega}_{T_0}$.

From the above results, we know that v_ε is monotone with respect to ε and is bounded from below to above. Thus we have

$$v(x,\tau) = \lim_{\varepsilon \to 0} v_\varepsilon(x,\tau). \qquad (2.5)$$

Theorem 2.8. Suppose that $v_0 \in C[0,l] \cap C^2(0,l)$ and satisfies(H_1')-(H_3'), then the function $v(x,\tau)$ defined by (2.5) is a unique solution of (2.1) in Ω_{T_0}.

Proof. It is required to prove that v belongs to$C(\overline{\Omega}_{T_0}) \cap C^{2,1}(\Omega_{T_0})$. Choose a point $(x_1,\tau_1) \in (0,l) \times (0,T_0)$. Then select a domain $D = (a_1,a_2) \times (0,\tau_2)$ such that

$$0 < a_1 < x_1 < a_2 < l \quad \text{and} \quad 0 < \tau_1 < \tau_2 < T_0.$$

Let $M_1 = \inf_{x \in [a_1,a_2]} v_0(x)$. By Lemma 2.7, we have that $v_\varepsilon \geq M_1 > 0$ in D, then $(v_\varepsilon)^{m_1} \geq M_1^{m_1}$. By Schauder interior estimate, we have $||v_\varepsilon||_{C^{2+\alpha}(D)} \leq M_2$, where M_2 depends only on $M_1^{m_1}, v_0, \alpha, D$.

Now an appeal to Ascoli-Arzelà Theorem show that $v \in C^{2+\alpha'}(D)(0 < \alpha' < \alpha < 1)$ with$||v_\varepsilon||_{C^{2+\alpha'}(D)} \leq M_2$. This shows that v is in $C^{2,1}$ at (x_1,τ_1). Notice that

$$0 \leq \lim_{x \to 0} v(x,\tau) \leq \lim_{x \to 0} v_\varepsilon(x,\tau) = 0, \quad 0 \leq \lim_{x \to l} v(x,\tau) \leq \lim_{x \to l} v_\varepsilon(x,\tau) = 0, \quad (\varepsilon \to 0).$$

we have that v is continuous on $\{0,l\} \times (0,T_0)$.

Suppose that $\psi(x,\tau), \omega(x,\tau)$ are two classical solution of (2.1). By using the same method used in Lemma 2.4, we can easily prove that $\psi \geq \omega$ and $\psi \leq \omega$, thus $\psi = \omega$.□

GLOBAL EXISTENCE AND BLOW-UP

The main object of this section is to show that v tends to infinity in finite time provided that v_0 is large enough.

Theorem 3.1. If $p_1 = 0,\ q_1 = r_1 > 1,\ l > k,\ v_0$ satisfies (H_1')-(H_3'). Let $\varphi(x)$ be the unique positive solution of the following linear elliptic problem

$$\begin{cases} -\varphi'' = 1, & x \in (0,l), \\ \varphi(0) = \varphi(l) = 0. \end{cases}$$

Then there exists constants $a_1 > 0$ such that the solution $v(x,\tau)$ of (2.1) exists globally when $v_0(x) \leq a_1\varphi(x)$.

Proof. Let $w(x) = a_1\varphi(x)$, where a_1 is chosen so that

$$-w''(x) = a_1 \geq a_1^{q_1}(a\int_0^l \varphi^{q_1}dx - k\varphi^{q_1}) = a\int_0^l w^{q_1}dx - kw^{q_1}, \quad x \in (0,l).$$

Thus

$$x^n w_\tau \geq (w+\varepsilon)^{m_1}(w_{xx} + a\int_0^l w^{q_1}dx - kw^{q_1}).$$

By Lemma 2.4 it follows that $v(x,\tau)$ exists globally provided that $v_0(x) \leq w(x) = a_1\varphi(x)$.

Theorem 3.2. Let $v(x,\tau)$ be the solution of problem (2.1). Assume that $p_1+q_1 > r_1 > 1$, v_0 satisfies (H_1')-(H_3'). Then the solution $v(x,\tau)$ of (2.1) exists globally when $v_0(x) \leq (\frac{k}{al})^{\frac{1}{p_1+q_1-r_1}}$.

Proof. Let $w(x) = (\frac{k}{al})^{\frac{1}{p_1+q_1-r_1}}$, then

$$\begin{aligned}
&x^n w_\tau = (w+\varepsilon)^{m_1}(w_{xx} + a\int_0^l w^{q_1}dx - kw^{q_1}) = 0,\\
&w(x) \geq 0, \quad \tau \in (0,T_0],\\
&w(x) \geq v_0(x), \quad x \in (0,l).
\end{aligned}$$

Thus $w(x)$ is a supersolution of (2.1), which means that (2.1) has a global solution.

Theorem 3.3. Let $u(x,t)$ be the solution of problem (1.1). Assume that $n > 1$, v_0 satisfies (H_1)-(H_3). Then the solution $v(x,t)$ of (1.1) blows up in finite time if $u_0(x)$ is sufficiently large.

Proof. Since problem (1.1) does not a prior make sense for negative values of u, we actually consider the following problem

$$\begin{cases}
x^n u_t - (u^m)_{xx} = au_+^p\int_0^l u_+^q dx - ku^r, & (x,t)\in(0,l)\times(0,T),\\
u(0,t) = u(l,t) = 0, & t\in(0,T),\\
u(x,0) = u_0(x), & x\in[0,l].
\end{cases} \tag{3.1}$$

We set

$$\psi(x,t) = \frac{1}{(T-t)^\gamma}V^{\frac{1}{m}}[\frac{|x|}{(T-t)^\sigma}], \qquad V(y) = 1 + \frac{A}{2} - \frac{y^2}{2A}, y \geq 0,$$

where $\gamma, \sigma > 0$, $A > 1$ and $0 < T < 1$ are to be determined.

First note that

$$suppz(t) = \overline{B(0,R(T-t)^\sigma)} \subset \overline{B(0,RT^\sigma)} \subset (0,l), \tag{3.2}$$

for sufficiently small $T > 0$ with $R = [A(2+A)]^{\frac{1}{2}}$.

Calculating directly, we obtain $-(z^m)_{xx} = \dfrac{N/A}{(T-t)^{m\gamma+2\sigma}}$.

For all $(x,t) \in (0,l)\times(0,T)$, we find $|z(x,t)| \leq \dfrac{1+A+4l^2}{(T-t)^{\gamma+2\sigma}}$.

The remaining terms are estimated in two different ways according to the size of $\frac{|x|}{(T-t)^\sigma}$. If $0 \leq y \leq A$, we have $1 \leq V(y) \leq 1+\frac{A}{2}$ and $V'(y) \leq 0$, thus we have

$$z_t(x,t) = \frac{m\gamma V^{\frac{1}{m}}(y) + \sigma y V'(y)V^{\frac{1-m}{m}}(y)}{m(T-t)^{\gamma+1}} \leq \frac{\gamma(1+\frac{A}{2})^{\frac{1}{m}}}{(T-t)^{\gamma+1}},$$

$$z_+^p\int_0^l z_+^q dx = \frac{V_+^{\frac{p}{m}}}{(T-t)^{\gamma(p+q)}}\int_{B(0,R(T-t)^\sigma)} V_+^{\frac{p}{m}}[\frac{|x|}{(T-t)^\sigma}] \geq \frac{M}{(T-t)^{\gamma(p+q)-N\sigma}},$$

in which $M = \int_{B(0,RT^\sigma)} V_+^{\frac{q}{m}}(|\xi|)d\xi$.

Hence,

$$x^n z_t - (z^m)_{xx} - az_+^p \int_0^l z_+^q dx + kz^r \le \frac{\gamma l^n (1+\frac{A}{2})^{\frac{1}{m}}}{(T-t)^{\gamma+1}} + \frac{N/A}{(T-t)^{m\gamma+2\sigma}}$$
$$-\frac{M}{(T-t)^{\gamma(p+q)-N\sigma}} + \frac{k(1+A+4l^2)^r}{(T-t)^{(\gamma+2\sigma)r}}. \qquad (3.3)$$

On the other hand, if $y \ge A$, we have $V(y) \le 1$ and $V'(y) \le -1$, thus we have

$$z_t(x,t) \le \frac{\gamma - \sigma A/m}{(T-t)^{\gamma+1}}.$$

Hence,

$$x^n z_t - (z^m)_{xx} - az_+^p \int_0^l z_+^q dx + kz^r \le \frac{l^n(\gamma - \sigma A/m)}{(T-t)^{\gamma+1}} + \frac{N/A}{(T-t)^{m\gamma+2\sigma}} + \frac{k(1+A+4l^2)^r}{(T-t)^{(\gamma+2\sigma)r}}. \quad (3.4)$$

Since $p+1 > r > 1$, we can choose $\sigma, \gamma > 0$ such that
$\gamma(p+q) - N\sigma > \gamma + 1 > (\gamma+2\sigma)r > m\gamma + 2\sigma.$
Select $A > \max\{1, \frac{m\gamma}{\sigma}\}$, then for $T > 0$ sufficiently small, (3.3) and (3.4) imply that

$$x^n z_t - (z^m)_{xx} - az_+^p \int_0^l z_+^q dx + kz^r \le 0. \qquad (3.5)$$

Let $\phi \in C^1(0,l)$, $\phi(x) \ge 0$, $\phi(x) \ne 0$, and $\phi(0) = \phi(l) = 0$. By translation, we may assume without loss of generality that $\phi(0) > 0$. Since $\phi(0) > 0$ and ϕ is continuous, there exist two positive numbers ρ and $\varepsilon > 0$, such that $\phi(x) > \varepsilon$, for all $x \in B(0,\rho) \subset (0,l)$. Taking T small enough to insure $B(0,RT^\sigma) \subset B(0,\rho)$, and hence $z \le 0 \text{ on } \{0,l\} \times (0,T)$. From (3.2), it follows that $z(x,0) \le \lambda\phi(x)$ for sufficiently large λ. By Lemma 2.4, we have $z \le u$ provided that $u_0(x) > \lambda\phi(x)$ and u can exist no later than $t = T$. This shows that u blows up in finite time.

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