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Hybrid Orthonormal Bernstein and Block-Pulse Functions for solving nonlinear Volterra integral equations

Mohamed A. Ramadan[1], Mohamed R. Ali[2]
[1]Department of Mathematics, Faculty of Science, Menoufia University, Shbeen El-Koom, Egypt
[2]Department of Mathematics, Faculty of Engineering, Benha University, Egypt

***Corresponding Author:**
Mohamed A. Ramadan
Email: mohamedredaabhit@yahoo.com

Abstract: In this paper, A numerical method based on an set of general, orthonormal Bernstein functions coupled with Block-Pulse Functions on the interval [0,1] to solve non linear Volterra integral equations of the second kind, numerically. First we introduced the proposed hybrid method, then we used it to transform the integral equations to the system of algebraic equations. The obtained numerical results of the proposed methods are compared with exact solution to show the convergence and advantages of the new method. the operational matrix of integration together with Newton-Cotes nodes are utilized to reduce the computation of nonlinear Volterra integral equations into some algebraic equations, the numerical example illustrate the efficiency and accuracy of this method.
Keywords: Orthonormal Bernstein functions; Block-pulse functions; nonlinear Volterra integral equations; integration of the cross product, product matrix, coefficient matrix

INTRODUCTION

The nonlinear Volterra integral equations arise from various physical and biological models. The essential features of these models are of wide applicable [1]. In recent years, much work has been done in the study of numerical solutions to Volterra integral equations using collocation methods [1-3]. Benitez and Bolos [4] pointed out that collocation methods have proven to be a very suitable technique for approximating solutions to nonlinear integral equations because of their stability and accuracy. Other authors such as [5–7] used quadrature rules like repeated trapezoidal and repeated Simpson's rule to solve linear Volterra integral equations.

Existence of solutions for nonlinear integral equations, which contain particular cases of important integral and functional equations such as nonlinear Volterra integral equation, Urysohn integral equation, and integral equations of Chandrasekhar type, have been considered in many papers and books [8-10].

In this study, the basic ideas of the previous works are developed and applied to the nonlinear Volterra integral equations:

$$y(x) = f(x) + \lambda_1 \int_a^x k(x,t)\; F(y(t))\, dt \qquad (1)$$

where $a \le x \le b, \lambda_1$ are scalar parameters, $f(x), k(x,t)$ are continuous functions and $u(x)$ is the unknown function to be determined. The advantage of this method to other existing methods is its simplicity of implementation besides some other advantages.

This paper is organized as follows: In section 2, we introduce Bernstein polynomials and their properties. Also we orthonormal these polynomials and hybrid them with Block-Pulse functions (HOBB) to obtain new basis. In section 3, these new basis together with collocation method are used to reduce the Volterra integral equation to a linear system that can be solved by various method. In section 4 we apply these set of HOBB for approximating the solution of non linear Volterra integral equations. Using the properties of HOBB together with collocation method, we reduce non linear Volterra integral equations to a non linear system of linear equations; these equation can be solved using Newton method. In section 5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally. Finally, section 6 concludes the paper.

Brief review of Hybrid Orthonormal Bernstein and Block-Pulse Functions

In this section we introduce Bernstein polynomials and their properties to get better approximation, we orthonormal these polynomials and hybrid them with Block-Pulse functions.

Bernstein polynomials

B-polynomials (Bernstein polynomials basis) of nth-degree were introduced in the approximation of continuous functions f(x) on an interval [0, 1] ;see [28].

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad 0 \le i \le n.$$

There are (n +1) nth-degree polynomials and for convenience,

we set $B_{i,n}(x) = 0$, if $i < 0$ or $i > n$.

A recursive definition also can be used to generate the B-polynomials over this interval, so that the ith nth degree B-polynomial can be written as;

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x) \qquad (2)$$

The explicit representation of the orthonormal Bernstein polynomials, denoted by ($OB_{i,n}(x)$) here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of varying degree n . For example, for $n = 4$, using the Gram-Schmidt process on $OB_{i,4}(x)$ normalizing, and simplifying the resulting functions, we get the following set of orthonormal polynomials;

$$OB_{3,4}(x) = \frac{\sqrt{3990}}{4}(4x^3(1-x) - \frac{4}{19} + \frac{68}{19}x(1-x)^3) - \frac{\sqrt{3990}}{4}(\frac{68}{19}x^2(1-x)^2)$$

We can see from these equations that the orthonormal Bernstein polynomials are, in general, a product of a factorable polynomial and a non-factorable polynomial. For the factorable part of these polynomials, there exists a pattern of the form

$$(\sqrt{2(n-i)+1})(1-t)^{n-i} \qquad i = 0,1,\ldots., n.$$

While it is less clear that there is a pattern in the non-factorable part of these polynomials, the pattern can be determined by analyzing the binomial coefficients present in Pascal's triangle. In doing this, we have determined the explicit representation for the orthonormal Bernstein polynomials to be

$$OB_{i,n}(x) = (\sqrt{2(n-j)+1})(1-t)^{n-i}\sum_{k=0}^{i}(-1)^K \binom{2n+1-k}{i-k}\binom{i}{k} t^{i-K} \qquad (3)$$

Block-Pulse functions (BPFs) and their properties

BPFs are studied by many authors and applied for solving different problems, for example see [12].

A k - set of BPFs over the interval $[0, T)$ are defined as

$$B_i(t) = \begin{cases} 1, & \frac{iT}{k} \le t < \frac{(i+1)T}{k} \\ 0, & elsewhere \end{cases}, i = 0,1,\ldots., k-1. \qquad (4)$$

with a positive integer value for k . In this paper, it is assumed that $T = 1$, so BPFs are defined over [0, 1) . BPFs have some main properties, the most important of these properties are disjointness, orthogonality, and completeness.

(1) The disjointness property can be clearly obtained from the definition of BPFs

$$B_i(t)B_j(t) = \begin{cases} B_i(t), & i = j \\ 0, & i \ne j \end{cases} \qquad i, j = 0,1,\ldots., k-1 \qquad (5)$$

(2) The orthogonality property of these functions is

$$\langle B_i(t), B_j(t)\rangle = \int_0^1 B_i(t)\,B_j(t)\,dt = \begin{cases} \frac{1}{k}, & i = j \\ 0, & i \neq j \end{cases} \qquad i, j = 0,1,\dots, k-1 \tag{6}$$

(3) The third property is completeness. For every $y \in L^2[0,1)$, when k approaches to the infinity, Parseval's identity holds, that is

$$\int_0^1 y^2(t)\,dt = \sum_{i=1}^{\infty} c_i^2 \left\| B_i(t) \right\|^2$$

where $$c_i = k\int_0^1 f(t)\,B_i(t)\,dt \tag{7}$$

Some properties of hybrid functions

Hybrid functions of block-pulse and Orthonormal Bernstein polynomials

We define $HOBB$ on the interval [0; 1] as follows:

$$HOBB_{i,j}(x) = \begin{cases} B_{j,n}(Mx - i + 1) & \frac{i-1}{M} \leq x < \frac{i}{M} \\ 0 & otherewise \end{cases} \tag{8}$$

where $i = 1,2,\dots M$ and $j = 0,1,2,\dots n$. Thus our new basis is $\{HOBB_{1,0}, HOBB_{1,1},\dots, HOBB_{M,n}\}$ and we can approximate function with this base. For example, for M = 2 and n = 1

$$HOBB_{1,0}(x) = \begin{cases} (-2x+1) & 0 \leq x < \frac{1}{2} \\ 0 & otherewise \end{cases}$$

$$HOBB_{2,0}(x) = \begin{cases} (2x) & \frac{1}{2} \leq x < 1 \\ 0 & otherewise \end{cases}$$

$$OBH_{1,1}(x) = \begin{cases} (-2x+2) & 0 \leq x < \frac{1}{2} \\ 0 & otherewise \end{cases}$$

$$HOBB_{2,1}(x) = \begin{cases} (2x-1) & \frac{1}{2} \leq x < 1 \\ 0 & otherewise \end{cases}$$

Function approximation by using OBH functions

Any function $y(t)$ which is square integrable in the interval $[0,1)$ can be expanded in a hybrid orthonormal Bernstein and Block-Pulse functions

$$y(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\,HOBB_{ij}(t), \quad i = 1,2,\dots,\infty, \quad j = 0,1,2,\dots,\infty, t \in [0,1), \tag{9}$$

where the hybrid orthonormal Bernstein and Block-Pulse coefficients

$$c_{nm} = \frac{(y(t), HOBB_{nm}(t))}{(HOBB_{nm}(t), HOBB_{nm}(t))} \quad (10)$$

In Eq. (10), $(.,.)$ denotes the inner product. Usually, the series expansion Eq. (9) contains an infinite number of terms for a smooth $y(t)$. If $y(t)$. is piecewise constant or may be approximated as piecewise constant, then the sum in Eq. (9) may be terminated after nm terms, that is

$$y(t) \cong \sum_{i=1}^{M} \sum_{j=0}^{n} c_{ij} HOBB_{ij}(t) = C^T HOBB(t) \quad (11)$$

where $HOBB(t) = [HOBB_{1,0}, HOBH_{1,1}, ..., HOBB_{M,n}]^T$,and $C = [c_{1,0}, c_{1,1}, ..., c_{M,n}]^T$

Therefore, we have

$C^T < HOBB(t), HOBB(t) >=< u(t), HOBB(t) >$

Then $C = D^{-1} < u(t), HOBB(t) >$, where

$$D =< HOBB(t), HOBB(t) >,$$

$$= \int_0^1 HOBB(t).HOBB^T(t)dt \quad (12)$$

$$= \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & D_M \end{pmatrix}$$

then by using (7) $D_i (i = 1,2,..., M)$ are defined as follows:

$$(D_n)_{i+1,j+1} = \int_{\frac{i-1}{M}}^{\frac{i}{M}} B_{i,n}(Mx - i + 1) B_{j,n}(Mx - j + 1)dx$$

$$= \frac{1}{M} \int_0^1 B_{i,n}(x) B_{j,n}(x)dx$$

$$= \frac{\binom{n}{i}\binom{n}{j}}{M(2n+1)\binom{2n}{i+j}}$$

We can also approximate the function $k(x,t) \in L[0,1]$ as follows:

$k(x,t) \approx HOBB^T(x) K HOBB(t)$, where K is an $M(n+1)$ matrix that we can obtain as:

$$K = D^{-1} < HOBB(x) < k(x,t), HOBB(t) >> D^{-1} \quad (13)$$

Integration of OBH functions

In HOBB function analysis for a dynamic system, all functions need to be transformed into HOBB functions. Since the differentiation of HOBB functions always results in impulse functions which must be avoided, the integration of HOBB functions is preferred. The integration of HOBB functions should be expandable into HOBB functions with the coefficient matrix P.

$$\int_0^t HOBB_{(n\times(m+1))}(\tau)d(\tau) \approx P_{n(m+1)\times n(m+1)} HOBB_{(n\times(m+1))}(t), t \in [0,1), \quad (14)$$

where the $n(m+1)$ -square matrix P is called the operational matrix of integration, and $HOBB_{(n\times(m+1))}(t)$ is defined in Eq. (8). A subscript $n(m+1)\times n(m+1)$ of P denotes its dimension and P is given as:

$$P_{n(m+1)\times n(m+1)} = \begin{bmatrix} H & G & G & \cdots & G \\ 0 & H & G & \cdots & G \\ 0 & 0 & H & \cdots & G \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H \end{bmatrix} \tag{15}$$

$$G_{n(m+1)\times n(m+1)} = \frac{1}{n(m+1)} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{16}$$

and H is the operational matrix of integration and can be obtained as:

$$H_{n(m+1)\times n(m+1)} = \frac{1}{2n(m+1)} \begin{bmatrix} \frac{1}{35} & \frac{263}{105} & \frac{263}{105} & \frac{71}{35} \\ \frac{-3}{35} & \frac{17}{35} & \frac{87}{35} & \frac{67}{35} \\ \frac{3}{35} & \frac{-17}{35} & \frac{53}{35} & \frac{73}{35} \\ \frac{-1}{35} & \frac{17}{105} & \frac{-53}{105} & \frac{69}{35} \end{bmatrix} \tag{17}$$

The integration of the cross product of two HOBB function vectors can be obtained as

$$D = \int_0^1 HOBB_{(n\times(m+1))}(t)\, HOBB^T_{(n\times(m+1))}(t)\; d(t) \tag{18}$$

$$\approx \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & L \end{bmatrix}$$

where L is an $M\times(n+1)$ diagonal matrix given by

$$L = \frac{1}{M(n+M)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{2} & \frac{3}{5} & \frac{9}{20} & \frac{1}{5} \\ \frac{1}{5} & \frac{9}{20} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{20} & \frac{1}{5} & \frac{1}{2} & 1 \end{bmatrix} \tag{19}$$

Eqs. (14-18) are very important for solving N Volterra integral equation, because the D and P matrix can increase the calculating speed, as well as save the memory storage.

Multiplication of HOBB functions

It is always necessary to evaluate the product of $\mathrm{HOBB}(x)$ and $\mathrm{HOBB}^T(x)$, that is called the product matrix of HOBB functions. Let

$$M(x) \cong \mathrm{HOBB}(x)\,\mathrm{HOBB}^T(x) \tag{20}$$

where $M(x)$ is $(M(n+1)\times M(n+1)$ matrix. Multiplying the matrix $M(x)$ by vector C we obtain

$$M(x)C = \tilde{C}\,\mathrm{HOBB}(x) \tag{21}$$

where $\tilde{C}$ is $(M(n+1)\times M(n+1)$ matrix and called the coefficient matrix. To illustrate the calculation procedure in Eq. (20), we consider that $M = 4,\ n = 3$. [12] we have

$$\tilde{C} = \begin{bmatrix} \tilde{C}_0 & 0 & 0 & 0 \\ 0 & \tilde{C}_1 & 0 & 0 \\ 0 & 0 & \tilde{C}_2 & 0 \\ 0 & 0 & 0 & \tilde{C}_3 \end{bmatrix} \tag{22}$$

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where C_i, $i = 0,1,2,3$ are 4×4 matrices given

$$\tilde{C}_i = \begin{bmatrix}
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i} \\
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i} \\
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i} \\
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i}
\end{bmatrix}$$

Let R is $(M(n+1)\times M(n+1)$ matrix. Multiplying the matrix R by vector $\mathrm{HOBB}(x)$ and multiplying the matrix $\mathrm{HOBB}(x)$ by the resulted matrix $R\,\mathrm{HOBB}(x)$ we obtain

$$\mathrm{HOBB}^T(x)\,R\,\mathrm{HOBB}(x) = \tilde{R}\,\mathrm{HOBB}(x) \tag{23}$$

where $\tilde{R}$ is $(1\times M(n+1)$ matrix and called the coefficient matrix With the powerful properties of Eq. (20) We can achieve $\tilde{R}$ by a way like $\tilde{C}$ we can convert the Volterra part of integral and Integro-Differential equations System equations to an algebraic equation.

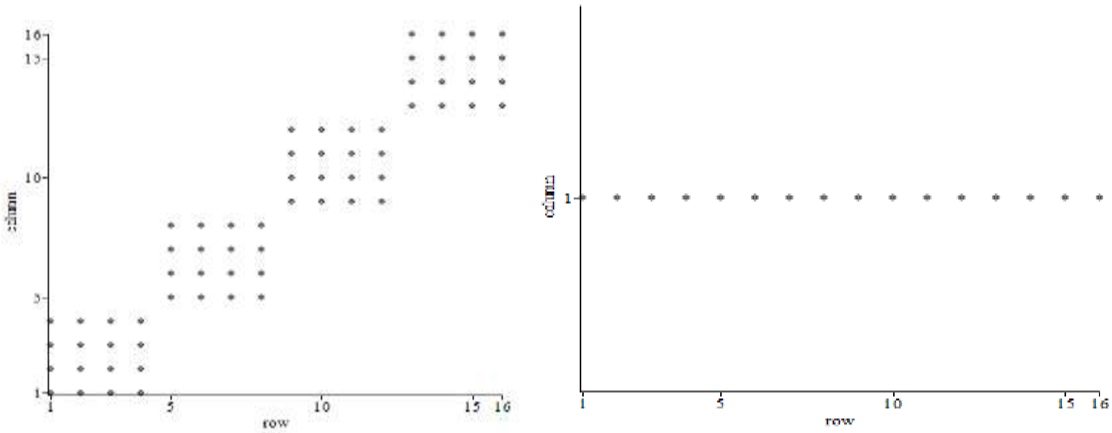

Fig-1: Patterns of the matrices R (right) and C (left).

Solution of nonlinear Volterra integral equation of the second kind via hybrid functions

Consider the following integral equation:

$$y(x) = f(x) + \lambda \int_0^x (k_1(x,t)\ (y(t))^p\, dt \tag{24}$$

$$y(x) \approx Y^T HOBB(x),$$

$$k(x,t) \approx HOBB^T(x)\, K\, HOBB(t) \quad , \quad f(x) \approx F^T HOBB(x)$$

Functions $u^q(x), v^q(x)$ can be expanded into the HOBB functions as

$$y^2(x) = [Y^T \text{HOBB}(x)]^2 = Y^T \text{HOBB}(x)\, \text{HOBB}(x)^T Y = \text{HOBB}(x)^T \tilde{Y}\, Y$$

$$y^3(t) = Y^T \text{HOBB}(t)[Y^T \text{HOBB}(t)]^2 = Y^T \text{HOBB}(t)\, \text{HOBB}(t)^T \tilde{Y}\, Y$$

$$= \text{HOBB}(t)^T \tilde{Y}\, \tilde{Y}\, Y = \text{HOBB}(t)^T (\tilde{Y})^2 Y$$

$$y^q(t) = \text{HOBB}(t)^T (\tilde{Y})^{q-1} Y \,,$$

with substituting in Eq. (24)

$$Y^T HOBB(x) = F^T HOBB(x)$$

$$+ \lambda \int_0^x HOBB^T(x)\, K\, HOBB(t)\ \text{HOBB}(t)^T (\tilde{Y})^{q-1} Y\, dt \tag{25}$$

$$Y^T HOBB(x) = F^T HOBB(x) + HOBB^T(x)\, K\, \lambda \int_0^x HOBB(t)\ \text{HOBB}(t)^T (\tilde{Y})^{q-1} Y\, dt$$

where $\int_0^x \text{HOBB}(t)\, \text{HOBB}^T(t)\, (\tilde{Y})^{P-1} Y\, dt = \int_0^x ((\tilde{Y})^{P-1} Y)\, \text{HOBB}(t)\, dt = ((\tilde{Y})^{P-1} Y)\, P\ \text{HOBB}(x)$

Applying the above Eqs. to Eq. (25) and Eq.(25) becomes

$$Y^T HOBB(x) = F^T HOBB(x) + \lambda\ \text{HOBB}^T(x)\ K\ ((\tilde{Y})^{P-1} Y)\, P\ \text{HOBB}(x) \tag{26}$$

If we approximate $\text{HOBB}^T(x)\, K_1((\tilde{Y})^{P-1} Y)\ P\ \text{HOBB}(x) \approx \tilde{R}\ \text{HOBB}(x)$,

We can achieve $(\tilde{R})$ by a way like $\tilde{C}$, and we see that for element of $\tilde{R}_1$ is obtained by the sum of column elements of $K_1((\tilde{U})^{P-1} U)\ P$ with respect to coefficient $\tilde{R}$ in Eq. (23) at each column. By using this property and omitting hybrid vector functions in Eq. (26), we will have

$$Y^T HOBB(x) = F^T HOBB(x) + \lambda\, \tilde{R}\ \text{HOBB}(x) \tag{27}$$

In order to find Y we collocate Eq. (27) in $M(n+1)$ nodal points of Newton-Cotes [13] as $t_i = \dfrac{2i-1}{2M(n+1)}$ (28)

From Eqs. (27) and (28), we have a system of $M(n+1)$ nonlinear equations and $M(n+1)$ unknowns. After solving above nonlinear system using Newton method, we can achieve the unknown vectors Y. The required approximated solution $y(x)$ for nonlinear Volterra integral Eq. (1) can be obtained by using Eqs.(22), (26) and (27) as follows

$$y(x) = Y^T HOBB(x)$$

Numerical Examples

We applied the presented schemes to the following Volterra integral equation of second kind. For this purpose, we consider an example.

Example 1

$$y(x) = f(x) + \int_0^x (xt+1)\, y^2(t)\, dt \qquad (25)$$

$$f(x) = -\frac{1}{4}x^5 - \frac{2}{3}x^4 - \frac{5}{6}x^3 - x^2 + 1$$

With the exact solution $y(x) = x+1$.

In Table 1 the exact and approximate solutions of Example 1 for M=4 and n=3 which confirms that the HOBB method gives almost higher accuracy than the other method. The average relative errors of our method 2.9872×10^{-6}. Better approximation is expected by choosing the higher values of M and n.

Table-1: The comparison among HOBB and analytic solutions for Example 1

x	HOBB solution	Analytic solution	Absolute error
0.1	1.10000103	1.1	1.03×10^{-6}
0.2	1.19999645	1.2	3.55×10^{-6}
0.3	1.30000453	1.3	4.53×10^{-6}
0.4	1.40005202	1.4	5.202×10^{-5}
0.5	1.49999654	1.5	3.458×10^{-6}
0.6	1.59996374	1.6	3.6255×10^{-5}
0.7	1.69997459	1.7	2.5401×10^{-5}
0.8	1.80000251	1.8	2.514×10^{-6}
0.9	1.90003121	1.9	3.121×10^{-5}

CONCLUSION

In this paper a combination of orthonormal Bernstein and Block-Pulse functions is proposed to obtain an approximate numerical solution of the nonlinear Volterra integral equations. The method is based upon reducing the system into a set of algebraic equations. The generation of this system needs just sampling of functions multiplication and addition of matrices and needs no integration. The main advantage of this method is its efficiency and simple applicability. The matrix D and P are sparse; hence are much faster than other functions and reduces the CPU time and the computer memory, at the same time keeping the accuracy of the solution. The numerical examples support this claim. Also we noted that when the degree of Hybrid Orthonormal Bernstein and Block-Pulse Functions is increasing the errors decreasing to smaller values. The advantages of these hybrid functions are that the values of n and m are adjustable as well as being able to yield more accurate numerical solutions than the piecewise constant orthogonal function, for the solutions of integral equations.

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