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More on gbsb*-Closed Sets in Topological Spaces

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Abstract: Using the concept of gbsb* -open sets and gbsb*-closed sets, we introduce and study the topological properties of gbsb*- interior and gbsb*-closure of a set, gbsb*-derived sets, gbsb*-border, gbsb*-frontier and gbsb*-exterior.
Keywords: gbsb*-interior, gbsb*-closure, gbsb*-limit point, gbsb*-derived set, gbsb*-border, gbsb*-frontier, gbsb*-exterior

INTRODUCTION

The notion of generalized b-strongly b*-closed set and its different characterizations are given in [4]. In this paper, we introduce the notions of gbsb*-limit points, gbsb*-derived sets, gbsb*-interior and gbsb*-closure of a set, gbsb*-interior points, gbsb*-border, gbsb*-frontier and gbsb*-exterior by using the concept of gbsb*-open sets and gbsb*-closed sets, and study their topological properties.

PRELIMINARIES

Throughout this paper (X, τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. (X, τ) will be replaced by X if there is no changes of confusion. For a subset A of a topological space X, cl(A) and int(A) denote the closure of A and the interior of A respectively. We recall the following definitions and results.

Definition 2.1.[1] Let (X, τ) be a topological space. A subset A of the space X is said to be b-open if $A \subseteq int(cl(A)) \cup cl(int(A))$ and b-closed if $int(cl(A)) \cap cl(int(A)) \subseteq A$.

Definition 2.2.[1] Let (X, τ) be a topological space and $A \subseteq X$. The b-closure of A, denoted by bcl(A) and is defined by the intersection of all b-closed sets containing A.

Definition 2.3.[1] Let (X, τ) be a topological space and $A \subseteq X$. The b-interior of A, denoted by bint(A) and is defined by the union of all b-open sets contained in A.

Definition 2.4.[2] A subset A of a topological space (X, τ) is said to be generalized b-closed (briefly gb-closed) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The collection of all gb-closed sets of X is denoted by $gb\text{-}C(X,\tau)$.

Definition 2.5.[2] Let (X, τ) be a topological space and $A \subseteq X$. The gb-closure of A, denoted by gb-cl(A) and is defined by the intersection of all gb-closed sets containing A.

Definition 2.6.[2] Let (X, τ) be a topological space and $A \subseteq X$. The gb-interior of A, denoted by gb-int(A) and is defined by the union of all gb-open sets contained in A.

Definition 2.7.[3] Let (X, τ) be a topological space. A subset Aof X is said to be strongly b*-closed (briefly sb*-closed) if $cl(int(A))) \subseteq U$ whenever $A \subseteq U$ and U is b-open in (X, τ).

Definition 2.8.[4] A subset A of a topological space (X, τ) is called a generalized b-strongly b*-closed set (briefly,gbsb*-closed) if bcl(A)$\subseteq$ U whenever A$\subseteq$ U and U is sb*-open in (X, τ). The collection of all gbsb*-closed sets of X is denoted by gbsb*-C(X,τ).

Theorem 2.9. [4]
(i) Every closed set is gbsb*-closed.
(ii) Every b-closed set is gbsb*-closed.
(iii) Every gbsb*-closed set is gb-closed.

Theorem 2.10.[4]
(i) Arbitrary intersection of gbsb*-closed sets is gbsb*-closed.
(ii) Arbitrary union of gbsb*-open sets is gbsb*-open.

Remark 2.11.[4]
(i) The union of gbsb*-closed sets need not be a gbsb*-closed set.
(ii) The intersection of gbsb*-open sets need not be a gbsb*-open sets.

The generalized b-strongly b*-interior operator
Definition 3.1. Let A be a subset of a topological space (X, τ). Then the union of all gbsb*-open sets contained in A is called the gbsb*-interior of A and it is denoted by gbsb*int(A). That is, gbsb*int(A)=$\cup\{V:V\subseteq A$ and $V\in$gbsb*-O(X)$\}$.

Remark 3.2. Since the union of gbsb*-open subsets of X is gbsb*-open in X, gbsb*int(A) is gbsb*-open in X.

Definition 3.3. Let A be a subset of a topological space X. A point x∈X is called a gbsb*-interior point of A if there exists a gbsb*-open set G such that $x\in G\subseteq A$.

Theorem 3.4. Let A be a subset of a topological space (X, τ). Then
(i) gbsb*int(A) is the largest gbsb*-open set contained in A.
(ii) A is gbsb*-open if and only if gbsb*int(A)=A.
(iii) gbsb*int(A) is the set of all gbsb*-interior points of A.
(iv) A is gbsb*-open if and only if every point of A is a gbsb*-interior point of A.

Proof:
(i) Being the union of all gbsb*-open sets, gbsb*int(A) is gbsb*-open and contains every gbsb*-open subset of A. Hence gbsb*int(A) is the largest gbsb*-open set contained in A.
(ii) Necessity: Suppose A is gbsb*-open. Then by Definition 3.1, A$\subseteq$gbsb*int(A). But gbsb*int(A)$\subseteq$A and therefore gbsb*int(A)=A. Sufficiency: suppose gbsb*int(A)=A. Then by Remark 3.2, gbsb*int(A) is gbsb*-open set. Hence A is gbsb*-open.
(iii) x∈gbsb*int(A)⇔there exists a gbsb*-open set G such that $x\in G\subseteq A$.
⇔x is a gbsb*-interior point of A.
Hence gbsb*int(A) is the set of all gbsb*-interior points of A.
(iv) It directly follows from (ii) and (iii).

Theorem 3.5. Let A and B be subsets of (X, τ). Then the following results hold.
(i) gbsb*int(ϕ)=ϕ and gbsb*int(X)=X.
(ii) If B is any gbsb*-open set contained in A, then B$\subseteq$gbsb*int(A) .
(iii) If A$\subseteq$B, then gbsb*int(A)$\subseteq$gbsb*int(B).
(iv) int(A)$\subseteq$bint(A)$\subseteq$gbsb*int(A)$\subseteq$gbint(A)$\subseteq$A.
(v) gbsb*int(gbsb*int(A))=gbsb*int(A).

Proof:
(i) Since ϕ is the only gbsb*-open set contained in ϕ, then gbsb*int(ϕ)=ϕ. Since X is gbsb*-open and gbsb*int(X) is the union of all gbsb*-open sets contained in X, gbsb*int(X)=X.
(ii) Suppose B is gbsb*-open set contained in A. Since gbsb*int(A) is the union of all gbsb*-open set contained in A, then we have B$\subseteq$gbsb*int(A).

(iii) suppose A⊆ B. Let x∈gbsb*int(A). Then x is a gbsb*-interior point of A and hence there exists a gbsb*-open set G such that x∈G⊆A. Since A⊆B, then x∈G⊆B. Therefore x is a gbsb*-interior point of B. Hence x∈gbsb*int(B). This proves (iii).

(iv) Since every gbsb*-open set is gb-open, gbsb*int(A)⊆gbint(A). Since every b-open set is gbsb*-open, bint(A)⊆gbsb*int(A). Every open set is b-open, int(A)⊆bint(A). Therefore int(A)⊆bint(A)⊆gbsb*int(A)⊆gbint(A)⊆A.

(v) By Remark 3.2, gbsb*int(A) is gbsb*-open and by Theorem 3.4(ii), gbsb*int(gbsb*int(A))=gbsb*int(A).

Theorem 3.6. Let A and B are the subsets of a topological space X. Then,
(i) gbsb*int(A)∪gbsb*int(B)⊆gbsb*int(A∪B).
(ii) gbsb*int(A∩B)⊆gbsb*int(A)∩gbsb*int(B).

Proof:
(i) Let A and B be subsets of X. By Theorem 3.5(iii), gbsb*int(A)⊆gbsb*int(A∪B) and gbsb*int(B)⊆gbsb*int(A∪B) which implies, gbsb*int(A)∪gbsb*int(B)⊆gbsb*int(A∪B).
(ii) By Theorem 3.5(iii), gbsb*int(A ∩ B)⊆gbsb*int(A) and gbsb*int(A ∩ B)⊆gbsb*int(B) which implies gbsb*int(A∩B)⊆gbsb*int(A)∩gbsb*int(B).

Theorem 3.7. For any subset A of X,
(i) int(gbsb*int(A))=int(A)
(ii) gbsb*int(int(A))=int(A).

Proof:(i) Since gbsb*int(A)⊆A, int(gbsb*int(A))⊆int(A). By Theorem 3.5(iv), int(A)⊆(gbsb*int(A)), and so int(A)=int(int(A))⊆int(gbsb*cl(A)). Hence int(gbsb*int(A))=int(A).
(ii)Since int(A) is open and hence gbsb*-open, by Theorem 3.4(ii), gbsb*int(int(A))=int(A).

The generalized b-strongly b*-closure operator

Definition 4.1. Let A be a subset of a topological space (X, τ). Then the intersection of all gbsb*-closed sets in X containing A is called the gbsb*-closure of A and it is denoted by gbsb*cl(A). That is, gbsb*cl(A)=∩{F: A⊆F and F∈gbsb*-C(X)}.

Remark 4.2. Since the intersection of gbsb*-closed set is gbsb*-closed, gbsb*cl(A) is gbsb*-closed.

Theorem 4.3. Let A be a subset of a topological space (X, τ). Then
(i) gbsb*cl(A) is the smallest gbsb*-closed set containing A.
(ii) A is gbsb*-closed if and only if gbsb*cl(A)=A.

Theorem 4.4. Let A and B be two subsets of a topological space (X, τ). Then the following results hold.
(i) gbsb*cl(ϕ)= $\phi$and gbsb*cl(X)=X.
(ii) If B is any gbsb*-closed set containing A, then gbsb*cl(A)⊆ B.
(iii) If A⊆B, then gbsb*cl(A)⊆gbsb*cl(B).
(iv) A⊆gb-cl(A)⊆gbsb*cl(A)⊆bcl(A)⊆cl(A).
(v) gbsb*cl(gbsb*cl(A))=gbsb*cl(A).

Theorem 4.5. Let A and B be subsets of a topological space (X, τ). Then,
(i) gbsb*cl(A)∪gbsb*cl(B)⊆gbsb*cl(A∪B).
(ii) gbsb*cl(A∩B)⊆gbsb*cl(A)∩gbsb*cl(B).

Theorem 4.6. For a subset A of X and x∈X, x∈gbsb*cl(A) if and only if V∩A≠ ϕ for every gbsb*-open set V containing x.

Proof: Necessity: Suppose x∈gbsb*cl(A). If there is agbsb*-open set V containing x such that V∩A= ϕ, then A⊆X\V and X\V is gbsb*-closed and hence gbsb*cl(A)⊆X\V. Since x∈gbsb*cl(A), then x∈X\V which contradicts to x∈V.

Sufficiency:Assume that $V \cap A \neq \phi$ for every gbsb*-open set V containing x. If $x \notin$ gbsb*cl(A), then there exists a gbsb*-closed set F such that $A \subseteq F$ and $x \notin F$. Therefore $x \in X \backslash F$, $A \cap (X \backslash F) = \phi$ and $X \backslash F$ is gbsb*-open. This is a contradiction to our assumption. Hence $x \in$ gbsb*cl(A).

Theorem 4.7. For any subset A of X,

(i) cl(gbsb*cl(A))=cl(A)

(ii) gbsb*cl(cl(A))=cl(A).

Generalized b-strongly b*-derived set

Definition 5.1. Let A be a subset of a topological space X. A point $x \in X$ is said to be a gbsb*-limit point of A if $G \cap (A \backslash \{x\}) \neq \phi$, for every gbsb*-open set G containing x.

Definition 5.2. The set of all gbsb*-limit points of A is called a gbsb*-derived set of A and is denoted by $D_{gbsb*}(A)$.

Remark 5.3. For a subset A of X, a point $x \in X$ is not a gbsb*-limit point of A if and only if there exists a gbsb*-open set G in X such that $G \cap (A \backslash \{x\}) = \phi$, equivalently $x \in G$ and $G \cap A = \phi$ or $G \cap A = \{x\}$.

Theorem 5.4. Let τ_1 and τ_2 be topologies on X such that gbsb*-O(X, τ_1) $\subseteq$gbsb*-O(X, τ_2). For any subset A of X, every gbsb*-limit point of A with respect to gbsb*-O(X, τ_2) is a gbsb*-limit of A with respect to gbsb*-O(X,τ_1).

Proof: Let A be any subset of X and $x \in X$ be a gbsb*-limit point of A with respect to gbsb*-O(X, τ_2).Then $G \cap (A \backslash \{x\}) \neq \phi$, for every gbsb*-open set G in (X, τ_2) containing x. But gbsb*-O(X, τ_1) $\subseteq$gbsb*-O(X, τ_2), so in particular $G \cap (A \backslash \{x\}) \neq \phi$, for every gbsb*-open set G in (X, τ_1) containing x. Hence x is a gbsb*-limit point of A with respect to gbsb*-O(X,τ_1).

Theorem 5.5. Let A and B be subsets of (X, τ). Then the following are valid:

(i) $D_{gbsb*}(\phi) = \phi$

(ii) $x \in D_{gbsb*}(A)$ implies $x \in D_{gbsb*}(A \backslash \{x\})$

(iii) If $A \subseteq B$, then $D_{gbsb*}(A) \subseteq D_{gbsb*}(B)$

(iv) $D_{gbsb*}(A) \cup D_{gbsb*}(B) \subseteq D_{gbsb*}(A \cup B)$ and $D_{gbsb*}(A \cap B) \subseteq D_{gbsb*}(A) \cap D_{gbsb*}(B)$

(v) $D_{gbsb*}(D_{gbsb*}(A)) \backslash A \subseteq D_{gbsb*}(A)$

(vi) $D_{gbsb*}(A \cup D_{gbsb*}(A)) \subseteq A \cup D_{gbsb*}(A)$

Proof:

(i) Obviously $D_{gbsb*}(\phi) = \phi$.

(ii) Let $x \in D_{gbsb*}(A)$. Then x is a gbsb*-limit point of A. That is, every gbsb*-open set containing x contains atleast one point of A other than x. Therefore x is a gbsb*-limit point of $A \backslash \{x\}$. This proves (ii).

(iii) Suppose$A \subseteq B$. Let $x \in D_{gbsb*}(A)$. Then $G \cap (A \backslash \{x\}) \neq \phi$, for every gbsb*-open set G in (X, τ) containing x. Since $A \subseteq B$, $A \backslash \{x\} \subseteq B \backslash \{x\}$ and hence $G \cap (B \backslash \{x\}) \neq \phi$. Therefore $x \in D_{gbsb*}(B)$. This proves (iii).

(iv) Let A and B be subsets of X. By part (iii), $D_{gbsb*}(A) \subseteq D_{gbsb*}(A \cup B)$ and $D_{gbsb*}(B) \subseteq D_{gbsb*}(A \cup B)$ which implies that, $D_{gbsb*}(A) \cup D_{gbsb*}(B) \subseteq D_{gbsb*}(A \cup B)$. Again by part(iii), $D_{gbsb*}(A \cap B) \subseteq D_{gbsb*}(A)$ and$D_{gbsb*}(A \cap B) \subseteq D_{gbsb*}(B)$ which implies $D_{gbsb*}(A \cap B) \subseteq D_{gbsb*}(A) \cap D_{gbsb*}(B)$.

(v) Let $x \in (D_{gbsb*}(D_{gbsb*}(A))) \backslash A$. Then $G \cap (D_{gbsb*}(A) \backslash \{x\}) \neq \phi$, for every gbsb*-open set G in (X, τ) containing x. Let $y \in G \cap (D_{gbsb*}(A) \backslash \{x\})$. This implies $y \in G$ and $y \in D_{gbsb*}(A)$ with $y \neq x$ and hence $G \cap (A \backslash \{y\}) \neq \phi$. Take $z \in G \cap (A \backslash \{y\})$, then $x \neq z$ because $x \notin A$. Hence $G \cap (A \backslash \{x\}) \neq \phi$ and therefore $x \in D_{gbsb*}(A)$. This proves (v).

(vi) Let $x \in D_{gbsb*}(A \cup D_{gbsb*}(A))$. If $x \in A$, then the result is obvious. Assume $x \notin A$. Since $x \in D_{gbsb*}(A \cup D_{gbsb*}(A))$, then $G \cap ((A \cup D_{gbsb*}(A)) \backslash \{x\}) \neq \phi$, for every gbsb*-open set G containing x. Hence either $G \cap (A \backslash \{x\}) \neq \phi$ or $G \cap (D_{gbsb*}(A) \backslash \{x\}) \neq \phi$. If $G \cap (A \backslash \{x\}) \neq \phi$, then $x \in D_{gbsb*}(A)$. If $G \cap (D_{gbsb*}(A) \backslash \{x\}) \neq \phi$, then $x \in (D_{gbsb*}(D_{gbsb*}(A))$. Since $x \notin A$ gives that $x \in (D_{gbsb*}(D_{gbsb*}(A))) \backslash A$ and by part (v), $x \in D_{gbsb*}(A)$. Hence $D_{gbsb*}(A \cup D_{gbsb*}(A)) \subseteq A \cup D_{gbsb*}(A)$

Theorem 5.6. For any subset A of X, we have $D_{gbsb*}(A) \subseteq$gbsb*cl(A).

Proof: Let $x \in D_{gbsb*}(A)$. Then $G \cap (A \backslash \{x\}) \neq \phi$, for every gbsb*-open set G in (X, τ) containing x. That is, $G \cap A \neq \phi$, for every gbsb*-open set G in (X, τ) containing x. By Theorem 4.6, $x \in$gbsb*cl(A). Hence $D_{gbsb*}(A) \subseteq$gbsb*cl(A).

Theorem 5.7. Let A be a subset of X. Then gbsb*cl(A)=A∪D_{gbsb*}(A).
Proof: Let x∈gbsb*cl(A). If x∈A, then the result is obvious. Suppose x∉A. Since x∈gbsb*cl(A), then by Theorem 4.6, G∩A≠ ϕ, for every gbsb*-open set G in (X, τ) containing x. Since x∉A, then we have G∩(A\{x})≠ ϕ and therefore x∈D_{gbsb*}(A). Hence gbsb*cl(A)⊆A∪D_{gbsb*}(A). On the other hand, we know that A⊆gbsb*cl(A) and by Theorem 5.6, D_{gbsb*}(A)⊆gbsb*cl(A), then we conclude that A∪D_{gbsb*}(A)⊆gbsb*cl(A). Hence gbsb*cl(A)=A∪D_{gbsb*}(A).

Theorem 5.8. Let A and B be subsets of X. If A ∈ gbsb*-O(X) and gbsb*-O(X) is topology on X, then A∩gbsb*cl(B)⊆gbsb*(A∩B).
Proof: Suppose A∈gbsb*-O(X) and gbsb*-O(X) is topology on X. Let x∈A∩gbsb*cl(B). Then x∈A and x∈gbsb*cl(B). By Theorem 5.7, x∈B∪D_{gbsb*}(B). Then we have two cases,

a) If x∈B, then x∈A∩B⊆gbsb*cl(A∩B).
b) If x∉B, then x∈D_{gbsb*}(B) and so G∩B≠ ϕ, for every gbsb*-open set G in (X, τ) containing x. Since A,G∈gbsb*-O(X) and gbsb*-O(X) is a topology on X, then A∩G is also a gbsb*-open set containing x. Hence G∩(A∩B)=(G∩A)∩B≠ ϕ. That implies, x∈D_{gbsb*}(A∩B). By Theorem 5.6, x∈gbsb*cl(A∩B)

Hence A∩gbsb*cl(B)⊆gbsb*(A∩B).

Remark 5.9. If gbsb*-O(X,τ) is not a topology on X, then the above theorem need not be true which is shown in the following
example.

Example 5.10. Let X={a,b,c,d} with a topology τ ={ ϕ, {a}, {b}, {a,b}, {a,b,c}, X}. Here gbsb*-O(X,τ)={ ϕ, {a}, {b}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, X}. Then gbsb*-O(X,τ) is not a topology on X. Let A={a, b, d} and B={a,c,d}. Then A∩gbsb*cl(B)={a, b, d} and gbsb*cl(A∩B)={a,d}. Hence A∩gbsb*cl(B)⊈gbsb*cl(A∩B).

Theorem 5.11. If A is a subset of a discrete topological space X, then D_{gbsb*}(A)= ϕ.
Proof: Let A be a subset of a discrete topological space X and x be any element of X. Since X is a discrete topology, every subset of X is open and so gbsb*-open. In particular the singleton set G={x} is gbsb*-open and therefore G∩(A\{x})=ϕ. Then we conclude that x is not a gbsb*-limit point of A. Since x∈X is arbitrary, then every element of X is not a gbsb*-limit point of A. Hence D_{gbsb*}(A)= ϕ.

Theorem 5.12. For any subset A of X, gbsb*int(A)=A\D_{gbsb*}(X\A).
Proof: Let x∈gbsb*int(A). Then there exists a gbsb*-open set G such that x∈G⊆A. That implies, G∩(X\A)= ϕ where G is a gbsb*-open set containing x and hence x∉D_{gbsb*}(X\A). Therefore x∈A\$D_{gbsb*}$(X\A). Hence gbsb*int(A)⊆A\$D_{gbsb*}$(A). Let x∈A\$D_{gbsb*}$(X\A). Then x∉$D_{gbsb*}$(X\A) and therefore there exists a gbsb*-open set G containing x such that G ∩ (X\A)= ϕ. That is, x ∈ G⊆A. Hence x is a gbsb*-interior point of A. Therefore x ∈ gbsb*int(A) and so A\$D_{gbsb*}$(X\A)⊆gbsb*int(A). Hence gbsb*int(A)=A\$D_{gbsb*}$(X\A).

Theorem 5.13. For any subset A of X,
(i) X\gbsb*int(A)=gbsb*cl(X\A)
(ii) X\gbsb*int(X\A)=gbsb*cl(A)
(iii) X\gbsb*cl(A)=gbsb*int(X\A)
(iv) X\gbsb*cl(X\A)=gbsb*int(A)

Proof. (i) X\gbsb*int(A)=X\(A\D_{gbsb*}(X\A)) = (X\A)∪ D_{gbsb*}(X\A)=gbsb*cl(X\A).
(ii) Replace A by X\A in part(i), X\gbsb*int(X\A)=gbsb*cl(A).
(iii) gbsb*int(X\A)=(X\A)\ D_{gbsb*}(A)=X\(A∪ D_{gbsb*}(A)) = X\gbsb*cl(A). This proves (iii).
(iv) Replace A by X\A in part(iii), X\gbsb*cl(X\A)=gbsb*int(A).

gbsb*-border and gbsb*-frontier of a set
Definition 6.1. Let A be a subset of X. Then the set B_{gbsb*}(A)=A\gbsb*int(A) is called the gbsb*-border of A and the set Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A) is called the gbsb*-frontier of A.

Example 6.2. Let X={a,b,c,d} with a topology τ ={ ϕ, {a}, {b}, {a,b}, {a,b,c}, X}. Here gbsb*-O(X)={ ϕ, {a}, {b}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, X}. Let A={a, b, c}. Then gbsb*cl(A)=X and gbsb*int(A)={a, b, c}. Therefore B_{gbsb*}(A)=A\gbsb*int(A)={a, b, c}\{a, b, c}= ϕ and Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A)=X\{a, b, c}={d}.

Theorem 6.3. If a subset A of X is gbsb*-closed, then B_{gbsb*}(A)=Fr_{gbsb*}(A).
Proof: Let A be a gbsb*-closed subset of X. Then by Theorem 4.3, gbsb*cl(A)=A. Now, Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A)=A\gbsb*int(A)=B_{gbsb*}(A).

Theorem 6.4. For a subset A of X, the following statements are hold:
(i) A=gbsb*int(A)∪B_{gbsb*}(A)
(ii) gbsb*int(A)∩B_{gbsb*}(A)= ϕ
(iii) A is gbsb*-open if and only if B_{gbsb*}(A)= ϕ
(iv) B_{gbsb*}(gbsb*int(A))= ϕ
(v) gbsb*int(B_{gbsb*}(A))= ϕ
(vi) B_{gbsb*}(B_{gbsb*}(A))=B_{gbsb*}(A)

Proof:
(i) Let x∈A. If x∈gbsb*int(A), then the result is obvious. If x∉gbsb*int(A), then by the definition of B_{gbsb*}(A), x ∈ B_{gbsb*}(A). Hence x ∈ gbsb*int(A) ∪ B_{gbsb*}(A) and so A⊆gbsb*int(A) ∪ B_{gbsb*}(A). On the other hand, since gbsb*int(A)⊆A and B_{gbsb*}(A)⊆A, then we have gbsb*int(A)∪B_{gbsb*}(A)⊆A. This proves (i).
(ii) Suppose gbsb*int(A)∩B_{gbsb*}(A)≠ ϕ. Let x∈gbsb*int(A)∩B_{gbsb*}(A). Then x∈gbsb*int(A) and x∈B_{gbsb*}(A). Since B_{gbsb*}(A)=A\gbsb*int(A), then x ∈ A but x ∈ gbsb*int(A), x ∉ A. There is a contradiction. Hence gbsb*int(A)∩B_{gbsb*}(A)= ϕ
(iii) Necessity: Suppose A is gbsb*-open. Then by Theorem 3.4, gbsb*int(A)=A. Now, B_{gbsb*}(A)=A\gbsb*int(A)=A\A= ϕ. Sufficiency: Suppose B_{gbsb*}(A)= ϕ. This implies, A\gbsb*int(A)= ϕ. Therefore A=gbsb*int(A) and hence A is gbsb*-open.
(iv) By the definition of gbsb*-border of a set, B_{gbsb*}(gbsb*int(A))=gbsb*int(A)\gbsb*int(gbsb*int(A)). By Theorem 3.4, gbsb*int(gbsb*int(A))=gbsb*int(A) and hence B_{gbsb*}(gbsb*int(A))= ϕ.
(v) Let x ∈ gbsb*int(B_{gbsb*}(A)). Since B_{gbsb*}(A)⊆A, by Theorem 3.5, gbsb*int(B_{gbsb*}(A))⊆gbsb*int(A). Hence x∈gbsb*int(A). Since gbsb*int(B_{gbsb*}(A))⊆B_{gbsb*}(A), then x∈B_{gbsb*}(A). Therefore x∈gbsb*int(A)∩B_{gbsb*}(A). By part (ii), x= ϕ. This proves (v).
(vi) By the definition of gbsb*-border of a set, B_{gbsb*}(B_{gbsb*}(A))=B_{gbsb*}(A)\gbsb*int(B_{gbsb*}(A)). By part (v), gbsb*int(B_{gbsb*}(A))= ϕ and hence B_{gbsb*}(B_{gbsb*}(A))=B_{gbsb*}(A).

Theorem 6.5. Let A be a subset of X. Then,
(i) B_{gbsb*}(A)=A∩gbsb*cl(X\A)
(ii) B_{gbsb*}(A)=A∩D_{gbsb*}(X\A).

Proof:
(i) Since B_{gbsb*}(A)=A\gbsb*int(A) and by Theorem 5.13, B_{gbsb*}(A)=A\(X\gbsb*cl(X\A)) =A∩(X\(X\gbsb*cl(X\A))=A∩gbsb*cl(X\A).
(ii) By Theorem 6.5 and Theorem 5.7, we have B_{gbsb*}(A)= A ∩ gbsb*cl(X\A)=A ∩ ((X\A) ∪ D_{gbsb*}(X\A))= (A∩(X\A))∪(A∩D_{gbsb*}(X\A))= ϕ ∪(A∩D_{gbsb*}(X\A))= A∩D_{gbsb*}(X\A).

Theorem 6.6. Let A be a subset of X. Then A is gbsb*-closed if and only if Fr_{gbsb*}(A)⊆A.
Proof: Necessity: Suppose A is gbsb*-closed. Then by Theorem 4.3, gbsb*cl(A)=A. Now, Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A)=A\gbsb*int(A)⊆A. Hence Fr_{gbsb*}(A)⊆A. Sufficiency: Assume that, Fr_{gbsb*}(A)⊆A. Then gbsb*cl(A)\gbsb*int(A)⊆A. Since gbsb*int(A)⊆A, then we conclude that gbsb*cl(A)⊆A. Also A⊆gbsb*cl(A). Therefore gbsb*cl(A)=A and hence A is gbsb*-closed.

Theorem 6.7. For a subset A of X, we have the following
(i) gbsb*cl(A)=gbsb*int(A)∪Fr_{gbsb*}(A).
(ii) gbsb*int(A)∩Fr_{gbsb*}(A)= ϕ.
(iii) B_{gbsb*}(A)⊆Fr_{gbsb*}(A).
(iv) Fr_{gbsb*}(A)=B_{gbsb*}(A)∪(D_{gbsb*}(A)\gbsb*int(A)).
(v) If A is gbsb*-open then Fr_{gbsb*}(A)=B_{gbsb*}(X\A).

(vi) $Fr_{gbsb*}(A)$=gbsb*cl(A)∩gbsb*cl(X\A).

Proof:

(i) Since gbsb*int(A)⊆gbsb*cl(A) and $Fr_{gbsb*}(A)$⊆gbsb*cl(A), then gbsb*int(A) ∪ $Fr_{gbsb*}(A)$⊆gbsb*cl(A). Let x∈gbsb*cl(A). Suppose x∉$Fr_{gbsb*}(A)$. Since, then x∈gbsb*int(A). Hence x∈gbsb*int(A)∪$Fr_{gbsb*}(A)$ and hence gbsb*cl(A)⊆gbsb*int(A)∪$Fr_{gbsb*}(A)$. This proves (i).

(ii) Suppose gbsb*int(A)∩$Fr_{gbsb*}(A)$≠ ϕ. Let x∈gbsb*int(A)∩$Fr_{gbsb*}(A)$. Then x∈gbsb*int(A) and x∈$Fr_{gbsb*}(A)$, which is impossible to x belongs to both gbsb*int(A) and $Fr_{gbsb*}(A)$, since$Fr_{gbsb*}(A)$=gbsb*cl(A)\gbsb*int(A). Hence gbsb*int(A)∩$Fr_{gbsb*}(A)$= ϕ.

(iii) Since A⊆gbsb*cl(A), then A\gbsb*int(A)⊆gbsb*cl(A)\gbsb*int(A). That implies, $B_{gbsb*}(A)$⊆$Fr_{gbsb*}(A)$.

(iv) Since $Fr_{gbsb*}(A)$=gbsb*cl(A)\gbsb*int(A) and by Theorem 5.7, $Fr_{gbsb*}(A)$=(A ∪ $D_{gbsb*}(A)$)\gbsb*int(A) =(A\gbsb*int(A))∪($D_{gbsb*}(A)$\gbsb*int(A))=$B_{gbsb*}(A)$∪($D_{gbsb*}(A)$\gbsb*int(A)). This proves (iv).

(v) Assume that a is gbsb*-open. Then by Theorem 3.4, gbsb*int(A)=A and by Theorem 6.4, $B_{gbsb*}(A)$= ϕ. By part (iv), $Fr_{gbsb*}(A)$=$B_{gbsb*}(A)$∪($D_{gbsb*}(A)$\gbsb*int(A)) =ϕ ∪($D_{gbsb*}(A)$\A)= $D_{gbsb*}(A)$\A=$D_{gbsb*}(A)$∩(X\A). Then by Theorem 6.5, $Fr_{gbsb*}(A)$=B_{gbsb*}(X\A).

(vi) Since gbsb*cl(X\A)=X\gbsb*int(A), gbsb*cl(A) ∩ gbsb*cl(X\A) =gbsb*cl(A) ∩ (X\gbsb*int(A)) =(gbsb*cl(A)∩X)\(gbsb*cl(A)∩gbsb*int(A)) =gbsb*cl(A)\gbsb*int(A) =$Fr_{gbsb*}(A)$.

Theorem 6.8. For a subset A of X, we have the following

(i) $Fr_{gbsb*}(A)$=Fr_{gbsb*}(X\A).
(ii) $Fr_{gbsb*}(A)$ is gbsb*-closed.
(iii) Fr_{gbsb*}($Fr_{gbsb*}(A)$)⊆$Fr_{gbsb*}(A)$.
(iv) Fr_{gbsb*}(gbsb*int(A))⊆$Fr_{gbsb*}(A)$.
(v) Fr_{gbsb*}(gbsb*cl(A))⊆$Fr_{gbsb*}(A)$.
(vi) gbsb*int(A)=A\$Fr_{gbsb*}(A)$.

Proof:

(i) By Theorem 6.7 (vi), $Fr_{gbsb*}(A)$=gbsb*cl(A)∩gbsb*cl(X\A)=Fr_{gbsb*}(X\A).

(ii) Now, gbsb*cl($Fr_{gbsb*}(A)$)=gbsb*cl(gbsb*cl(A) ∩ gbsb*cl(X\A))⊆gbsb*cl(A) ∩ gbsb*cl(X\A) =$Fr_{gbsb*}(A)$. That is, gbsb*cl($Fr_{gbsb*}(A)$)⊆$Fr_{gbsb*}(A)$. Also $Fr_{gbsb*}(A)$⊆gbsb*($Fr_{gbsb*}(A)$). Therefore gbsb*cl($Fr_{gbsb*}(A)$)=$Fr_{gbsb*}(A)$ and hence $Fr_{gbsb*}(A)$ is gbsb*-closed.

(iii) By part (ii), $Fr_{gbsb*}(A)$ is gbsb*-closed and by Theorem 6.6, Fr_{gbsb*}($Fr_{gbsb*}(A)$)⊆$Fr_{gbsb*}(A)$.

(iv) By the definition of gbsb*-frontier, Fr_{gbsb*}(gbsb*int(A))= gbsb*cl(gbsb*int(A))\gbsb*int(gbsb*int(A))⊆ gbsb*cl(A)\gbsb*int(A) =$Fr_{gbsb*}(A)$. Hence Fr_{gbsb*}(gbsb*int(A))⊆$Fr_{gbsb*}(A)$.

(v) By the definition of gbsb*-frontier, Fr_{gbsb*}(gbsb*cl(A))= gbsb*cl(gbsb*cl(A))\gbsb*int(gbsb*cl(A))⊆ gbsb*cl(A)\gbsb*int(A) =$Fr_{gbsb*}(A)$. Hence Fr_{gbsb*}(gbsb*cl(A))⊆$Fr_{gbsb*}(A)$.

(vi) Now A\$Fr_{gbsb*}(A)$=A\(gbsb*cl(A)\gbsb*int(A)) =A∩(X\gbsb*cl(A))∪gbsb*int(A)=φ ∪gbsb*int(A)=gbsb*int(A).

gbsb*-exterior and gbsb*-kernel

Definition 7.1. Let A be a subset of a topological space (X, τ). Then the gbsb*-interior of X\A is called the gbsb*-exterior of A and it is denoted by $Ext_{gbsb*}(A)$. That is, $Ext_{gbsb*}(A)$=gbsb*int(X\A).

Example 7.2. Let X={a,b,c,d} with a topology τ ={ ϕ, {a,b}, {a,b,c},X}. Here gbsb*-O(X)={ ϕ, {a}, {b}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, X}. Let A={c, d}. Then $Ext_{gbsb*}(A)$=gbsb*int(X\A)={a, b}.

Theorem 7.3. For a subsets A and B of X, the following are valid.

(i) $Ext_{gbsb*}(A)$=X\gbsb*cl(A).
(ii) Ext_{gbsb*}(Ext_{gbsb*} (A))=gbsb*int(gbsb*cl(A))⊇gbsb*int(A).
(iii) If A⊆B, then Ext_{gbsb*} (B)⊆ Ext_{gbsb*} (A).
(iv) Ext_{gbsb*} (A∪B)_ Ext_{gbsb*} (A)∩ Ext_{gbsb*} (B).
(v) Ext_{gbsb*}(A∩B)_ Ext_{gbsb*} (A)∪ Ext_{gbsb*} (B).
(vi) Ext_{gbsb*}(X)= ϕ and Ext_{gbsb*} (ϕ)=X.
(vii) $Ext_{gbsb*}(A)$=Ext_{gbsb*}(X\$Ext_{gbsb*}(A)$).

Proof.

(i) Since, X\gbsb*cl(A)=gbsb*int(X\A), Ext_{gbsb*}(A)=X\gbsb*cl(A).

(ii) Ext_{gbsb*}(Ext_{gbsb*}(A))= Ext_{gbsb*}(gbsb*int(X\A))=gbsb*int(X\gbsb*int(X\A))=gbsb*int(gbsb*cl(A))⊇gbsb*int(A).

(iii) Suppose A⊆B. Then, Ext_{gbsb*}(B)=gbsb*int(X\B)⊆gbsb*int(X\A)= Ext_{gbsb*}(A).

(iv) Ext_{gbsb*}(A∪B)=gbsb*int(X\(A∪B))=gbsb*int((X\A)∩(X\B)) ⊆ gbsb*(X\A)∩gbsb*cl(X\B)= Ext_{gbsb*}(A)∩ Ext_{gbsb*}(B).

(v) Ext_{gbsb*} (A∩B)=gbsb*int(X\(A∩B))=gbsb*int((X\A)∪(X\B)) ⊇ gbsb*(X\A)∪gbsb*cl(X\B)= Ext_{gbsb*}(A)∪ Ext_{gbsb*}(B).

(vi) Ext_{gbsb*}(X)=gbsb*int(X\X)=gbsb*int(ϕ) and Ext_{gbsb*}(ϕ)=gbsb*int(X\ ϕ)=gbsb*int(X). Since ϕ and X are gbsb*-open sets, then gbsb*int(ϕ)= ϕ and gbsb*int(X)=X. Hence Ext_{gbsb*}(ϕ)=X and Ext_{gbsb*}(X)= ϕ.

(vii) Ext_{gbsb*}(X\ Ext_{gbsb*}(A))= Ext_{gbsb*}(X\gbsb*int(X\A))=gbsb*int(X\(X\gbsb*int(X\A)))=gbsb*int(gbsb*int(X\A)) =gbsb*int(X\A)= Ext_{gbsb*}(A).

Definition 7.4. Let A be a subset of a topological space X. Then the intersection of all gbsb*-open sets containing A is called the gbsb*-kernel of A. It is denoted by gbsb*ker(A). That is, gbsb*ker(A)=∩{U∈gbsb*-O(X,τ) and A⊆U}.

Theorem 7.5. Let A and B be subsets of (X, τ) . Then the following results hold.

(i) A⊆gbsb*ker(A).

(ii) If A⊆B, then gbsb*ker(A) ⊆gbsb*ker(B).

(iii) If A is gbsb*-open, then gbsb*ker(A)=A.

(iv) gbsb*ker(gbsb*ker(A))=gbsb*ker(A).

Proof:

(i) Since gbsb*ker(A) is the intersection of all gbsb*-open sets containing A, then we have A⊆gbsb*ker(A).

(ii) Suppose A⊆B. Let U be any gbsb*-open set containing B. Since A⊆B, then A⊆U and hence by the definition of gbsb*ker(A), gbsb*ker(A)⊆U. Therefore, gbsb*ker(A)⊆∩{U/B⊆U and U is gbsb*-open}=gbsb*ker(B). This proves (ii).

(iii) Suppose A is gbsb*-open. Then by Definition, gbsb*ker(A) ⊆A. But A⊆gbsb*ker(A) and therefore gbsb*ker(A)=A.

(iv) By part (i) and (ii), gbsb*ker(A)⊆gbsb*ker(gbsb*ker(A)). If x∉gbsb*ker(A), then there exists a gbsb*-open set U such that A⊆U and x∉U. This implies that, gbsb*ker(A) ⊆U, and so we have x∉gbsb*ker(gbsb*ker(A)). Thus gbsb*ker(gbsb*ker(A))=gbsb*ker(A).

Theorem 7.7. Let (X, τ) be a topological space. Then ∩{gbsb*cl({x})/x∈X} = ϕ if and only if gbsb*ker({x})≠X for every x∈X.

Proof. Necessity: Suppose that ∩{gbsb*cl({x}): x∈X} = ϕ. Suppose there is a point y in X such that gbsb*ker({y}) = X. Let x be any point of X. Then x∈gbsb*ker({y}) and therefore x∈V for every gbsb*-open set V containing y. That is, every gbsb*-closed set containing x must contain y and hence y∈gbsb*cl({x}) for any x∈X. This implies that y∈∩{gbsb*cl({x}): x∈X}. This is a contradiction to our assumption. Hence gbsb*ker({x})≠X for every x∈X.
Sufficiency: Assume that gbsb*ker({x})≠X, for every x∈X. If there exists a point y in X such that y∈∩{gbsb*cl(x): x∈X}, then every gbsb*-open set containing y must contain every point of X. This implies, the space X is the unique gbsb*-open set containing y. Hence gbsb*ker({y}) = X, which is a contradiction. Therefore, ∩{gbdb*cl({x}): x∈X}= ϕ.

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