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On The Positive Pell Equation $y^2 = 40x^2 + 1$

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Abstract: The binary quadratic equation respected by the positive pellian $y^2 = 40x^2 + 1$ is analysed for its distinct integer solutions. A few interesting among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained of other choices of hyperbolas, parabolas and special Pythagorean triangle.
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INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 40x^2 + 1$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS

Consider the binary quadratic equation

$$y^2 = 40x^2 + 1$$

The least positive integer solutions $x_0 = 3, y_0 = 19$

The general solution (x_n, y_n) of (1) is given by

where,

$$f_n = \left(19 + 3\sqrt{40}\right)^{n+1} + \left(19 - 3\sqrt{40}\right)^{n+1}$$

$$g_n = \left(19 + 3\sqrt{40}\right)^{n+1} - \left(19 - 3\sqrt{40}\right)^{n+1}$$

where n = -1,0,1,2,…

The recurrence relations satisfied by the solutions (2) are given by

$$y_{n+2} - 38y_{n+1} + y_n = 0$$

$$x_{n+2} - 38x_{n+1} + x_n = 0$$

Some numerical examples of x & y are satisfying (1) are given in the table below.

Numerical examples

n	x_n	y_n
0	3	19
1	114	721
2	4329	27379
3	164388	1039681
4	6242415	39480499

From the above, we observe some interesting relations among the solutions which are presented below:

1. The x_n values are alternatively odd and even
2. The y_n values are always odd
3. Each of the following expressions is a nasty number
   - ❖ $8y_{2n+1}+12$
   - ❖ $\dfrac{1368y_{2n+2}-36y_{2n+4}+36}{3}$
   - ❖ $\dfrac{12y_{2n+2}-720x_{2n+1}+228}{19}$
   - ❖ $228y_{2n+2}-1440x_{2n+2}+12$
   - ❖ $\dfrac{8652y_{2n+2}-1440x_{2n+3}+228}{19}$
   - ❖ $\dfrac{12y_{2n+3}-54720x_{2n+1}+8652}{721}$
   - ❖ $\dfrac{228y_{2n+3}-54720x_{2n+2}+228}{19}$
   - ❖ $\dfrac{12x_{2n+3}-8652x_{2n+1}+1368}{114}$
   - ❖ $\dfrac{228x_{2n+3}-8652x_{2n+2}+36}{3}$
   - ❖ $8652y_{2n+3}-54720x_{2n+2}+12$
   - ❖ $12x_{2n+2}-228x_{2n+1}+12$
4. Each of the following expressions is a cubical integer :
   - ❖ $2y_{3n+2}+6y_n$
   - ❖ $9[2(114y_{3n+3}-3y_{3n+4})+6(114y_{n+1}-3y_{n+2})]$
   - ❖ $2(y_{3n+3}-120x_{3n+2})+6(y_{n+1}-120x_n)$
   - ❖ $2(19y_{3n+3}-120x_{3n+3})+6(19y_{n+1}-120x_{n+1})$
   - ❖ $2(721y_{3n+3}-120x_{3n+4})+6(721y_{n+1}-120x_{n+2})$
   - ❖ $\dfrac{1}{721}[2(y_{3n+4}-4560x_{3n+2})+6(y_{n+2}-4560x_n)]$
   - ❖ $2(y_{3n+4}-240x_{3n+3})+6(y_{n+2}-240x_{n+1})$

- $2(721y_{3n+4}-4560x_{3n+4})+6(721y_{n+2}-4560x_{n+2})$
- $9[2(x_{3n+3}-19y_{3n+2})+6(x_{n+1}-19x_n)]$
- $12996[2(x_{3n+4}-721x_{3n+2})+6(x_{n+2}-721x_n)]$
- $9[2(19x_{3n+4}-721x_{3n+3})+6(19x_{n+2}-721x_{n+1})]$

5. Relations among the solutions:

- $2y_{n+2}=76y_{n+1}-2y_n$
- $120x_{n+1}=19y_{n+1}-y_n$
- $114y_{n+1}=3y_{n+2}+3y_n$
- $y_{n+2}=240x_{n+1}+y_n$
- $19x_{n+2}=721x_{n+1}+3y_n$
- $721y_{n+1}=120x_{n+2}+19y_n$
- $721y_{n+2}=4560x_{n+2}+y_n$
- $721x_n=x_{n+2}-114y_n$
- $120x_{n+1}=y_{n+2}-19y_{n+1}$
- $19y_n=y_{n+1}-120x_n$
- $19y_{n+2}=721(y_{n+1}-4440x_n)$
- $19x_{n+1}=x_n+3y_{n+1}$
- $19x_{n+2}=19x_n+114y_{n+1}$
- $x_n=19x_{n+1}-3y_{n+1}$
- $19y_{n+2}=y_{n+1}+120x_{n+2}$
- $19x_n=19x_{n+2}-114y_{n+1}$
- $19x_{n+1}=x_{n+2}-3y_{n+1}$
- $721y_n=y_{n+2}-4560x_n$
- $721y_{n+1}=19y_{n+2}-120x_n$
- $721x_{n+1}=3y_{n+2}+19x_n$
- $721x_{n+2}=114y_{n+2}+x_n$
- $19y_n=19y_{n+2}-4560x_{n+1}$
- $19x_{n+2}=x_{n+1}+3y_{n+2}$
- $3y_n=x_{n+1}-19x_n$
- $3y_{n+1}=19x_{n+1}+341x_n$
- $x_{n+2}=38x_{n+1}-x_n$
- $6y_{n+1}=x_{n+2}-x_n$
- $3y_n=19x_{n+2}-721x_{n+1}$
- $3y_{n+1}=x_{n+2}-19x_{n+1}$
- $3y_{n+2}=19x_{n+2}-x_{n+1}$

- $3x_n = 114x_{n+1} - 3x_{n+2}$
- $2y_n = 2(721y_{n+2} - 4560x_{n+2})$
- $2y_{n+1} = 38y_{n+2} - 240x_{n+2}$

REMARKABLE OBSERVATION

I. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of hyperbola which are presented in the table1 below:

Table-1:Hyperbolas

S.no	(X,Y)	Hyperbola
1	$(y_{n+1} - y_n, 2y_n)$	$90Y^2 - X^2 = 360$
2	$(y_{n+2} - 721y_n, 2y_n)$	$129960Y^2 - X^2 = 4$
3	$(x_{n+1} - 3y_n, 2y_n)$	$361Y^2 - 160X^2 = 4$
4	$(x_{n+2} - 114y_n, 2y_n)$	$519841Y^2 - 160X^2 = 2079364$
5	$(38y_{n+2} - 1442y_{n+1}, 114y_{n+1} - 3y_{n+2})$	$160Y^2 - X^2 = 1440$
6	$(x_n, y_{n+1} - 120x_n)$	$Y^2 - 14440X^2 = 361$
7	$(19x_{n+2} - 114y_{n+1}, 721y_{n+1} - 120x_{n+2})$	$Y^2 - 40X^2 = 361$
8	$(x_n, y_{n+2} - 4560x_n)$	$Y^2 - 20793640X^2 = 519841$
9	$(721x_{n+1} - 3y_{n+2}, 19y_{n+2} - 4560x_{n+1})$	$Y^2 - 40X^2 = 361$
10	$(721x_{n+2} - 114y_{n+2}, 721y_{n+2} - 4560x_{n+2})$	$Y^2 - 80X^2 = 2$
11	$(x_n, x_{n+1} - 19x_n)$	$Y^2 - 240X^2 = 6$
12	$(x_n, x_{n+2} - 721x_n)$	$Y^2 - 519840X^2 = 12996$
13	$(114x_{n+1} - 3x_{n+2}, 19x_{n+2} - 721x_{n+1})$	$Y^2 - 40X^2 = 9$

II. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of parabola which are presented in the table 2 below:

Table-2: Parabolas

S.no	(X,Y)	Parabola
1	$(y_{n+1} - y_n, 2y_{2n+1})$	$X^2 = 90Y - 180$
2	$(y_{n+2} - 721y_n, 2y_{2n+1})$	$X^2 = 129960Y - 259920$
3	$(x_{n+1} - 3y_n, 2y_{2n+1})$	$160X^2 = 361Y - 722$
4	$(x_{n+2} - 114y_n, 2y_{2n+1})$	$160X^2 = 519841Y - 1039682$
5	$(38y_{n+2} - 1442y_{n+1}, 114y_{2n+2} - 3y_{2n+4})$	$X^2 = 240Y - 720$
6	$(x_n, y_{2n+2} - 120x_{2n+1})$	$3040X^2 = 2Y - 38$
7	$(19x_{n+1} - 3y_{n+1}, 19y_{2n+2} - 120x_{2n+2})$	$80X^2 = Y - 1$
8	$(19x_{n+2} - 114y_{n+1}, 721y_{2n+2} - 120x_{2n+4})$	$80X^2 = 19Y - 361$

9	$(x_n, y_{2n+3} - 4560x_{2n+1})$	$57680X^2 = Y - 721$
10	$(721x_{n+1} - 3y_{n+2}, 19y_{n+2} - 4560x_{n+1})$	$80X^2 = 19Y - 361$
11	$(721x_{n+2} - 114y_{n+2}, 721y_{2n+3} - 4560x_{2n+3})$	$80X^2 = Y - 1$
12	$(x_n, x_{2n+2} - 19x_{2n+1})$	$240X^2 = Y - 3$
13	$(x_n, x_{2n+3} - 721x_{2n+1})$	$9120X^2 = Y - 114$
14	$(114x_{n+1} - 3x_{n+2}, 19x_{2n+3} - 721x_{2n+2})$	$80X^2 = 3Y - 9$

III. Consider $m = x_{n+1} + y_{n+1}, n = x_{n+1}$.Observe that $m > n > 0$.Treat m, n as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, $\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2$

Then the following interesting relations are observed:

a) $\alpha - 20\beta + 19\gamma = -1$

b) $21\alpha - \gamma = 80\frac{A}{P} - 1$

c) $11\alpha - 10\beta + 9\gamma + 1 = 40\frac{A}{P}$

d) $\frac{2A}{P} = x_{n+1}y_{n+1}$

CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 40x^2 + 1$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive pell equations and determine their integer solutions along with suitable properties.

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