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Marcinkiewicz Integral Associated with Schrödinger Operator on Weighted Herz Spaces with Variable Exponent

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Abstract: Set $L=-\Delta+V$ be a Schrödinger operator, where V is a non-negative potential belonging to some reverse Hölder class. In this paper, using the properties of the Marcinkiewicz integrals associated with operators and the estimate of the classical inequalities and Muckenhoupt weighted functions, the authors obtained the boundedness of Marcinkiewicz integrals associated with Schrödinger operators on $L^{p(\cdot)}(\omega)$, where $p(\cdot)\in[1,\infty)$ and $\omega\in A_{p(\cdot)}$. Then, the boundedness of Marcinkiewicz integrals associated with Schrödinger operator μ_j^L on weighted Herz spaces with variable exponent is established.
Keywords: Schrödinger operator, Marcinkiewicz integral, variable exponent, weighted, boundedness, Herz space

INTRODUCTION

Function spaces with variable exponent are studied with keen interest not in real analysis, but also in partial differential equations. They are applicable to the image restoration and modeling for electror- heological fluids. We know that the theory of function spaces with variable exponent and its applications has made rapidly progress in the past twenty years. In particular, Muckenhoupt [1] has obtained the theory on weights called the Muckenhoupt A_p theory in the study of weighted function spaces.

In this paper, we will consider the Schrödinger operator $L=-\Delta+V(x)$, in $\mathbb{R}^n$, and $V(x)$ is a non-negative potential belonging to the reverse Hölder class RH_q, where $q\ge n/2$, and exists a constant $C>0$, such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q dx\right)^{\frac{1}{q}}\le C\left(\frac{1}{|B|}\int_B V dx\right),$$

holds for every ball in $\mathbb{R}^n$.

The Marcinkiewicz integral operator μ is defined by

$$\mu(f)=\left(\int_0^\infty\left|\int_{|x-y|<t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

Stein [2] first introduced the operator μ for higher dimension and proved that μ are of type (p,p) $(1<p\le 2)$ and of weak type (1,1), in the case of $\Omega\in \mathrm{Lip}_\gamma(S^{n-1})$. Then, Benedek *et al.* [3] extended Stein's results, proved that $\Omega\in C^1(S^{n-1})$, and then μ is of type (p,p) $(1<p\le\infty)$. Similarly, the Marcinkiewicz integral associated with the Schrödinger operator L is defined by

$$\mu_j^L f(x)=\left(\int_0^\infty\left|\int_{|x-y|<t}K_j^L(x,y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}},$$

Where, $K_j^L(x,y)=\tilde{K}_j^L(x,y)|x-y|$, $\tilde{K}_j^L(x,y)$ is the kernel of $R_j=\left(\partial/\partial x_j\right)L^{-1/2}$, $j=1\cdots n$. In particular, when $V=0$, $K_j^\Delta(x,y)=\tilde{K}_j^\Delta(x,y)|x-y|=\left(\left(x_j-y_j\right)/|x-y|\right)/|x-y|^{n-1}$, and $\tilde{K}_j^\Delta(x,y)$ is the kernel of $R_j^\Delta=\left(\partial/\partial x_j\right)\Delta^{-\frac{1}{2}}$, $j=1,\cdots,n$. Therefore

$$\mu_j^\Delta f(x)=\left(\int_0^\infty\left|\int_{|x-y|<t}K_j^\Delta(x,y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

We write $K_j(x,y)=K_j^\Delta(x,y)$ and $\mu_j=\mu_j^\Delta$. Obviously, $\mu_j f(x)$ is the classic Marcinkiewicz integral.

For a given potential $V\in RH_q$, where $q>n/2$. we introduce the auxiliary function

$$\rho(x)=\frac{1}{m_V(x)}=\sup_{r>0}\left\{r:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)dy\le 1\right\},\qquad x\in\mathbb{R}^n.$$

The above assumption $\rho(x)$ are finite, for all $x\in\mathbb{R}^n$.

Throughout this paper, we denote the Lebesgue measure, and the characteristic function for a measurable set $E\subset\mathbb{R}^n$, by $|E|$ and χ_E, respectively. C denotes the positive constant which is independent of the main parameters involved, but the value may change from line to line.

Preliminaries

A measurable function $p(x)$, $x\in\mathbb{R}^n$ is said to be a variable exponent, if $0<p(x)<\infty$. Denote by $P(\mathbb{R}^n)$ to be the set of all variable exponents $p(\cdot)$ such that

$$1<p^-=\operatorname{essinf}\left\{p(x):x\in\mathbb{R}^n\right\}\le\operatorname{esssup}\left\{p(x):x\in\mathbb{R}^n\right\}=p^+<\infty.$$

In what follow, for any $p(\cdot)\in P(\mathbb{R}^n)$, we use $p'(\cdot)$ to denote its conjugate variable exponent, that is to say, for any $x\in\mathbb{R}^n$, $1/p(x)+1/p'(x)=1$. The variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f defined on $\mathbb{R}^n$ satisfying that there exists a constant $\lambda>0$ such that $\rho_{p(\cdot)}(f/\lambda)<\infty$. Where $\rho_{p(\cdot)}$ associated with $p(\cdot)$ is given by

$$\rho_{p(\cdot)}(f)=\int_{\mathbb{R}^n}|f(x)|^{p(x)}dx.$$

For $p(\cdot)\in P(\mathbb{R}^n)$, $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space equipped with the norm.

Definition 2.1 A measurable function $\alpha(\cdot):\mathbb{R}^n\to(0,\infty)$, is said to be globally log-Hölder continuous, if there exists $C>0$, such that

$$|\alpha(x)-\alpha(y)|\le\frac{-C}{\log(|x-y|)}\qquad\left(|x-y|<1/2\right),$$

$$|\alpha(x)-\alpha(\infty)|\le\frac{C}{\log(e+|x|)}\qquad\left(x\in\mathbb{R}^n\right),$$

for some consant $\alpha(\infty)$. The sets of globally log-Hölder continuous is denoted by $LH(\mathbb{R}^n)$.

Shen [4] gave the following kernel estimate that we needed .

Lemma 2.2 If $V\in RH_d$, then, for every N, there exists a constant C, such that

$$|K_j^L(x,z)|\le\frac{C(1+(|x-z|/\rho(x)))^{-N}}{|x-z|^{n-1}}.$$

The classical Marcinkiewicz integral has the following bounded result (Ding et al [5]).

Lemma 2.3 Set $\Omega \in L^q(S^{n-1})$ and $1 < q \le \infty$. Let $1 < p \le \infty$ and $\omega \in A_{p/q'}$. Then μ_j^L are bounded on $L^p(\omega, \mathbb{R}^n)$, this is

$$\left\| \mu_j^L f(x) \right\|_{L^p(\omega, \mathbb{R}^n)} \le C \left\| f(x) \right\|_{L^p(\omega, \mathbb{R}^n)}.$$

The following lemma is very important for our main results (Gao and Tang [6]).

Lemma 2.4 $\mu_j^L f(x) \le CM(f)(x) + \mu_j f(x)$, there M is the Hardy-Littlewood maximal operator.

Izuki [7] introduce the definition of the Muckenhoupt weight with variable exponent.

Definition 2.5 Set $p(\cdot) \in \mathrm{P}(\mathbb{R}^n)$. A weight ω is said to be an $A_{p(\cdot)}$ weight, if

$$\sup_{\text{B:ball}} \frac{1}{|B|} \left\| \omega^{1/p(\cdot)} \chi_B \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left\| \omega^{-1/p(\cdot)} \chi_B \right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty.$$

The sets of $A_{p(\cdot)}$ weight is denoted by $A_{p(\cdot)}$.

Definition 2.6 Set $p(\cdot) \in \mathrm{P}(\mathbb{R}^n)$. A weight ω is said to be an $\tilde{A}_{p(\cdot)}$ weight, if

$$\sup_{\text{B:ball}} |B|^{-pB} \left\| \omega \chi_B \right\|_{L^1(\mathbb{R}^n)} \left\| \omega^{-1/p(\cdot)} \chi_B \right\|_{L^{p'(\cdot)/p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Where, p_B is defined by

$$p_B = (1/|B| \int_B 1/p(x)\, dx)^{-1}.$$

The sets of $\tilde{A}_{p(\cdot)}$ weight is denoted by $\tilde{A}_{p(\cdot)}$.

Definition 2.7 Let $p(\cdot) \in \mathrm{P}(\mathbb{R}^n)$ and ω be a weight. The weight Lebesgue space with variable exponent $L^{p(\cdot)}(\omega)$ is defined by

$$L^{p(\cdot)}(\omega) = L^{p(\cdot)}(\mathbb{R}^n, \omega^{1/p(\cdot)}).$$

Namely, the space $L^{p(\cdot)}(\omega)$ is a Banach function space equipped with the norm

$$\left\| f \right\|_{L^{p(\cdot)}(\omega)} = \left\| f \omega^{1/p(\cdot)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The following lemmas can be found in (Izuki [7]).

Lemma 2.8 Set $p(\cdot) \in \mathrm{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$. then the following three conditions are equivalent:

(i) $\omega \in A_{p(\cdot)}$.

(ii) $\omega \in \tilde{A}_{p(\cdot)}$.

(iii) The Hardy-Littlewood maximal operator is bounded on the weighted variable Lebesgue space.

Lemma 2.9 Suppose that X is a Banach function space. Let the Hardy-Littlewood maximal operator M be weakly on X. Then, for all balls $B \subset \mathbb{R}^n$ and all measurable sets $E \subset B$,

$$\frac{|E|}{|B|} \le C \frac{\left\| \chi_E \right\|_X}{\left\| \chi_B \right\|_X}.$$

Next, we introduce the definition of the weighted Herz spaces with variable exponent. Suppose $k \in \mathbb{Z}$, we write $B_k = \left\{ |x| \le 2^k \right\}$, $D_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{D_k}$. The symbol $\mathbb{N}_0$ is the set of all non-negative integers. For $k \in \mathbb{N}_0$, we denote $\tilde{\chi}_k = \chi_{D_k}$, if $k \ge 1$, and $\tilde{\chi}_0 = \chi_{D_0}$.

Definition 2.10 Let $\alpha \in \mathbb{R}$, $p(\cdot) \in \mathrm{P}(\mathbb{R}^n)$, $0 < q < \infty$ and ω be a weight. The homogeneous Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)$ and non-homogeneous Herz spaces $K_{p(\cdot)}^{\alpha,q}(\omega)$ are defined, respectively, by

$$\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)=\left\{f\in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n\setminus\{0\},\omega^{1/p(\cdot)}):\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)}<\infty\right\}$$

and

$$K_{p(\cdot)}^{\alpha,q}(\omega)=\left\{f\in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n,\omega^{1/p(\cdot)}):\|f\|_{K_{p(\cdot)}^{\alpha,q}(\omega)}<\infty\right\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)}=\left(\sum_{k=-\infty}^{\infty}\left\|2^{\alpha k}f\chi_k\right\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q},$$

$$\|f\|_{K_{p(\cdot)}^{\alpha,q}(\omega)}=\left(\sum_{k=0}^{\infty}\left\|2^{\alpha k}f\tilde{\chi}_k\right\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}.$$

The following lemma is essential for our main result (Izuki [8]).

Lemma 2.11 Suppose that X is a Banach function space. Let the Hardy-Littlewood maximal operator M be weakly on X. Then

$$\sup_{B:\text{ball}}\frac{1}{|B|}\|\chi_B\|_X\,\|\chi_B\|_{X'}<\infty.$$

The following lemma can be found in (Cruz-Uribe and Wang [9]).

Lemma 2.12 Suppose that there exists a constant $1<p_0<\infty$, such that for every $\omega_0\in A_{p_0}$

$$\|f\|_{L^{p_0}(\omega_0)}\le C\|g\|_{L^{p_0}(\omega_0)},$$

Where, for all $f\in L^{p_0}(\omega_0)$ and all measurable functions g. Soppose $p(\cdot)\in \mathrm{P}(\mathbb{R}^n)$ and ω is a weight. If the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\omega)$ and on $L^{p'(\cdot)}(\omega^{-1/p(\cdot)-1})$. then, for all $f\in L^{p(\cdot)}(\omega)$ and all measurable functions g

$$\|f\|_{L^{p(\cdot)}(\omega)}\le C\|g\|_{L^{p(\cdot)}(\omega)}.$$

Proposition 2.13 Set $p(\cdot)\in \mathrm{P}(\mathbb{R}^n)\cap LH(\mathbb{R}^n)$ and $\omega\in A_{p(\cdot)}$. Then, the Marcinkiewicz integral associated with Schrödinger operator μ_j^L is bounded on $L^{p(\cdot)}(\omega)$.

Proof: It is easy to see that $\left(x_j-y_j\right)/|x-y|\in L^{\infty}(S^{n-1})$. Thus, for the classical Marcinkiewicz integral operator μ_j, by applying Lemma 2.3, we have μ_j are bounded on weighted Lebesgue space $L^p(\omega)$. By using Lemma 2.8 and Lemma 2.12, we obtain that μ_j are bounded on $L^{p(\cdot)}(\omega)$. Thanks to Lemma 2.4 we have that the Marcinkiewicz integral associated with Schrödinger operator is bounded on $L^{p(\cdot)}(\omega)$.

THE MAIN RESULT

In this section, we prove that the Marcinkiewicz integral associated with Schrödinger operator μ_j^L are bounded on $\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)$ and $K_{p(\cdot)}^{\alpha,q}(\omega)$.

Theorem 3.1 Suppose $p(\cdot)\in \mathrm{P}(\mathbb{R}^n)\cap LH(\mathbb{R}^n)$ and $0<q<\infty$. Let $1/p_-<r<1$, $\omega\in A_{rp(\cdot)}$ and $n\delta<\alpha<n(1-r)$, where $0<\delta<1$ is a constant, satisfying

$$\frac{\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(\omega)}}{\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}(\omega)}}\le C2^{\delta n(k-1)},$$

for all $k,l\in\mathbb{Z}$ and $k\le 1$. Then, μ_j^L are bounded on $\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)$ and $K_{p(\cdot)}^{\alpha,q}(\omega)$.

Proof: We only prove the homogeneous case. The non-homogeneous case can be proved in the similar way. We decompose $f\in\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)$ as

$$f=f\chi_{B_{k+1}\setminus B_{k-2}}+f\chi_{B_{k-2}}+f\chi_{\mathbb{R}^n\setminus B_{k+1}}.$$

Thus, we have that

$$\begin{aligned}\left\|\mu_j^L f\right\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)}\le C&\left\{\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left\|\mu_j^L(f\chi_{B_{k+1}\setminus B_{k-2}})\chi_k\right\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}\right.\\&+\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left\|\mu_j^L(f\chi_{B_{k-2}})\chi_k\right\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}\\&\left.+\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left\|\mu_j^L(f\chi_{\mathbb{R}^n\setminus B_{k-2}})\chi_k\right\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}\right\}\\&\le C(I_1+I_2+I_3).\end{aligned}$$

Firstly, we estimate I_1. Using the boundedness of μ_j^L on $L^{p(\cdot)}(\omega)$, we obtain

$$I_1\le C\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left\|(f\chi_{B_{k+1}\setminus B_{k-2}})\right\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}\le C\left\|f\right\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)}.$$

Secondly, we estimate I_2.

$$I_2\le\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{l=-\infty}^{k-2}\left\|\mu_j^L(f\chi_l)\chi_k\right\|_{L^{p(\cdot)}(\omega)}\right)^q\right)^{1/q}.$$

Thus, for $x\in\mathbb{Z}$, $l\le k-2$, $x\in D_k$ and Lemma 2.2, we deduce that $|x-y|\sim|x|$ and

$$|\mu_j^L(f\chi_l)|\le C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-1}}\chi_l(y)\left(\int_{|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\le C|x|^{-n}\left(\int_{D_l}|f(y)|\,dy\right).$$

Thus, for $x\in D_k$, by the generalized Hölder inequality and Lemma 2.11, we have that

$$\begin{aligned}|\mu_j^L(f\chi_l)|&\le C|x|^{-n}\left\|f\omega^{1/p(\cdot)}\chi_l\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\omega^{-1/p(\cdot)}\chi_l\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\le C|x|^{-n}\left\|f\omega^{1/p(\cdot)}\chi_l\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_l}\right\|_{\left(L^{p(\cdot)}(\omega)\right)'}\\&\le C\frac{|B_l|}{|B_k|}\left\|f\chi_l\right\|_{L^{p(\cdot)}(\omega)}\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}(\omega)}^{-1}.\end{aligned}$$

Then

$$I_2\le C\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{l=-\infty}^{k-2}\frac{|B_l|}{|B_k|}\frac{\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(\omega)}}{\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}(\omega)}}\left\|f\chi_l\right\|_{L^{p(\cdot)}(\omega)}\right)^q\right)^{1/q}.$$

For every $k,l\in\mathbb{Z}$, $k\ge l+2$ and $rp(\cdot)\in\mathrm{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$. Applying Lemma 2.9, we obtain

$$\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(\omega)}\Big/\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}(\omega)}=\left\|\chi_{B_k}\right\|_{L^{rp(\cdot)}(\omega)}\Big/\left\|\chi_{B_l}\right\|_{L^{rp(\cdot)}(\omega)}\le C\left(|B_k|/|B_l|\right)^r=C2^{(k-l)nr}.$$

Then we deduce that

$$I_2\le C\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{l=-\infty}^{k-2}2^{n(k-l)(r-1)}\left\|f\chi_l\right\|_{L^{p(\cdot)}(\omega)}\right)^q\right)^{1/q}.$$

When $0<q\le 1$, by the Jeason's inequality and $\alpha<n(1-r)$, we have that

$$I_2\le C\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\sum_{l=-\infty}^{k-2}2^{nq(k-l)(r-1)}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}$$

$$\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\sum_{k=l+2}^{\infty}2^{q(k-l)(\alpha-n(1-r))}\right)^{1/q}$$

$$\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}\le C\|f\|_{\dot K_{p(\cdot)}^{\alpha,q}(\omega)}.$$

When $1<q<\infty$, take $1<s<\infty$, such that $\alpha<n/s\cdot(1-r)$. Using the Hölder inequality, for every $k\in\mathbb{Z}$, we obtain

$$\left(\sum_{l=-\infty}^{k-2}2^{n(k-l)(r-1)}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}\right)^q$$

$$\le C\left(\sum_{l=-\infty}^{k-2}2^{\frac{nq}{s'}(k-l)(r-1)}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\right)\left(\sum_{l=-\infty}^{k-2}2^{\left(\frac{n}{s}(r-1)(k-l)-\alpha l\right)q'}\right)^{q/q'}$$

$$\le C\left(\sum_{l=-\infty}^{k-2}2^{\frac{nq}{s'}(k-l)(r-1)}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\right)\left(\sum_{l=-\infty}^{k-2}2^{\left(l\left(\frac{n}{s}(r-1)-\alpha\right)-k\frac{n}{s}(r-1)\right)q'}\right)^{q/q'}$$

$$\le C2^{-\alpha kq}\left(\sum_{l=-\infty}^{k-2}2^{\frac{nq}{s'}(k-l)(r-1)}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\right).$$

Thus, we have that

$$I_2\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\sum_{k=l+2}^{\infty}2^{\frac{nq}{s'}(k-l)(r-1)}\right)^{1/q}$$

$$\le C\left(\sum_{l=-\infty}^{\infty}2^{\alpha lq}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}^q\right)^{1/q}\le C\|f\|_{\dot K_{p(\cdot)}^{\alpha,q}(\omega)}.$$

Finally, to estimate I_3

$$I_3\le\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{l=k+2}^{\infty}\left\|\mu_j^L(f\chi_l)\chi_k\right\|_{L^{p(\cdot)}(\omega)}\right)^q\right)^{1/q}.$$

Take $k\in\mathbb{Z},\ l\ge k+2,\ x\in D_k$ and $x\in D_l$. Then

$$t\ge|x-y|\ge|x|-|y|\ge 2^{l-1}-2^k\ge 2^{l-2}.$$

Thus, thanks to Lemma 2.2, we have that

$$|\mu_j^L(f\chi_l)|\le C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-1}}\chi_l(y)\left(\int_{|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\le C|B_l|^{-1}\left(\int_{D_l}|f(y)|dy\right).$$

By the Hölder inequality and Lemma 2.11, we deduce that

$$|\mu_j^L(f\chi_l)|\le C|B_l|^{-1}\left\|f\omega^{1/p(\cdot)}\chi_l\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\omega^{-1/p(\cdot)}\chi_l\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C\|f\chi_l\|_{L^{p(\cdot)}(\omega)}\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}(\omega)}^{-1}.$$

Thus, we conclude that

$$I_3\le C\left(\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{l=k+2}^{\infty}\frac{\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(\omega)}}{\left\|\chi_{B_l}\right\|_{L^{p(\cdot)}(\omega)}}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}\right)^q\right)^{1/q}$$

$$\leq C\left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq}\left(\sum_{l=k+2}^{\infty} 2^{\delta n(k-l)}\|f\chi_l\|_{L^{p(\cdot)}(\omega)}\right)^q\right)^{1/q}.$$

When $0<q\leq 1,$ applying the Jeason's inequality and $-n\delta<\alpha,$ we obtain that

$$I_3 \leq C\left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq}\sum_{l=k+2}^{\infty} 2^{\delta nq(k-l)}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\right)^{1/q}$$

$$\leq C\left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\sum_{k=-\infty}^{l-2} 2^{q(\delta n+\alpha)(k-l)}\right)^{1/q}$$

$$\leq C\left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\right)^{1/q} \leq C\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\omega)}.$$

When $1<q<\infty.$ Using the Hölder inequality and $-n\delta<\alpha,$ we conclude that

$$I_3 \leq C\left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq}\left(\sum_{l=k+2}^{\infty} 2^{\alpha l}\|f\chi_l\|_{L^{p(\cdot)}(\omega)} 2^{\delta n(k-l)/\mu'}2^{-\alpha l}2^{\delta n(k-l)/\mu}\right)^q\right)^{1/q}$$

$$\leq C\left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq}\left(\sum_{l=k+2}^{\infty} 2^{\alpha lq}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)} 2^{q\delta n(k-l)/\mu'}\right)\left(\sum_{l=k+2}^{\infty} 2^{-\alpha lq'}2^{q'\delta n(k-l)/\mu}\right)^{q/q'}\right)^{1/q}$$

$$\leq C\left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\sum_{k=-\infty}^{l-2} 2^{q\delta n(k-l)/\mu'}\right)^{1/q}$$

$$\leq C\left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\right)^{1/q} \leq C\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\omega)}.$$

This completes the proof of Theorem3.1.

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