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# The Eigenvalues and the Eigenfunctions of the Sturm-Liouville Fuzzy Boundary Value Problem According To the Generalized Differentiability

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**Abstract:** In this paper, the eigenvalues and the eigenfunctions of the fuzzy Sturm-Liouville fuzzy boundary value problem equation is examined under the approach of generalized differentiability. Different examples are solved for these problems.
**Keywords:** Fuzzy boundary value problems, generalized differentiability, eigenvalue, eigenfunction

## 1. Introduction

Fuzzy differential equations are studied by many researchers. Fuzzy differential equations and fuzzy boundary value problems are one of the major applications of fuzzy number arithmetic. The first approach is the use of Hukuhara differentiability to solve fuzzy differential equations. This approach has a drawback: the solution becomes fuzzier as time goes by [2, 5]. The second approach is the generalized differentiability. The generalized differentiability was introduced [1] and studied in [2-4, 7, 10].

Gültekin Çitil and Altınışık [6] have defined the fuzzy Sturm-Liouville equation and they have examined eigenvalues and eigenfunctions of the fuzzy Sturm-Liouville problem under the approach of the Hukuhara differentiability.

In this paper, a investigation is made on the eigenvalues and the eigenfunctions of the fuzzy Sturm-Liouville problem by using generalized differentiability.

## 2. Preliminaries

**Definition 2.1.** [7] A fuzzy number is a function $u: \mathbb{R} \to [0,1]$ satisfying the following properties:
u is normal, convex fuzzy set, upper semi-continuous on $\mathbb{R}$ and $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

**Definition 2.2.** [8] Let $u \in \mathbb{R}_F$. The $\alpha$ -level set of u, denoted $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}$. If $\alpha$ =0, the support of u is defined $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$. The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the $\alpha$ -level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid $\alpha$ -level set.

**Remark 2.1.** [7] The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

**1)** $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

**2)** $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

**3)** $\underline{u}_\alpha \le \overline{u}_\alpha$, $0 \le \alpha \le 1$.

**Definition 2.3.** [9] If A is a symmetric triangular number with support $[\underline{a}, \overline{a}]$, the $\alpha$ -level set of A is

$$[A]^\alpha = \left[ \underline{a} + \left( \frac{\overline{a} - \underline{a}}{2} \right) \alpha, \overline{a} - \left( \frac{\overline{a} - \underline{a}}{2} \right) \alpha \right].$$

**Definition 2.4.** [8] For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda u$ are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max \left\{ \left| \underline{u}_\alpha - \underline{v}_\alpha \right|, \left| \overline{u}_\alpha - \overline{v}_\alpha \right| \right\} \text{ [7].}$$

**Definition 2.5.** [9] Let $u, v \in \mathbb{R}_F$. If there exist $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted $u \underset{H}{-} v$.

**Definition 2.6** [8] Let $f: [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$. We say that f is (1)-differentiable at $t_0$, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small near to 0, exist $f(t_0 + h) - f(t_0)$, $f(t_0) - f(t_0 - h)$ and the limits

$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) - f(t_0 - h)}{h} = f'(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small near to 0, exist $f(t_0) - f(t_0 + h)$, $f(t_0 - h) - f(t_0)$ and the limits

$$\lim_{h \to 0} \frac{f(t_0) - f(t_0 + h)}{-h} = \lim_{h \to 0} \frac{f(t_0 - h) - f(t_0)}{-h} = f'(t_0).$$

**Theorem 2.1.** [7] Let $f: [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[ \underline{f}_\alpha(t), \overline{f}_\alpha(t) \right]$, for each $\alpha \in [0,1]$.

$(i)$ If f is (1)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[ f'(t) \right]^\alpha = \left[ \underline{f}'_\alpha(t), \overline{f}'_\alpha(t) \right]$,

$(ii)$ If f is (2)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[ f'(t) \right]^\alpha = \left[ \overline{f}'_\alpha(t), \underline{f}'_\alpha(t) \right]$.

**Theorem 2.2.** [7] Let $f': [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[ \underline{f}_\alpha(t), \overline{f}_\alpha(t) \right]$, for each $\alpha \in [0,1]$, f is (1)-differentiable or (2)-differentiable.

$(i)$ If f and $f'$ are (1)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and

$$\left[ f''(t) \right]^\alpha = \left[ \underline{f}''_\alpha(t), \overline{f}''_\alpha(t) \right],$$

$(ii)$ If f is (1)-differentiable and $f'$ is (2)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and

$$\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right],$$

$(iii)$ If f is (2)-differentiable and $f'$ is (1)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and

$$\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right],$$

$(iv)$ If f and $f'$ are (2)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and

$$\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right].$$

**3. The Eigenvalues and The Eigenfunctions of The Sturm-Liouville Fuzzy Boundary Value Problem According to The Generalized Differentiability**

Consider the eigenvalues and the eigenfunctions of the fuzzy boundary value problem

$$Ly = p(x)y'' + q(x)y$$

$$Ly + \lambda y = 0,\ x \in (a,b), \tag{3.1}$$

$$B_1(y) := Ay(a) + By'(a) = 0, \tag{3.2}$$

$$B_2(y) := Cy(b) + Dy'(b) = 0 \tag{3.3}$$

using the generalized differentiability, where $p(x)$, $q(x)$ are continuous functions and are positive on $[a,b]$, $p'(x) = 0$, $\lambda > 0$, $A, B, C, D \geq 0$, $A^2 + B^2 \neq 0$ and $C^2 + D^2 \neq 0$. Here, (i,j) solution means that y is (i) differentiable and $y'$ is (j) differentiable, i,j=1,2.

For (1,1) solution,

$$p(x)\left[\underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right] + q(x)\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] + \lambda\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] = 0,$$
$$A\left[\underline{y}_\alpha(a), \overline{y}_\alpha(a)\right] + B\left[\underline{y}'_\alpha(a), \overline{y}'_\alpha(a)\right] = 0,$$
$$C\left[\underline{y}_\alpha(b), \overline{y}_\alpha(b)\right] + D\left[\underline{y}'_\alpha(b), \overline{y}'_\alpha(b)\right] = 0,$$

for (1,2) solution,

$$p(x)\left[\overline{y}''_\alpha(x), \underline{y}''_\alpha(x)\right] + q(x)\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] + \lambda\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] = 0,$$
$$A\left[\underline{y}_\alpha(a), \overline{y}_\alpha(a)\right] + B\left[\underline{y}'_\alpha(a), \overline{y}'_\alpha(a)\right] = 0,$$
$$C\left[\underline{y}_\alpha(b), \overline{y}_\alpha(b)\right] + D\left[\underline{y}'_\alpha(b), \overline{y}'_\alpha(b)\right] = 0,$$

for (2,2) solution;

$$p(x)\left[\underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right] + q(x)\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] + \lambda\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] = 0,$$
$$A\left[\underline{y}_\alpha(a), \overline{y}_\alpha(a)\right] + B\left[\overline{y}'_\alpha(a), \underline{y}'_\alpha(a)\right] = 0,$$
$$C\left[\underline{y}_\alpha(b), \overline{y}_\alpha(b)\right] + D\left[, \overline{y}'_\alpha(b), \underline{y}'_\alpha(b)\right] = 0,$$

for (2,1) solution;

$$p(x)\left[\overline{y}''_\alpha(x), \underline{y}''_\alpha(x)\right] + q(x)\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] + \lambda\left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right] = 0,$$

$$A\left[\underline{y}_{\alpha}(a),\overline{y}_{\alpha}(a)\right]+B\left[\overline{y}'_{\alpha}(a),\underline{y}'_{\alpha}(a)\right]=0,$$
$$C\left[\underline{y}_{\alpha}(b),\overline{y}_{\alpha}(b)\right]+D\left[,\overline{y}'_{\alpha}(b),\underline{y}'_{\alpha}(b)\right]=0,$$

are solved.

**Example 3.1.** Consider the fuzzy Sturm-Liouville problem

$$y''+\lambda y=0,\quad y(0)=0,\quad y(1)=0\ . \tag{3.4}$$

Let be $\lambda=k^2,\ k>0$ . For (1,1) and (2,2) solution,

$$\left[\underline{y}''_{\alpha},\overline{y}''_{\alpha}\right]+\lambda\left[\underline{y}_{\alpha},\overline{y}_{\alpha}\right]=0$$
$$\left[\underline{y}_{\alpha}(0),\overline{y}_{\alpha}(0)\right]=0,\ \left[\underline{y}_{\alpha}(1),\overline{y}_{\alpha}(1)\right]=0$$

is solved. Then, the solution of fuzzy differential equation is

$$\underline{y}_{\alpha}(x)=c_1(\alpha)cos(kx)+c_2(\alpha)sin(kx), \tag{3.5}$$
$$\overline{y}_{\alpha}(x)=c_3(\alpha)cos(kx)+c_4(\alpha)sin(kx), \tag{3.6}$$
$$[y(x)]^{\alpha}=\left[\underline{y}_{\alpha}(x),\overline{y}_{\alpha}(x)\right]. \tag{3.7}$$

From the first boundary condition $c_1(\alpha)$=$c_3(\alpha)=0$, from the second boundary condition, $c_2(\alpha)sin(k)=0$, $c_4(\alpha)sin(k)=0$ are obtained. Then,

$$c_2(\alpha)\neq 0,\ c_4(\alpha)\neq 0,\ \ sin(k)=0\Rightarrow k_n=n\pi, n=1,2,...$$
$$[y(x)]^{\alpha}=[c_2(\alpha)sin(n\pi x),c_4(\alpha)sin(n\pi x)].$$

As $\frac{\partial(c_2(\alpha)sin(n\pi x))}{\partial\alpha}\geq 0$, $\frac{\partial(c_4(\alpha)sin(n\pi x))}{\partial\alpha}\leq 0$ and $c_2(\alpha)sin(n\pi x)\leq c_4(\alpha)sin(n\pi x)$ $[y(x)]^{\alpha}$ is a valid $\alpha-$ level set. Let be $n\pi x\in\left[(n-1)\pi,n\pi\right]$, $n=1,2,\ldots$ .

**i)** If n is single, $\sin(n\pi x)\geq 0$ . Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\leq 0$ and $c_2(\alpha)\leq c_4(\alpha)$, $[y(x)]^{\alpha}$ is a valid $\alpha$-level set.

**ii)** If n is double, $\sin(n\pi x)\leq 0$ . Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\leq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\geq 0$ and $c_2(\alpha)\geq c_4(\alpha)$, $[y(x)]^{\alpha}$ is a valid $\alpha$-level set.

Consequently; $n\pi x\in\left[(n-1)\pi,n\pi\right]$, $n=1,2,\ldots$

**i)** If n is single, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\leq 0$ and $c_2(\alpha)\leq c_4(\alpha)$, the eigenvalues are $\lambda_n=n^2\pi^2$, with associated eigenfunctions

$$[y_n(x)]^{\alpha}=[c_2(\alpha)sin(n\pi x),c_4(\alpha)sin(n\pi x)],$$

**ii)** If n is double, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\leq 0$ , $\frac{\partial(c_4(\alpha))}{\partial\alpha}\geq 0$ and $c_2(\alpha)\geq c_4(\alpha)$, the eigenvalues are $\lambda_n=n^2\pi^2$, with associated eigenfunctions

$$[y_n(x)]^{\alpha}=[c_2(\alpha)sin(n\pi x),c_4(\alpha)sin(n\pi x)],$$

**iii)** If $\alpha=1$, the eigenvalues are $\lambda_n=n^2\pi^2$, with associated eigenfunctions

$$\left[y_n(x)\right]^{\alpha}=\sin(n\pi x).$$

For (1,2) and (2,1) solution,

$$\left[\overline{y}''_{\alpha},\underline{y}''_{\alpha},\right]+\lambda\left[\underline{y}_{\alpha},\overline{y}_{\alpha}\right]=0$$
$$\left[\underline{y}_{\alpha}(0),\overline{y}_{\alpha}(0)\right]=0,\ \left[\underline{y}_{\alpha}(1),\overline{y}_{\alpha}(1)\right]=0$$

is solved. Then, the solution of fuzzy differential equation is

$$\underline{y}_{\alpha}(x)=-a_1(\alpha)e^{kx}-a_2(\alpha)e^{-kx}+a_3(\alpha)sin(kx)+a_4(\alpha)cos(kx), \tag{3.8}$$
$$\overline{y}_{\alpha}(x)=a_1(\alpha)e^{kx}+a_2(\alpha)e^{-kx}+a_3(\alpha)sin(kx)+a_4(\alpha)cos(kx), \tag{3.9}$$

$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right]. \tag{3.10}$$

From the boundary conditions $a_1(\alpha) = a_2(\alpha) = a_4(\alpha) = 0$, $a_3(\alpha)sin(k) = 0$ are obtained.
From here,

$$a_3(\alpha) \neq 0,\ sin(k) = 0 \Rightarrow k_n = n\pi, n = 1,2, ...$$

The eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = sin(n\pi x).$$

**Example 3.2.** If we take

$$y(0) = 0, \quad y'(1) = 0 \tag{3.11}$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

$$\left[\underline{y}''_\alpha, \overline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}_\alpha(0), \overline{y}_\alpha(0)\right] = 0,\ \left[\underline{y}'_\alpha(1), \overline{y}'_\alpha(1)\right] = 0$$

is solved. Then, the solution of fuzzy differential equation is (3.5)-(3.7). From the first boundary condition $c_1(\alpha)$=$c_3(\alpha) = 0$, from the second boundary condition,

$$kc_2(\alpha)cos(k) = 0,\ kc_4(\alpha)cos(k) = 0$$

are obtained.
Then, $c_2(\alpha) \neq 0$, $c_4(\alpha) \neq 0$, $cos(k) = 0 \Rightarrow k_n = \frac{(2n-1)}{2}\pi, n = 1,2, ...$

$$[y(x)]^\alpha = \left[c_2(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right), c_4(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right].$$

As

$$\frac{\partial\left(c_2(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right)}{\partial\alpha} \geq 0, \qquad \frac{\partial\left(c_4(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right)}{\partial\alpha} \leq 0$$

and

$$c_2(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right) \leq c_4(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)$$

$[y(x)]^\alpha$ is a valid $\alpha$ – level set. Let be $\left(\frac{2n-1}{2}\right)\pi x \in [(n-1)\pi, n\pi], n = 1,2, ...$

**i)** If n is single, $sin\left(\left(\frac{2n-1}{2}\right)\pi x\right) \geq 0$. Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \leq 0$ and $c_2(\alpha) \leq c_4(\alpha)$, $[y(x)]^\alpha$ is a valid $\alpha$-level set.

**ii)** If n is double, $sin\left(\left(\frac{2n-1}{2}\right)\pi x\right) \leq 0$. Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \leq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \geq 0$ and $c_2(\alpha) \geq c_4(\alpha)$, $[y(x)]^\alpha$ is a valid $\alpha$-level set.

Consequently; $\left(\frac{2n-1}{2}\right)\pi x \in \left[(n-1)\pi, n\pi\right]$, $n=1,2,...$

**i)** If n is single, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \leq 0$ and $c_2(\alpha) \leq c_4(\alpha)$, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$[y(x)]^\alpha = \left[c_2(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right), c_4(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right],$$

**ii)** If n is double, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \leq 0$,

$\frac{\partial(c_4(\alpha))}{\partial\alpha} \geq 0$ and $c_2(\alpha) \geq c_4(\alpha)$, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$[y(x)]^\alpha = \left[c_2(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right), c_4(\alpha)sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right],$$

**iii)** If $\alpha = 1$, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$\left[y_n(x)\right]^\alpha = \sin\left(\left(\frac{2n-1}{2}\right)\pi x\right).$$

For (2,2) solution,

$$\left[\underline{y}''_\alpha, \overline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}_\alpha(0), \overline{y}_\alpha(0)\right] = 0, \left[\overline{y}'_\alpha(1), \underline{y}'_\alpha(1)\right] = 0$$

is solved. Eigenvalues and eigenfunctions are same with (1,1) solution.
For (1,2) solution,

$$\left[\overline{y}''_\alpha, \underline{y}''_\alpha,\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}_\alpha(0), \overline{y}_\alpha(0)\right] = 0, \left[\underline{y}'_\alpha(1), \overline{y}'_\alpha(1)\right] = 0$$

is solved. Then, the solution of fuzzy differential equation is (3.8)-(3.10). From the boundary conditions $a_1(\alpha) = a_2(\alpha) = a_4(\alpha) = 0$, $a_3(\alpha)cos(k) = 0$,

$$a_3(\alpha) \neq 0, cos(k) = 0 \Rightarrow k_n = \left(\frac{2n-1}{2}\right)\pi, n = 1,2, \ldots$$

are obtained. The eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$\left[y_n(x)\right]^\alpha = \sin\left(\left(\frac{2n-1}{2}\right)\pi x\right).$$

For (2,1) solution,

$$\left[\overline{y}''_\alpha, \underline{y}''_\alpha,\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}_\alpha(0), \overline{y}_\alpha(0)\right] = 0, \left[\overline{y}'_\alpha(1), \underline{y}'_\alpha(1)\right] = 0$$

is solved. Eigenvalues and eigenfunctions are same with (1,2) solution.

**Example 3.3.** If we take

$$y'(0) = 0, \quad y(1) = 0 \tag{3.12}$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

$$\left[\underline{y}''_\alpha, \overline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}'_\alpha(0), \overline{y}'_\alpha(0)\right] = 0, \left[\underline{y}_\alpha(1), \overline{y}_\alpha(1)\right] = 0$$

is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the first boundary condition $c_2(\alpha)$=$c_4(\alpha) = 0$, from the second boundary condition,

$$\underline{y}_\alpha(1) = c_1(\alpha)cos(k) = 0, \overline{y}_\alpha(1) = c_3(\alpha)cos(k) = 0$$

are obtained.
Then, $c_1(\alpha) \neq 0$, $c_3(\alpha) \neq 0$, $cos(k) = 0 \Rightarrow k_n = \frac{(2n-1)}{2}\pi, n = 1,2, \ldots$

$$[y(x)]^\alpha = \left[c_1(\alpha)cos\left(\left(\frac{2n-1}{2}\right)\pi x\right), c_3(\alpha)cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right].$$

As

$$\frac{\partial\left(c_1(\alpha)cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right)}{\partial\alpha}\geq 0,\qquad \frac{\partial\left(c_3(\alpha)cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)\right)}{\partial\alpha}\leq 0$$

and

$$c_1(\alpha)cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)\leq c_3(\alpha)cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)$$

$[y(x)]^\alpha$ is a valid $\alpha-$ level set.

**i)** If $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-1)-1)}{2}\pi,\frac{(2(n-1)+1)}{2}\pi\right)$, $n=1,3,5,\dots$, $cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)>0$. Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\leq 0$ and $c_2(\alpha)\leq c_4(\alpha)$, $[y(x)]^\alpha$ is a valid $\alpha$-level set.

**ii)** If $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-2)+1)}{2}\pi,\frac{(2(n-2)+3)}{2}\pi\right)$, $n=2,4,6,\dots$ , $cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)<0$. Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\leq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\geq 0$ and $c_2(\alpha)\geq c_4(\alpha)$, $[y(x)]^\alpha$ is a valid $\alpha$-level set.

Consequently;

**i)** If $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-1)-1)}{2}\pi,\frac{(2(n-1)+1)}{2}\pi\right)$, $n=1,3,5,\dots$, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\leq 0$ and $c_2(\alpha)\leq c_4(\alpha)$, the eigenvalues are $\lambda_n=\frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$[y(x)]^\alpha=\left[c_1(\alpha)cos\left(\left(\tfrac{2n-1}{2}\right)\pi x\right),c_3(\alpha)cos\left(\left(\tfrac{2n-1}{2}\right)\pi x\right)\right],$$

**ii)** $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-2)+1)}{2}\pi,\frac{(2(n-2)+3)}{2}\pi\right)$, $n=2,4,6,\dots$ for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha}\leq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha}\geq 0$ and $c_2(\alpha)\geq c_4(\alpha)$, the eigenvalues are $\lambda_n=\frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$[y(x)]^\alpha=\left[c_1(\alpha)cos\left(\left(\tfrac{2n-1}{2}\right)\pi x\right),c_3(\alpha)cos\left(\left(\tfrac{2n-1}{2}\right)\pi x\right)\right],$$

**iii)** If $\alpha=1$, the eigenvalues are $\lambda_n=\frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha=cos\left(\left(\frac{2n-1}{2}\right)\pi x\right).$$

For (2,2) solution,

$$\left[\underline{y}''_\alpha,\overline{y}''_\alpha\right]+\lambda\left[\underline{y}_\alpha,\overline{y}_\alpha\right]=0,$$
$$\left[\overline{y}'_\alpha(0),\underline{y}'_\alpha(0)\right]=0,\left[\underline{y}_\alpha(1),\overline{y}_\alpha(1)\right]=0$$

is solved. Eigenvalues and eigenfunctions are same with (1,1) solution.

For (1,2) solution,

$$\left[\overline{y}''_\alpha,\underline{y}''_\alpha,\right]+\lambda\left[\underline{y}_\alpha,\overline{y}_\alpha\right]=0$$
$$\left[\underline{y}'_\alpha(0),\overline{y}'_\alpha(0)\right]=0\left[\underline{y}_\alpha(1),\overline{y}_\alpha(1)\right]=0$$

is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions $a_1(\alpha) = a_2(\alpha) = a_3(\alpha) = 0$, $a_4(\alpha)cos(k) = 0$ are obtained. From here,

$$a_4(\alpha) \neq 0, cos(k) = 0 \Rightarrow k_n = \left(\frac{2n-1}{2}\right)\pi, n = 1,2, ...$$

The eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \cos\left(\left(\frac{2n-1}{2}\right)\pi x\right).$$

For (2,1) solution,

$$\left[\overline{y}''_\alpha, \underline{y}''_\alpha,\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\overline{y}'_\alpha(0), \underline{y}'_\alpha(0)\right] = 0 \left[\underline{y}_\alpha(1), \overline{y}_\alpha(1)\right] = 0,$$

Eigenvalues and eigenfunctions are same with (1,2) solution.

**Example 3.4.** If we take

$$y'(0) = 0, \quad y'(1) = 0 \tag{3.13}$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

$$\left[\underline{y}''_\alpha, \overline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\underline{y}'_\alpha(0), \overline{y}'_\alpha(0)\right] = 0, \ \left[\underline{y}'_\alpha(1), \overline{y}'_\alpha(1)\right] = 0$$

is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the first boundary condition $c_2(\alpha)$=$c_4(\alpha) = 0$, from the second boundary condition,

$$\underline{y}'_\alpha(1) = -kc_1(\alpha)sin(k) = 0, \overline{y}'_\alpha(1) = -kc_3(\alpha)sin(k) = 0$$

are obtained.

Then, $c_1(\alpha) \neq 0$, $c_3(\alpha) \neq 0$, $sin(k) = 0 \Rightarrow k_n = n\pi, n = 1,2, ...$

$$[y(x)]^\alpha = [c_1(\alpha)cos(n\pi x), c_3(\alpha)cos(n\pi x)].$$

As

$$\frac{\partial\left(c_1(\alpha)cos(n\pi x)\right)}{\partial\alpha} \geq 0, \qquad \frac{\partial\left(c_3(\alpha)cos(n\pi x)\right)}{\partial\alpha} \leq 0$$

and

$$c_1(\alpha)cos(n\pi x) \leq c_3(\alpha)cos(n\pi x)$$

$[y(x)]^\alpha$ is a valid $\alpha$-level set.

**i)** If $n\pi x \in \left(\frac{(2(n-2)-1)}{2}\pi, \frac{(2(n-2)+1)}{2}\pi\right)$, $n=2,4,6,...$, $\cos(n\pi x) > 0$. Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \leq 0$ and $c_2(\alpha) \leq c_4(\alpha)$, $[y(x)]^\alpha$ is a valid $\alpha$-level set.

**ii)** If $n\pi x \in \left(\frac{(2(n-1)+1)}{2}\pi, \frac{(2(n-1)+3)}{2}\pi\right)$, $n=1,3,5,...$, $\cos(n\pi x) < 0$. Then, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \leq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \geq 0$ and $c_2(\alpha) \geq c_4(\alpha)$, $[y(x)]^\alpha$ is a valid $\alpha$-level set.

Consequently;

i) If $n\pi x \in \left(\frac{(2(n-2)-1)}{2}\pi, \frac{(2(n-2)+1)}{2}\pi\right)$, $n=2,4,6,...$, for $c_2(\alpha)$, $c_4(\alpha)$ satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \geq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \leq 0$ and $c_2(\alpha) \leq c_4(\alpha)$, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfunctions

$$[y(x)]^\alpha = [c_1(\alpha)cos(n\pi x), c_3(\alpha)cos(n\pi x)].$$

ii) If $n\pi x \in \left(\frac{(2(n-1)+1)}{2}\pi, \frac{(2(n-1)+3)}{2}\pi\right)$, $n=1,3,5,\ldots$, for $c_2(\alpha)$, $c_4(\alpha)$

satisfying the inequality $\frac{\partial(c_2(\alpha))}{\partial\alpha} \leq 0$, $\frac{\partial(c_4(\alpha))}{\partial\alpha} \geq 0$ and $c_2(\alpha) \geq c_4(\alpha)$, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfunctions

$$[y(x)]^\alpha = [c_1(\alpha)cos(n\pi x), c_3(\alpha)cos(n\pi x)].$$

iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \cos(n\pi x).$$

For (2,2) solution,

$$\left[\underline{y}''_\alpha, \overline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\overline{y}'_\alpha(0), \underline{y}'_\alpha(0)\right] = 0,\ \left[\overline{y}'_\alpha(1), \underline{y}'_\alpha(1)\right] = 0,$$

Eigenvalues and eigenfunctions are same with (1,1) solution.
For (1,2) solution,

$$\left[\overline{y}''_\alpha, \underline{y}''_\alpha,\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\underline{y}'_\alpha(0), \overline{y}'_\alpha(0)\right] = 0, \qquad \left[\underline{y}'_\alpha(1), \overline{y}'_\alpha(1)\right] = 0$$

is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions $a_1(\alpha) = a_2(\alpha) = a_3(\alpha) = 0$, $a_4(\alpha)ksin(k) = 0$,

$$a_4(\alpha) \neq 0,\ sin(k) = 0 \Rightarrow k_n = n\pi, n = 1,2,\ldots$$

are obtained. The eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \cos(n\pi x).$$

For (2,1) solution,

$$\left[\overline{y}''_\alpha, \underline{y}''_\alpha,\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\overline{y}'_\alpha(0), \underline{y}'_\alpha(0)\right] = 0,\ \left[\overline{y}'_\alpha(1), \underline{y}'_\alpha(1)\right] = 0,$$

Eigenvalues and eigenfunctions are same with (1,2) solution.

**Example 3.5.** If we take

$$y(0) + y'(0) = 0, \quad y(1) = 0 \tag{3.14}$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

$$\left[\underline{y}''_\alpha, \overline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\underline{y}_\alpha(0) + \underline{y}'_\alpha(0), \overline{y}_\alpha(0) + \overline{y}'_\alpha(0)\right] = 0,\ \left[\underline{y}_\alpha(1), \overline{y}_\alpha(1)\right] = 0$$

is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the boundary conditions,

$$c_1(\alpha) + kc_2(\alpha) = 0, \qquad c_1(\alpha)cos(k) + c_2(\alpha)sin(k) = 0$$

$$c_3(\alpha) + kc_4(\alpha) = 0, \qquad c_3(\alpha)cos(k) + c_4(\alpha)sin(k) = 0$$

are obtained. If

$$\begin{vmatrix} 1 & k \\ cos(k) & sim(k) \end{vmatrix} = 0,$$

there is the nontrival solution of the fuzzy differential equation in (3.4) with the boundary conditions (3.14).

$$\Longrightarrow sin(k) - kcos(k) = 0 \Longrightarrow \tan(k) = k$$

$$k_1 = 2.63008 \times 10^8,\ k_2 = 4.49341,\ k_3 = 7.72525,\ k_4 = 10.9041, \dots$$
$$\underline{y}_\alpha(x) = c_1(\alpha)cos(k_n x) + c_2(\alpha)sin(k_n x), \qquad \overline{y}_\alpha(x) = c_3(\alpha)cos(k_n x) + c_4(\alpha)sin(k_n x)$$
$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right].$$

For $c_1(\alpha),\ c_2(\alpha),\ c_3(\alpha), c_4(\alpha)$ satisfying the inequality $\frac{\partial\left(\underline{y}_\alpha(x)\right)}{\partial\alpha} \geq 0$, $\frac{\partial\left(\overline{y}_\alpha(x)\right)}{\partial\alpha} \leq 0$ and $\underline{y}_\alpha(x) \leq \overline{y}_\alpha(x)$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions
$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right].$$

For (1,2) solution,
$$\left[\overline{y}_\alpha'', \underline{y}_\alpha''\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}_\alpha(0) + \underline{y}_\alpha'(0), \overline{y}_\alpha(0) + \overline{y}_\alpha'(0)\right] = 0,\ \left[\underline{y}_\alpha(1), \overline{y}_\alpha(1)\right] = 0$$
is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions, we have
$$a_3(\alpha) + ka_4(\alpha) = 0, \qquad (1+k)a_1(\alpha) + (1-k)a_2(\alpha) = 0,$$
$$a_3(\alpha)cos(k) + a_4(\alpha)sin(k) = 0, \quad a_1(\alpha)e^k + a_2(\alpha)e^{-k} = 0.$$
Thus, it must be
$$sin(k) - kcos(k) = 0 \text{ or } e^{2k} = \frac{1+k}{1-k}.$$
If $sin(k) - kcos(k) = 0$,
$$k_1 = 2.63008 \times 10^8,\ k_2 = 4.49341,\ k_3 = 7.72525,\ k_4 = 10.9041, \dots$$
and if $e^{2k} = \frac{1+k}{1-k}$
$$k_1 = 6.04989 \times 10^{-6},\ k_2 = 5.05302 \times 10^{-6},\ k_3 = 7.73022 \times 10^{-6}, \dots$$
are obtained. So, if $sin(k) - kcos(k) = 0$,
$$\underline{y}_\alpha(x) = \overline{y}_\alpha(x) = a_3(\alpha)sin(k_n x) + a_4(\alpha)cos(k_n x),$$
and if $e^{2k} = \frac{1+k}{1-k}$,
$$\underline{y}_\alpha(x) = -a_1(\alpha)e^{k_n x} - a_2(\alpha)e^{-k_n x},\ \overline{y}_\alpha(x) = a_1(\alpha)e^{k_n x} + a_2(\alpha)e^{-k_n x}.$$
$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right].$$
Consequently,

**i)** if $sin(k) - kcos(k) = 0$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions
$$y_n(x) = a_3(\alpha)sin(k_n x) + a_4(\alpha)cos(k_n x),$$

**ii)** if $e^{2k} = \frac{1+k}{1-k}$, for $a_1(\alpha),\ a_2(\alpha)$ satisfying the inequality $\frac{\partial\left(\underline{y}_\alpha(x)\right)}{\partial\alpha} \geq 0$, $\frac{\partial\left(\overline{y}_\alpha(x)\right)}{\partial\alpha} \leq 0$ and $\underline{y}_\alpha(x) \leq \overline{y}_\alpha(x)$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions
$$\underline{y}_\alpha(x) = -a_1(\alpha)e^{k_n x} - a_2(\alpha)e^{-k_n x},\ \overline{y}_\alpha(x) = a_1(\alpha)e^{k_n x} + a_2(\alpha)e^{-k_n x},$$
$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right].$$

For (2,2) solution,
$$\left[\underline{y}_\alpha'', \overline{y}_\alpha''\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$
$$\left[\underline{y}_\alpha(0) + \overline{y}_\alpha'(0), \overline{y}_\alpha(0) + \underline{y}_\alpha'(0)\right] = 0,\ \left[\underline{y}_\alpha(1), \overline{y}_\alpha(1)\right] = 0$$
is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the boundary conditions, we have
$$\frac{\sin(k)}{k}c_1(\alpha) + kcos(k)c_2(\alpha) = 0, \quad \frac{\sin(k)}{k}c_3(\alpha) + kcos(k)c_4(\alpha) = 0,$$
$$cos(k)c_1(\alpha) + sin(k)c_2(\alpha) = 0, \quad cos(k)c_3(\alpha) + sin(k)c_4(\alpha) = 0$$

So, from these equation systems, it must be

$$\frac{sin^2(k)}{k} - kcos^2(k) = 0 \Rightarrow tan^2(k) = k^2.$$

$$k_1 = 3.51833 \times 10^{-8},\ k_2 = 2.02876,\ k_3 = 4.91318, \ldots$$

Thus, for $c_1(\alpha),\ c_2(\alpha), c_3(\alpha), c_4(\alpha)$ satisfying the inequality $\frac{\partial\left(\underline{y}_{n\alpha}(x)\right)}{\partial\alpha} \geq 0,\ \frac{\partial\left(\overline{y}_{n\alpha}(x)\right)}{\partial\alpha} \leq 0$ and $\underline{y}_{n\alpha}(x) \leq \overline{y}_{n\alpha}(x)$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$\underline{y}_{n\alpha}(x) = c_1(\alpha)cos(k_nx) + c_2(\alpha)sin(k_nx), \overline{y}_{n\alpha}(x) = c_3(\alpha)cos(k_nx) + c_4(\alpha)sin(k_nx)$

$$[y_n(x)]^\alpha = \left[\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)\right].$$

For (2,1) solution,

$$\left[\overline{y}''_\alpha, \underline{y}''_\alpha\right] + \lambda\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = 0$$

$$\left[\underline{y}_\alpha(0) + \overline{y}'_\alpha(0), \overline{y}_\alpha(0) + \underline{y}'_\alpha(0)\right] = 0,\ \left[\underline{y}_\alpha(1), \overline{y}_\alpha(1)\right] = 0$$

is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions, we have

$$a_3(\alpha) + ka_4(\alpha) = 0, \qquad (1-k)a_1(\alpha) + (1+k)a_2(\alpha) = 0,$$

$$a_3(\alpha)cos(k) + a_4(\alpha)sin(k) = 0, \qquad a_1(\alpha)e^k + a_2(\alpha)e^{-k} = 0.$$

Thus, it must be

$$sin(k) - kcos(k) = 0 \text{ or } e^{2k} = \frac{1-k}{1+k}.$$

If $sin(k) - kcos(k) = 0$,

$$k_1 = 2.63008 \times 10^{8},\ k_2 = 4.49341,\ k_3 = 7.72525,\ k_4 = 10.9041, \ldots$$

and if $e^{2k} = \frac{1-k}{1+k}$

$$k_1 = 3.98993 \times 10^{-17},\ k_2 = 2.5154 \times 10^{-17},\ k_3 = 9.54151 \times 10^{-18}, \ldots$$

are obtained. So, if $sin(k) - kcos(k) = 0$,

$$\underline{y}_{n\alpha}(x) = \overline{y}_{n\alpha}(x) = a_3(\alpha)sin(k_nx) + a_4(\alpha)cos(k_nx),$$

and if $e^{2k} = \frac{1+k}{1-k}$,

$$\underline{y}_{n\alpha}(x) = -a_1(\alpha)e^{k_nx} - a_2(\alpha)e^{-k_nx}, \overline{y}_{n\alpha}(x) = a_1(\alpha)e^{k_nx} + a_2(\alpha)e^{-k_nx}.$$

$$[y_n(x)]^\alpha = \left[\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)\right].$$

Consequently,

**i)** if $sin(k) - kcos(k) = 0$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$y_n(x) = a_3(\alpha)sin(k_nx) + a_4(\alpha)cos(k_nx),$$

**ii)** if $e^{2k} = \frac{1-k}{1+k}$, for $a_1(\alpha),\ a_2(\alpha)$ satisfying the inequality $\frac{\partial\left(\underline{y}_{n\alpha}(x)\right)}{\partial\alpha} \geq 0$, $\frac{\partial\left(\overline{y}_{n\alpha}(x)\right)}{\partial\alpha} \leq 0$ and $\underline{y}_{n\alpha}(x) \leq \overline{y}_{n\alpha}(x)$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$\underline{y}_{n\alpha}(x) = -a_1(\alpha)e^{k_nx} - a_2(\alpha)e^{-k_nx}, \overline{y}_{n\alpha}(x) = a_1(\alpha)e^{k_nx} + a_2(\alpha)e^{-k_nx},$$

$$[y_n(x)]^\alpha = \left[\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)\right].$$

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