

## Common Fixed Point Results for Weakly Compatible Map in Digital Metric Spaces

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### Article History

Received: 14.11.2017

Accepted: 05.12.2017

Published: 30.12.2017

### DOI:

10.21276/sjpms.2017.4.4.7



**Abstract:** This paper aims at proving fixed point results for weakly compatible maps in the setting of digital metric spaces. Also, an application and conclusion is cited in the end of this note.

**Keywords:** Weakly compatible maps, fixed point results, digital metric space, metric space

**Subject Classification:** 47H10, 54H25

### INTRODUCTION

Digital topology is an emerging area based on general topology and functional analysis and focuses on studying digital topological properties of  $n$  – dimensional digital spaces, where as Euclidean topology deals with topological properties of subspaces of the  $n$  – dimensional real space, which has contributed to the study of some areas of computer sciences such as computer graphics, Image processing, approximation theory, mathematical morphology, optimization theory etc. Rosenfield [17] was the first to consider digital topology as a tool to study digital images. Boxer [4], then introduced the digital fundamental group of a discrete object and produced the digital versions of the topological concepts [2], also later studied digital continuous functions [3]. Ege and Karaca [7,8,9] established relative and reduced Lefschetz fixed point theorem for digital images and proposed the notion of a digital metric space and proved the famous Banach Contraction Principle for digital images.

Fixed point theory extends a lots of applications in mathematics, computer science, engineering, game theory, fuzzy theory, image processing and so forth [4, 7, 16]. In metric spaces, this theory begins with the Banach fixed-point theorem which provides a constructive method of finding fixed points and an essential tool for solution of some problems in mathematics and engineering and consequently has been generalized in many ways. Up to now, several developments have contributed [7, 11, 16, 21, 22]. A major shift in the arena of fixed point theory occurred in 1976 when Jungck [12], defined the concept of commutative maps and proved the common fixed point results for such maps. After which, Sessa [16] gave the concept of weakly compatible, and Jungck [13, 14] gave the concepts of compatibility and weak compatibility. Certain altercations of commutativity and compatibility can also be found in [1, 6, 16, 23].

This paper is organized as follows. In the first part, we give the required background about the digital topology and fixed point theory. In the next section, we state and prove main results for weakly compatible compatible mappings in digital metric spaces. Our results improve and generalize many other results [1, 6, 10-23]. Finally, we give an important application of fixed point theorems for digital images. Lastly, we make some conclusions.

Our improvement in this paper is four-fold:

1. to relax the continuity requirement of maps completely,
2. to minimize the commutativity requirement of the maps to the point of coincidence,
3. to weaken the completeness requirement of the space to four alternative conditions,
4. to employ a more general contraction condition in proving our results.

### Preliminaries

Let  $X$  be subset of  $Z^n$  for a positive integer  $n$  where  $Z^n$  is the set of lattice points in the  $n$ - dimensional Euclidean space and  $\rho$  represent an adjacency relation for the members of  $X$ . A digital image consists of  $(X, \rho)$ .

Definition 2.1 [4]: Let  $l, n$  be positive integers,  $1 \leq l \leq n$  and two distinct points

$a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in Z^n$ , and  $b$  are  $k_l$ -adjacent if there are at most  $l$  indices  $i$  such that  $|a_i - b_i| = 1$  and for all other indices  $j$  such that  $|a_j - b_j| \neq 1, a_j = b_j$ .

A  $\rho$ -neighbour [4] of  $a \in Z^n$  is a point of  $Z^n$  that is  $\rho$ -adjacent to  $a$  where  $\rho \in \{2, 4, 6, 8, 18, 26\}$  and  $n \in \{1, 2, 3\}$ . The set

$N_\rho(a) = \{b / b \text{ is adjacent to } a\}$  is called the  $\rho$ -neighbourhood of  $a$ .

A digital interval [13] is defined by

$[p, q]_z = \{z \in Z / p \leq z \leq q\}$ , where  $p, q \in Z$  and  $p < q$ .

A digital image  $X \subset Z^n$  is  $\rho$ -connected [1] if and only if for every pair of different points

$u, v \in X$ , there is a set  $\{u_0, u_1, \dots, u_r\}$  of points of digital image  $X$  such that  $u = u_0, v = u_r$  and  $u_i$  and  $u_{i+1}$  are  $\rho$ -neighbours where  $i = 0, 1, \dots, r-1$ .

Definition 2.2: Let  $(X, \rho_0) \subset Z^{n_0}, (Y, \rho_1) \subset Z^{n_1}$  be digital images and  $T : X \rightarrow Y$  be a function, then

- (i)  $T$  is said to be  $(\rho_0, \rho_1)$ -continuous [5], if for all  $\rho_0$ -connected subset  $E$  of  $X$ ,  $f(E)$  is a  $\rho_1$ -connected subset of  $Y$ .
- (ii) For all  $\rho_0$ -adjacent points  $\{u_0, u_1\}$  of  $X$ , either  $T(u_0) = T(u_1)$  or  $T(u_0)$  and  $T(u_1)$  are  $\rho_1$ -adjacent in  $Y$  if and only if  $T$  is  $(\rho_0, \rho_1)$ -continuous [5].
- (iii) If  $f$  is  $(\rho_0, \rho_1)$ -continuous, bijective and  $T^{-1}$  is  $(\rho_0, \rho_1)$ -continuous, then  $T$  is called  $(\rho_0, \rho_1)$ -isomorphism [6] and denoted by  $X \cong_{(\rho_0, \rho_1)} Y$ .

A  $(2, \rho)$ -continuous function  $T$ , is called a digital  $\rho$ -path [5] from  $u$  to  $v$  in a digital image  $X$  if

$T : [0, m]_z \rightarrow X$  such that  $T(0) = u$  and  $T(m) = v$ . A simple closed  $\rho$ -curve of  $m \geq 4$  points [5] in a digital image  $X$  is a sequence  $\{T(0), T(1), \dots, T(m-1)\}$  of images of the  $\rho$ -path

$T : [0, m-1]_z \rightarrow X$  such that  $T(i)$  and  $T(j)$  are  $\rho$ -adjacent if and only if  $j = i \pm \text{mod } m$ .

Definition 2.3 [7]: A sequence  $\{x_n\}$  of points of a digital metric space  $(X, d, \rho)$  is a Cauchy sequence if for all  $\varepsilon > 0$ , there exists  $\delta \in \mathbb{N}$  such that for all  $n, m > \delta$ , then  $d(x_n, x_m) < \varepsilon$ .

Definition 2.4 [7]: A sequence  $\{x_n\}$  of points of a digital metric space  $(X, d, \rho)$  converges to a limit  $p \in X$  if for all  $\varepsilon > 0$ , there exists  $\alpha \in \mathbb{N}$  such that for all  $n > \delta$ , then  $d(x_n, p) < \varepsilon$ .

Definition 2.5 [7]: A digital metric space  $(X, d, \rho)$  is a complete digital metric space if any Cauchy sequence  $\{x_n\}$  of points of  $(X, d, \rho)$  converges to a point  $p$  of  $(X, d, \rho)$ .

Definition 2.6 [7]: Let  $(X, d, \rho)$  be any digital metric space and  $T : (X, d, \rho) \rightarrow (X, d, \rho)$  be a self digital map. If there exists  $\alpha \in (0, 1)$  such that for all  $x \in X$ ,  $d(Tx, Ty) \leq \alpha(x, y)$ , then  $T$  is called a digital contraction map.

Proposition 2.8 [7]: Every digital contraction map is digitally continuous.

Theorem 2.9 [7]: (Banach Contraction principle) Let  $(X, d, \rho)$  be a complete metric space which has a usual Euclidean metric in  $Z^n$ . Let,  $T : X \rightarrow X$  be a digital contraction map. Then  $T$  has a unique fixed point, i.e. there exists a unique  $p \in X$  such that  $T(p) = p$

## MAIN RESULTS

Definition 3.1 [10]: Suppose that  $(X, d, \rho)$  is a complete digital metric space and  $S, T : X \rightarrow X$  be maps defined on  $X$ . Then  $S$  and  $T$  are said to be commutative if  $S \circ T x = T \circ S \forall x \in X$ .

Definition 3.2[10]: The self maps  $S$  and  $T$  of a digital metric space  $(X, d, \rho)$  are said to be weakly commutative iff  $d(S(T(x)), T(S(x))) \leq d(S(x), T(x))$  for all  $x \in X$ .

Remark 3.3 [1]: Every pair of commutative maps is weakly commutative but the converse is not true.

Definition 3.4: Let  $S$  and  $T$  be self maps of a digital metric space  $(X, d, \rho)$  and  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} S(x_n) = \lim_{n \rightarrow \infty} T(x_n) = t$  for some  $t$  in  $X$ . Then

$S$  and  $T$  are said to be compatible if  $\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0$

Remark 3.5 : Note that the maps which are commutative are clearly compatible but the converse is not true.

Definition 3.6 : Two self mappings  $S$  and  $T$  of a digital metric space  $(X, d, \rho)$  are called weakly compatible if they commute at coincidence points. That is, if  $Sx = Tx, \forall x \in X \Rightarrow STx = TSx, \forall x \in X$ .

Theorem 3.1: Let  $A, B, S$  and  $T$  be four self-mappings of a complete digital metric space  $(X, d, \rho)$  satisfying the following conditions:

- (a)  $S(X) \subset B(X)$  and  $T(X) \subset A(X)$ ;
- (b) the pairs  $(A, S)$  and  $(B, T)$  are coincidentally commuting ;
- (c) one of  $S(X), T(X), A(X)$  and  $B(X)$  is complete subspace of  $X$  ;
- (d)  $d(Sx, Ty) \leq \left[ \phi \left\{ \max \left( d(Ax, By), d(Sx, Ax), d(Sx, By), d(By, Ty) \right) \right\} \right]$ ,

$\forall x, y \in X$ , where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous and monotone increasing function such that  $\phi(t) < t, \forall t > 0$ . Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .

*Proof.* Since  $S(X) \subset B(X)$ , we can consider a point  $x_0 \in X$ , there exists  $x_1 \in X$  such that

$Sx_0 = Bx_1 = y_0$ . Also, for this point  $x_1$ , there exists  $x_2 \in X$  such that  $Tx_1 = Ax_2 = y_1$ . Continuing in this way, we can construct a sequence  $\{y_n\}$  in  $X$  such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n+2} \text{ for each } n \geq 0.$$

Now, we have to show that  $\{y_n\}$  is Cauchy sequence in  $(X, d, \rho)$ . Indeed, it follows that, for all  $n \geq 1$ ,

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \left[ \phi \left\{ \max \left( d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Sx_{2n}, Bx_{2n+1}), d(Bx_{2n+1}, Tx_{2n+1}) \right) \right\} \right]$$

$$= \left[ \phi \left\{ \max (d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n}), d(y_{2n}, y_{2n+1})) \right\} \right]$$

If  $d(y_{2n+1}, y_{2n}) > d(y_{2n}, y_{2n-1})$ , then a contradiction and so,

$$d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n}),$$

therefore, the sequence  $\{d(y_{2n}, y_{2n+1})\}$  is strictly decreasing. Then there exists  $r \geq 0$  such that

$\lim_{n \rightarrow \infty} d(y_{2n}, y_{2n+1}) = r$ . Suppose that  $r > 0$ , then, letting  $n \rightarrow \infty$  in above equation,  $r \leq \phi(r) < r$ , which is impossible. Hence,  $r = 0$ , that is,

$\lim_{n \rightarrow \infty} d(y_{2n}, y_{2n+1}) = 0$ . Thus  $\{y_n\}$  is Cauchy sequence in  $X$ .

Now, suppose  $B(X)$  is a complete subspace of  $X$ , then the subsequence  $y_{2n} = Bx_{2n+1}$  must get a limit in  $B(X)$ .

Call it to be  $u$  and  $v \in B^{-1}u$ . Then  $Bv = u$ . As  $\{y_n\}$  is a Cauchy sequence containing a convergent subsequence  $\{y_{2n+1}\}$ , therefore the sequence  $\{y_n\}$  also converges implying there by the convergence of  $\{y_{2n}\}$  being a subsequence of the convergent sequence  $\{y_n\}$ . On setting  $y = v$  and  $x = x_{2n+1}$  in (d) one gets

$$d(Sx_{2n+1}, Tv) \leq \left[ \phi \left\{ \max (d(Ax_{2n+1}, Bv), d(Sx_{2n+1}, Ax_{2n+1}), d(Sx_{2n+1}, Bv), d(Bv, Tv)) \right\} \right],$$

which on letting  $n \rightarrow \infty$ , gives

$d(u, Tv) \leq \phi(d(u, Tv)) < d(u, Tv)$ , a contradiction and thus  $Tv = u$ , shows that  $u$  is the point of coincidence for the pair  $(B, T)$ .

As  $T(X) \subseteq A(X)$ ,  $Tv = u$  implies that  $u \in A(X)$ . Let  $w \in A^{-1}u$ . Then  $Aw = u$ . Now using (d)

$$d(Sw, Tx_{2n}) \leq \left[ \phi \left\{ \max (d(Aw, Bx_{2n}), d(Sw, Aw), d(Sw, Bx_{2n}), d(Bx_{2n}, Tx_{2n})) \right\} \right],$$

taking  $n \rightarrow \infty$ , we get  $Sw = u$ , shows that  $u$  is the point of coincidence for the pair  $(A, S)$ .

Thus we have shown  $u = Tv = Bv = Aw = Sw$  which amounts to say that  $u$  is point of coincidence for both pairs.

If one assumes  $T(X)$  to be complete, then an analogous argument follows claim. The remaining two cases pertain essentially to the previous cases. Indeed if  $A(X)$  is complete, then  $u \in T(X) \subseteq A(X)$  and if  $B(X)$  is complete, then  $u \in S(X) \subset B(X)$  and the claim is completely established.

Since the pairs  $(A, S)$  and  $(B, T)$  are weakly compatible and hence,

$$Au = A(Sv) = S(Av) = Su \text{ and}$$

$$Bw = B(Tw) = T(Bw) = Tw.$$

If  $Su \neq u$ , then

$$d(Su, u) = d(Su, Tw) \leq \left[ \phi \left\{ \max (d(Au, Bw), d(Su, Au), d(Su, Bw), d(Bw, Tw)) \right\} \right],$$

a contradiction. Therefore  $Su = u$ . Similarly, one can show that  $Bu = u$ . Thus  $u$  is a common fixed point of  $A, B, S$  and  $T$ . The uniqueness of a common fixed point follows easily.

**Theorem 3.2** Let  $A, B, S$  and  $T$  be four self-mappings of a complete digital metric space  $(X, d, \rho)$  satisfying the following conditions:

- (a)  $S(X) \subset B(X)$  and  $T(X) \subset A(X)$ ;
- (b) the pairs  $(A, S)$  and  $(B, T)$  are weakly compatible ;
- (c) one of  $S(X), T(X), A(X)$  and  $B(X)$  is complete subspace of  $X$  ;
- (d)  $d(Sx, Ty) \leq \lambda \{ \max(d(Ax, By), d(Sx, Ax), d(Sx, By)) \}, \forall x, y \in X,$   
 $\forall x, y \in X, \lambda \in [0, 1)$ . Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .

*Proof.* If we set  $\phi(t) = \lambda t$ , we get the result.

### Applications of Common Fixed Point Theorems in Digital Metric Space

In this section, we give an application of digital contractions to solve the problem related to image compression. The aim of image compression is to reduce redundant image information in the digital image. When we store an image we may come across certain type of problems like either memory data is usually too large or stored image has not more information than original image. Also, the quality of compressed image can be poor. For this reason, we must pay attention to compress a digital image. Fixed point theorem can be used for image compression of a digital image.

### CONCLUSION

The aim of this paper is to introduce common fixed point theorems for the digital metric spaces, using compatible maps and its variants. This concept may come handy in the image processing and redefinition of the image storage. The redefinition of the same slot of memory is a proposed application of the proposed concept of the paper.

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