Scholars Journal of Physics, Mathematics and Statistics
Sch. J. Phys. Math. Stat. 2017; 4(4):172-177
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

Zero Factorial

Venkateswara Rao Mudunuru[1], M. Vidya Bhargavi[2], Ravindharan Ethiraj[3]
[1]Visiting Instructor, Department of Mathematics and Statistics, University of South Florida, Tampa, FL USA
[2]Assistant Professor,Department of Mathematics, Stanley College of Engineering and Technology for Women, Hyderabad, India.
[3]Professor, Department of Electrical Engineering, Director, R & D Wing, Stanley College of Engineering and Technology for Women, Hyderabad, India.

***Corresponding author**
Venkateswara Rao Mudunuru

Article History
Received: 14.10.2017
Accepted: 25.10.2017
Published: 30.10.2017

DOI:
10.21276/sjpms.2017.4.4.4

Abstract: Factorials play a major role in many applications. In this work we will discuss the concepts of factorial and provide the proofs related to 0!=1. Finally we present the new variations of factorials with few examples
Keywords: Undergraduate Statistics; Euler; Proofs; Combinatorics; Statistics and Probability; Gamma Function

INTRODUCTION

Many of the present days research in any discipline involves Mathematics and Statistics. These two are potentially applicable in every discipline. Training received by students and researchers in the areas of Mathematics and Statistics vary greatly from discipline to discipline , but one common aspect appears to be that all of them acquire some default beliefs which are accepted without asking for proofs, one such being, “0!=1”. In this article we look into a deeper understanding of the conventional definition of factorial and present various alternate proofs of “0!=1” from different perspectives.

The use of factorials can be seen in every field of science. According to N.L.Biggs [5] and Dukta Jacques [15], the concept of factorial was known as early as the 12th century to Indian scholars. Daniel Bernoulli in the year 1729 gave an interpolating function of factorials as an infinite product. Christian Kramp initially called it by the name 'faculty', which was substituted by the name ‘factorial’ (*factorielle* in French) by Louis François Antoine Arbogast (1759-1803) which conveys what it really is. Later in 1768 Euler defined factorial in terms of Gamma function. Research on the interpolation of factorials started with correspondence among Leonhard Euler, Daniel Bernoulli and Christian Goldbach in the year 1729 [2, 26].

Factorials are very easy to understand as they just involve multiplication of positive numbers. Since factorials have a unique property of recursion they are frequently used in computer programs involving recursions. Factorials were initially denoted using “ ∠ ” as suggested by Rev. Thomas Jarrett but Christian Kramp introduced the notation “!” to represent factorial because it makes typing easy [12, 14]. There is been a lot of research using factorials which resulted in the development of concepts like Multifactorials, Hyperfactorials, Superfactorials, etc.

Our interest started when our students started asking us where to find the proof for “0!”. One way to answer this question is to say that it is by definition, or one may say an extraction from combinatorics, or so on. In this article we focus in proving 0! = 1 for every level of students ranging from school to research. There are so many sources available for these proofs but are mostly incomplete or scattered and are very hard to find. This article aims to bring all the possible proofs for “0!” together and make it easy to understand. We will first start by defining factorials along with some examples, properties, and applications and then proceed with proofs for “0!”

Definitions

There are several definitions of factorial in literature. Without loss of generality, we wish to define factorial as follows. However, there are several definitions in literature mentioned below.

$$n! = \begin{cases} 1, & if \ \ n = 0 \\ n, & if \ \ n = 1, 2 \\ 2 \times 3 \times 4 \times \ldots \times (n-2) \times (n-1) \times n, & if \ \ n = 3,4,5,\ldots \end{cases} \quad (1)$$

Following the recurrence relation, the factorials of positive integers are defined as,

$$n! = n \times (n-1)! \qquad for \quad n = 1, 2, 3, \ldots$$
$$(n+1)! = (n+1) \times n! \qquad for \quad n = 0, 1, 2, \ldots \quad (2)$$

In 1730 Euler introduced an integral function [2] that gives the factorial of *n*. He called it as a "Pi function", $\Pi(n)$ which also follows the recursive relation of factorial for all positive real numbers.

$$\Pi(n) = \int_0^1 (-\ln x)^n dx = \int_0^\infty x^n e^{-x} dx = n!, \ \ n \geq 0$$

recursively, (3)

$$\Pi(n+1) = (n+1)\Pi(n), \quad n \geq 0$$

In 1768, Euler extended the concept of factorials to all real negative numbers, except zero and negative integers by introducing gamma function. Gamma function Γ(x), also known as Euler's integral of second kind is an extension of the Pi function. More information about Euler work on Gamma function can be found elsewhere [2,3,11].

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$$
$$\Pi(n) = \Gamma(n+1) = n! \quad (4)$$

Some of the definitions of factorial from different sources are:

- According to Wikipedia [6], the factorial of a non-negative integer *n*, denoted by *n!*, is the product of all positive integers less than or equal to *n*. The value of 0! is 1, according to the convention for an empty product.
- The factorials n! for n=1, 2, 3, … are 1, 2, 6, 24, 120, 720,… [22]
- Stapel [24] says factorials are very simple things. They're just products, indicated by an exclamation mark. 0! is defined to be equal to 1, not 0. Memorize this now: 0! = 1.
- Weisstein [27] defines that the factorial of a positive integer *n* denoted by n! as n!=n(n-1)...2·1. The special case 0! is defined to have value 0!=1, consistent with the combinatorial interpretation of there being exactly one way to arrange zero objects.

Few properties of factorials

1. $(n+1)! = (n+1) \times n!$
2. $(n \times m)! \neq n! \times m!$
3. $(n \pm m)! \neq n! \pm m!$
4. $1 = 1!$
5. $2 = 2!$
6. $145 = 1! + 4! + 5!$
7. $40585 = 4! + 0! + 5! + 8! + 5!$

The last four properties above are true only those four integers satisfying this property. They are called as Factorions. Pickover [17] in his article titled *"The Loneliness of the Factorions"* defined 'factorion' as an integer which is equal to the sum of factorials of its digits. These are exactly four such numbers satisfying this property.

Proofs for 0! = 1

0! (Zero factorial) is defined as 1. This definition is useful in expressing many mathematical identities in simple form. Here in this section we will provide all the possible proofs in increasing order of complexities.

Level 1
Using Convention:
Accept that 0!=1 in good faith. There is no proof.

Level 2
Using Logic:

We can rewrite the recursive definition of factorial of n from Equation (2) as, $(n-1)! = \frac{n!}{n}$. For $n = \ldots, 4, 3, 2, 1$ we have the following results respectively.

$$For \quad n = 4 : (4-1)! = 3! = \frac{4!}{4} = 6$$

$$For \quad n = 3 : (3-1)! = 2! = \frac{3!}{3} = 2$$

$$For \quad n = 2 : (2-1)! = 1! = \frac{2!}{2} = 1$$

$$For \quad n = 1 : (1-1)! = 0! = \frac{1!}{1} = 1$$

Hence the proof, 0! = 1.

Level 3
Using Combinations:

The number of ways of selecting r objects out of n objects regardless of order, $r \le n$, is given by $nC_r = \frac{n!}{r!(n-r)!}$. Now given n objects, in how many ways one can choose 0 objects is just one way, i.e., one way is just no way {recollect from set theory, *"null set is a subset of every set"*}.

$i.e.,$ we have , for $r = 0$,

$$nC_0 = 1 \Rightarrow nC_0 = \frac{n!}{0!(n-0)!} = 1 \Rightarrow \frac{1}{0!} = 1 \Rightarrow 0! = 1$$

Philosophically, we would like to say something here. *"'No way' is also a 'way'."*

The same argument is also true for how many ways one can choose all n objects when given with n objects, is also one way.

$i.e.,$ we have , for $r = n$,

$$nC_n = 1 \Rightarrow nC_n = \frac{n!}{n!(n-n)!} = 1 \Rightarrow \frac{1}{0!} = 1 \Rightarrow 0! = 1$$

Level 4
Using recursive definition:
The recursive definition of factorial from Equation (2) above, we have $n! = n \times (n-1)!$

$$For \quad n = 1, we \quad get \quad 1! = 1 \times (1-1)!$$

$$\Rightarrow 1! = 0! \tag{5}$$

Also we can also write the *n!* as

$$n! = [(n-1)+1]! = n \times (n-1)!$$

$$i.e., 5! = [4+1]! = 5 \times 4! = 120$$

$$4! = [3+1]! = 4 \times 3! = 24$$

$$3! = [2+1]! = 3 \times 2! = 6$$

$$2! = [1+1]! = 2 \times 1! = 2$$

Finally, $1! = [0+1]! = 1 \times 0! = 1$ (6)

From the Equation (5) and Equation (6) we have $0! = 1$

Level 5
Using Gamma function:

As mentioned above, Gamma function for any $x > 0$ is defined as,

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y}\, dy \tag{7}$$

Using integration by parts we can compute $\Gamma(x+1)$, which gives the result as $\Gamma(x+1) = x\Gamma(x)$. Performing this integration recursively, we get,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \dots$$

$$or$$

$$\Gamma(x+1) = x \times (x-1) \times (x-2) \times \dots 3 \times 2 \times 1 = x!$$

$$or$$

$$\Gamma(x) = (x-1) \times (x-2) \times \dots 3 \times 2 \times 1 = (x-1)! \tag{8}$$

Substituting x=1 in the Equation (8) above, we get $\Gamma(x) = \Gamma(1) = (1-1)! = 0!$ (9)

Now from Equation (7) above, for x=1 we get, $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y}\, dy \Rightarrow \Gamma(1) = \int_0^\infty y^{1-1} e^{-y}\, dy$

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = -[e^{-\infty} - e^0] = 1 \tag{10}$$

Comparing Equation (9) and Equation (10) we get $\Gamma(1) = 0! = 1$. Hence the proof.

Variations on factorials

Now we will extend our discussion on factorials by defining few variations on factorials and their examples. Given below are the few of several other integer sequences similar to that of a factorial and are frequently used in various applications.

Multifactorials

By using multiple exclamation points (!) one can define a *'Multifactorial'* [1,22] as the product of integers in steps of two, three, or more. For suppose, "*n!!*" denotes the *'double factorial'* of *n*, and so on. One should not misinterpret n!! as factorial of n!, i.e., $n!! \neq (n!)!$. Also by definition of double factorial [20], *-1!! = 0!! = 1*.

Arfken [1] has expressed the double factorial using Gamma function as $\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}\,(2n-1)!!}{2^n}$

$$n!! = \begin{cases} 1, & for \quad n = 0 \\ n, & for \quad n = 1 \\ n(n-2)!! & for \quad n > 1 \end{cases}$$

$$n!!! = \begin{cases} 1, & for \quad n = 0 \\ n, & for \quad n = 1, 2 \\ n(n-3)!!! & for \quad n > 2 \end{cases}$$

This can be generalized as $n!^{(m)} = \begin{cases} 1, & for \quad n = 0 \\ n, & for \quad 0 < n < m \\ n(n-m)!^{(m)} & for \quad n > (m-1) \end{cases}$

Example:

$$4!! = 4(4-2) = 8$$
$$5!! = 5(5-2)(5-4) = 15$$
$$5!!! = 5(5-3) = 10$$
$$7!!! = 7(7-3)(7-6) = 28$$

Hyperfactorials

The *'Hyperfactorial'* function [8,23] is similar to the factorial, but produces larger numbers. The hyperfactorial written as *H(n)* is the function defined as follows.

$$H(n) = n^n \times (n-1)^{n-1} \times \ldots \times 3^3 \times 2^2 \times 1^1 = \prod_{p=1}^{n} p^p$$

Example: $H(3) = 3^3 \times 2^2 \times 1^1 = 108$
$H(4) = 4^4 \times 3^3 \times 2^2 \times 1^1 = 27648$

Superfactorials

The *'Superfactorial'* of n, written as n$ (looks like a dollar sign which actually is a factorial sign (!) written over S) is a factorial-based function with differing definitions.

Clifford Pickover [16] defined it as $n\$ = n!^{n!^{n!^{\cdots}}}$. Using this definition, the first two values for *n* = 1 and *n* = 2 are 1 and 4 respectively, but from *n* = 3 the value will grow larger very rapidly.
N.J.A. Sloane and Simon Plouffe [19] defined $n\$$ as the product of the first *n* factorials, i.e., $n\$ = \prod_{k=1}^{n} k! = 1! \times 2! \times \ldots \times (n-1)! \times n!$

Example: $4\$ = 4! \times 3! \times 2! \times 1! = 288$
$5\$ = 5! \times 4! \times 3! \times 2! \times 1! = 34560$

Primorial

The *'Primorial'* [4, 7, 18, 21] is similar to the factorial, but with the product taken only over the prime numbers. For the nth prime number the primordial, *prime (n)#* is defined as the product of the first *n* primes.

Example: $prime\ (3)\# = 2 \times 3 \times 5 = 30$
$prime\ (4)\# = 2 \times 3 \times 5 \times 7 = 210$
$prime\ (7)\# = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 = 510510$

CONCLUSION

Factorials have their place in every discipline. Factorials are important in combinatorics because there are *n!* different ways of arranging *n* distinct objects in a sequence. Factorials also play an important role in formulas such as in expansions of trigonometric functions, exponential functions, hyperbolic functions, etc., in calculus. Consider for example, the nth derivative of the function x^n *is* $n!$. Also when *n* happens to be large, *n!* can be estimated using Stirling's approximation. Due to their recursive nature, factorials play a major role in computer programming. In this article we have provided all the different ways of proving 0! =1 and also presented various research results in developments of factorials with some examples.

ACKNOWLEDGMENTS

We would like to dedicate this article to Dr. Leslaw A. Skrzypek, Chair of Department of Mathematics and Statistics, University of South Florida, Tampa, USA for encouraging us to work on this article.

REFERENCES

1. Arfken G. *Mathematical Methods for Physicists* 3rd edn (Orlando, FL: Academic). 1967; Academic Press, pp. 544-545 and 547-548.
2. Dartmouth College. *Correspondance between Leonhard Euler and Chr. Goldbach*. 2014; 1729–1763, pp 1–59
3. Davis PJ. *Leonhard Euler's integral: A Historical Profile of the Gamma Function:* In memoriam: Milton abramowitz. The American Mathematical Monthly. 1959 Dec 1; 66(10):849-869.
4. Dubner H. *Factorial and Primorial Primes*. J. Rec. Math. 1987;19(3):197-203
5. Dutka J. *The early history of the factorial function*. Archive for history of exact sciences. 1991 Sep 1;43(3):225-249.
6. Factorial. In Wikipedia, the Free Encyclopedia. 2015; May 18: 23-26.
7. Finch SR. "*Mathematical Constants*" Cambridge, England: Cambridge University Press. 2003.
8. Fletcher A, Miller JC, Rosenhead L, Comrie LJ. *An Index of Mathematical Tables*. Reading, MA: Addison-Wesley, p. 50; 1962.
9. Gardner M. *Factorial Oddities*. Scientific American. 1978; pp 61 and 64.
10. Graham RL. *Concrete Mathematics: A Foundation for Computer Science*.1994; Addison-Wesley, p. 477.
11. Gronau D. *Why is the gamma function so as it is?* Teaching Mathematics and Computer Science. 2003;1: 43-53.

12. Lewin L. *Dilogarithms and Associated Functions*.1958.
13. Madachy JS. *Madachy's Mathematical Recreations*. Dover Publications; 1979.
14. Mellin HJ. *Abrißeiner einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen*. Mathematische Annalen. 1910 Sep 1;68(3):305-337.
15. Biggs NL. *The roots of Combinatorics*. Historia Mathematica. 1979 May 1;6(2):109-136.
16. Pickover CA. *Keys to infinity*. Wiley; 1995.
17. Pickover CA. *The Loneliness of the Factorions*. Wiley; 1995; pp. 169-171 and 319-320.
18. Ruiz SM. 81.27 *A Result on Prime Numbers*. The Mathematical Gazette. 1997 Jul;81(491):269.
19. Sloane NJ. The on-line encyclopedia of integer sequences.2003; Sequence A000178/M2049.
20. Sloane NJ. The on-line encyclopedia of integer sequences.2010; Sequence A002110/M1691 and A034386.
21. Sloane NJ. The on-line encyclopedia of integer sequences, 2007; Sequence A034386.
22. Sloane NJ. The encyclopedia of integer sequences. 1995; Sequences A000142/M1675, A006882/M0876, A007661/M0596, and A007662/M0534.
23. Sloane NJ. The encyclopedia of integer sequences. 2003; Sequences A002109/M3706, A143475, and A143476
24. Stapel, Elizabeth. "*Factorials*", Purplemath. 2015.
25. Stedman, Fabian, "*Campanalogia*", London, 1677; pp. 6–9.
26. Thukral AK. *Factorials of Real Negative and Imaginary Numbers-A new perspective*. SpringerPlus. 2014 Dec 1;3(1):658.
27. Weisstein, Eric W. "*Factorial*." From MathWorld--A Wolfram Web Resource.
28. Kaiser HF. *An index of factorial simplicity*. Psychometrika. 1974 Mar 27;39(1):31-6.