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# A Note on Existence of Positive Solutions for the Sturm-Liouville Boundary Value Problems

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**Abstract:** In this paper, we prove a maximum principle for the Sturm-Liouville problem, and use it and the fixed point theorem in Banach spaces to prove a new result of positive solutions for the Sturm-Liouville problem under superlinear conditions, our assumptions on $f$ and $p$ are weaker than usual ones.

**Keywords:** Sturm-Liouville, positive solutions, maximum principle, fixed point, existence

## INTRODUCTION

We investigate the existence of positive solutions for the Sturm-Liouville problem

$$(p(t)z'(t))' + f(t, z(t)) = 0 \quad \text{a.e. on } [0,1] \tag{1.1}$$

subject to the boundary condition

$$z(0) = 0 = z(1). \tag{1.2}$$

It is well-known that (1.1) and (1.2) is widely used in many fields, what people are interested in is the existence of positive solutions. There have been many papers studying the existence of positive solutions via the various methods and a great deal of results have been obtained under various assumptions.

For the positone case and the semipositone case, the well-known fixed theorems in cone [1] has been widely used, for example, see [2, 3, 4] and the references therein. For the case that $f$ has a functional lower bound, Li [5] obtained some results for the sublinear case and superlinear case where some usual limit conditions such as $f_\infty = \lim_{z\to\infty} \inf_{t\in[0,1]} \frac{f(t,z)}{z}$ and $f_0 = \lim_{z\to 0} \inf_{t\in[0,1]} \frac{f(t,z)}{z}$ are bounded below, and $p \in C^1[0,1]$; Yao [6] extended the limits to consider that $f$ satisfies $\int_a^b \lim_{z\to\infty} \inf \frac{f(t,z)}{z} dt = \infty$ ( $0 < a < b < 1$ ). And there are other articles that different limit conditions were considered, for example, see [7, 8].

Utilizing the Leray-Schauder fixed point theorem in Banach Space, Yang and Zhou [9] proved the existence of positive solutions of (1.1)-(1.2) for the sublinear case, they abandoned the condition that $f$ has numerical or functional lower bounds and just needed there exists a constant $r_0 > 0$ such that $f(t,z) \geq 0$ on $[0,1]\times[0,r_0]$, see Theorem 2.1 [9]. Yang and Feng [10] investigated the superlinear case, where usual limit conditions are not required to be bounded below [9] .

In [9, 10], a key assumption is $f(t,0) \geq 0$ for almost every $t \in [0,1]$. In this paper, we relax this assumption. We shall prove a maximum principle for the Sturm-Liouville problem, and utilize it and some inequalities [10] to obtain a new existence result of (1.1)-(1.2).

## SOME PRELIMINARIES

In this paper, we make the following assumptions on $f$ and $p$ :

$(A_1)$ $f:[0,1]\times R^+ (R^+=[0,\infty))\to R$ is a Carathéodory function, that is, $f(\cdot,z)$ is measurable for each fixed $z\in R^+$, $f(t,\cdot)$ is continuous for almost every $t\in[0,1]$, and for each $r>0$, there exists $g_r\in L_+[0,1]$ such that

$$|f(t,z)|\le g_r(t) \quad \text{for a.e. } t\in[0,1] \text{ and all } z\in[0,r],$$

where $L_+[0,1]=\{g\in L[0,1]: g(s)\ge 0 \ a.e. [0,1]\}$.

$(A_2)$ There exists $w(t)\in L[0,1]$ such that

$$f(t,z)\ge w(t) \quad \text{for a.e. } t\in[0,1] \text{ and all } z\in[0,\infty).$$

$(A_3)$ $p:[0,1]\to R^+\setminus\{0\}$, and $p\in C[0,1]$.

First, we consider the following problem

$$-(p(t)z'(t))'\ge w(t) \text{ a.e. on } [0,1], \tag{2.1}$$

$$z(0)=0=z(1), \tag{2.2}$$

and prove a maximum principle.

**Notation**

$$\|W\|=\max\left\{|W(t)|: W(t)=\int_0^t w(s)ds,\ t\in[0,1]\right\},$$

$$p_{\min}=\min\{p(t):t\in[0,1]\},\quad p_{\max}=\max\{p(t):t\in[0,1]\}.$$

**Lemma 2.1 (Maximum principle)**

*If there exists* $z$ *satisfies (2.1)-(2.2) and*

$$\|z\|>\frac{2\|W\|}{p_{\min}}, \text{ then } z(t)>0 \text{ for all } t\in(0,1).$$

***Proof***

Take $\varepsilon>0$ sufficiently small such that

$$\|z\|>\frac{2\|W\|+2\varepsilon}{p_{\min}}:=L$$

and $\tilde{w}(t):=w(t)-\varepsilon\not\equiv 0$. Set

$$\tilde{W}(t):=\int_0^t \tilde{w}(s)ds=W(t)-\varepsilon t.$$

Then we can write (2.1) in the following form

$$-\left[p(t)z'(t)+\tilde{W}(t)\right]'\ge\varepsilon>0 \quad \text{for } t\in(0,1). \tag{2.3}$$

We now define

$$y(t):=p(t)z'(t)+\tilde{W}(t) \quad \text{for } t\in(0,1). \tag{2.4}$$

From (2.3) and (2.4) we know that $y$ is strictly decreasing on (0,1) and

$$z'(t)=\frac{y(t)-\tilde{W}(t)}{p(t)} \quad \text{for } t\in(0,1).$$

Let $t_0\in(0,1)$ be such that $|z(t_0)|=\|z\|>L$, which implies that $z'(t_0)=0$. And from (2.4) we know

$$|y(t_0)| = |p(t_0)z'(t_0) + \tilde{W}(t_0)| = |\tilde{W}(t_0)| \le \|\tilde{W}\|.$$

Let $[t_1, t_2] \subset [0,1]$ be the maximal interval that containing $t_0$ and such that $|y(t)| \le \|\tilde{W}\|$ for all $t \in (t_1, t_2)$.
For $t \in (t_1, t_2)$, we have

$$|z'(t)| = \left|\frac{y(t) - \tilde{W}(t)}{p(t)}\right| \le \frac{|y(t)| + |\tilde{W}(t)|}{p(t)} \le \frac{\|\tilde{W}\| + \|\tilde{W}\|}{p(t)} \le \frac{2\|\tilde{W}\|}{p_{\min}}.$$

Hence, we obtain

$$\|z\| = |z(t_0)| = \left|z(t) - \int_{t_0}^{t} z'(s)ds\right| \le |z(t)| + \left|\int_{t_0}^{t} z'(s)ds\right|$$
$$\le |z(t)| + \left|\int_{t_0}^{t} |z'(s)|ds\right| \le |z(t)| + \left|\int_{t_0}^{t} \frac{2\|\tilde{W}\|}{p_{\min}} ds\right|$$
$$\le |z(t)| + \frac{2\|\tilde{W}\|}{p_{\min}}.$$

By $\tilde{W}(t) = W(t) - \varepsilon t$, we obtain easily $\|\tilde{W}\| \le \|W\| + \varepsilon$. Then we have

$$\|z\| \le |z(t)| + \frac{2\|\tilde{W}\|}{p_{\min}} \le |z(t)| + \frac{2\|W\| + 2\varepsilon}{p_{\min}} = |z(t)| + L.$$

Thus, we have proved that
$|z(t)| \ge \|z\| - L$ for $t \in (t_1, t_2)$.
Since $z$ is continuous on $[0,1]$, then we conclude that

$$|z(t)| \ge \|z\| - L > 0 \quad \text{for } t \in [t_1, t_2]. \tag{2.5}$$

As $z(0) = 0 = z(1)$, from (2.5) we obtain that $0 < t_1 \le t_2 < 1$. By the maximality of $[t_1, t_2]$, the continuity of $y$ and the fact that $y$ is strictly decreasing on $(0,1)$, we also have

$$y(t) > \|\tilde{W}\| \quad \text{for } t \in (0, t_1), \tag{2.6}$$
$$y(t_1) = \|\tilde{W}\|, \qquad |y(t)| \le \|\tilde{W}\| \text{ for } t \in (t_1, t_2), \qquad y(t_2) = -\|\tilde{W}\|,$$
$$y(t) < -\|\tilde{W}\| \quad \text{for } t \in (t_2, 1). \tag{2.7}$$

For $t \in (0, t_1)$, from (2.6), we obtain

$$z'(t) = \frac{y(t) - \tilde{W}(t)}{p(t)} \ge \frac{y(t) - |\tilde{W}(t)|}{p(t)} \ge \frac{y(t) - \|\tilde{W}\|}{p(t)} > 0.$$

For $t \in (t_2, 1)$, from (2.7), we also have

$$z'(t) = \frac{y(t) - \tilde{W}(t)}{p(t)} \le \frac{y(t) + |\tilde{W}(t)|}{p(t)} \le \frac{y(t) + \|\tilde{W}\|}{p(t)} < 0.$$

Using the continuity of $z$ and the above information of $z'$, we obtain easily that
$z$ is strictly increasing on $[0, t_1]$, and $z$ is strictly decreasing on $[t_2, 1]$.
Hence, we have that $z: [0,1] \to R$ is a continuous function with $z(0) = 0$, $z(t)$ strictly increasing on $[0, t_1]$, $|z(t)| > 0$ on $[t_1, t_2]$, $z(t)$ strictly decreasing on $[t_2, 1]$, $z(1) = 0$, which implies that $z(t) > 0$ for $t \in (0,1)$.

This completes the proof. □

**Remark**

Lemma 2.1 extends the maximum principle of the one-dimensional $p-$ Laplacian equations [11] to the Sturm-Liouville problem.

A function $z$ is said to be a positive solution of (1.1)-(1.2) if $z \in C^1[0,1]$ with $z(t) \geq 0$ on $[0,1]$, $z \not\equiv 0$, $p(t)z'(t) \in AC[0,1]$ and $z$ satisfies (1.1)-(1.2), where $AC[0,1]$ is the space of all absolutely continuous functions on $[0,1]$.

Let $C[0,1]$ be a continuous function space with norm $\|z\| = \max\{|z(t)| : t \in [0,1]\}$. It is well-known that $z$ is a positive solution of (1.1)-(1.2) if and only if $z \in C[0,1]$ with $z \not\equiv 0$ and $z(t) \geq 0$ on $[0,1]$ satisfies the following integral equation [2, 3, 6]:

$$z(t) = \int_0^1 G(t,s) f(s, z(s))\,ds := Az(t) \quad \text{for } t \in [0,1], \tag{2.8}$$

where $G(t,s)$ is Green function to $-(p(t)z'(t))' = 0$ associated with the boundary conditions (1.2) defined by

$$G(t,s) = \frac{1}{\int_0^1 \frac{1}{p(\mu)}d\mu} \begin{cases} \int_t^1 \frac{1}{p(\mu)}d\mu \int_0^s \frac{1}{p(\mu)}d\mu, & s \leq t, \\ \int_0^t \frac{1}{p(\mu)}d\mu \int_s^1 \frac{1}{p(\mu)}d\mu, & s > t. \end{cases}$$

Letting $z \in C[0,1]$ and $z^+(t) = \max\{z(t), 0\}$, we define a map $A^*$ from $C[0,1]$ to $C[0,1]$ by

$$A^* z(t) = \int_0^1 G(t,s) f(s, z^+(s))\,ds .$$

The following theorem plays a key role in the study of the existence of positive solutions of (1.1)-(1.2).

**Theorem 2.1** *Assume that* $(A_1)-(A_3)$ *hold. Let* $0 < a < b < 1$, $w_0 \in L[0,1]$, $w_0(t) \geq 0$ *on* $[0,1]$ *and* $w^*(t) = \int_a^b G(t,s) w_0(s)\,ds$. *If* $z = \lambda A^* z + \mu w^*$ *has a solution* $z \in C[0,1]$ *for some* $0 < \lambda \leq 1$, $\mu \geq 0$ *and* $\|z\| > \frac{2\|w\|}{p_{\min}}$, *then* $z(t) > 0$ *for* $t \in (0,1)$.

***Proof***

Let

$$w_1(t) = \begin{cases} w_0(t) & \text{if } a \leq t \leq b, \\ 0 & \text{if } 0 \leq t < a \text{ or } b < t \leq 1. \end{cases}$$

Then $w^*(t) = \int_a^b G(t,s) w_0(s)\,ds = \int_0^1 G(t,s) w_1(s)\,ds$, and

$$\begin{aligned} z &= \lambda A^* z + \mu w^* \\ &= \int_0^1 G(t,s)(\lambda f(s, z^+(s)) + \mu w_1(s))\,ds . \end{aligned}$$

Differentiating $z$ with $t$ twice, we have

$$-(p(t)z'(t))' = \lambda f(t, z^+(t)) + \mu w_1(t) \quad \text{on } (0,1).$$

Since $w_0(t) \geq 0$ on $[0,1]$ and $(A_2)$ holds, then

$$-(p(t)z'(t))' = \lambda f(t, z^+(t)) + \mu w_1(t) \geq \lambda w(t) \quad \text{on } (0,1).$$

By $0 < \lambda \leq 1$, we know $\|z\| > \frac{2\|W\|}{p_{\min}} \geq \frac{2\lambda\|W\|}{p_{\min}}$. This, together with Lemma 2.1, implies $z(t) > 0$ for $t \in (0,1)$.

This completes the proof. □

Next, we recall some important lemmas which have been proved in the references.

Let $g, h \in L_+[0,1]$ and $\int_0^1 h(s)ds > 0$, then we have a lemma as follows.

**Lemma 2.2** *([10], Theorem 2.1) Assume that* $(A_3)$ *holds. Then there exist* $0 < a_0 < b_0 < 1$ *such that*

$$\int_a^b G(t,s)h(s)ds \geq \int_0^a G(t,s)g(s)ds + \int_b^1 G(t,s)g(s)ds \quad on\ [0,1],$$

*for all* $0 < a \leq a_0$ *and* $b_0 \leq b < 1$.

Letting $g_0 \in L_+[0,1]$ be a function that satisfies

$$f(t,z) + g_0(t) \geq 0 \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [0,\infty), \tag{2.9}$$

and

$$g_*(t) = \int_0^1 G(t,s)g_0(s)ds .$$

Let $z \in C[0,1]$ satisfy

$$z(t) = A^* z(t) + \mu w^*(t) .$$

We define a function $\alpha \in C[0,1]$ by

$$\alpha(t) = z(t) + g_*(t) = A^* z(t) + \mu w^*(t) + g_*(t), \tag{2.10}$$

where $\mu \geq 0$ and $w^*(t)$ has the properties as in Theorem 2.1.

**Lemma 2.3** ([10], *Lemma 2.1) Assume that* $(A_1) - (A_3)$ *hold. Let* $\rho > 0$ *and* $\|\alpha\| > (\frac{p_{\max}}{p_{\min}} + 1)(\rho + \|g_*\|)$.

*Then there exist* $0 \leq a_1 < b_1 \leq 1$ *such that* $z(t) \geq \rho$ *on* $[a_1, b_1]$ *and*

$$a_1 \leq \frac{p_{\max}(\rho + \|g_*\|)}{p_{\min}(\|\alpha\| - \rho - \|g_*\|)}, b_1 \geq 1 - \frac{p_{\max}(\rho + \|g_*\|)}{p_{\min}(\|\alpha\| - \rho - \|g_*\|)} .$$

Let $K = \{z \in C[0,1] : z(t) \geq 0 \ \ on \ \ [0,1]\}$, then $K$ is the standard positive cone of $C[0,1]$ and $K$ is a total cone. $K$ defines the partial order "$\leq$" of $C[0,1]$ by $x \leq y$ if and only if $y - x \in K$.

Letting $g \in L_+[0,1]$ with $\int_0^1 g(s)ds > 0$ and $z \in C[0,1]$, we define two linear maps by

$$L_g z(t) = \int_0^1 G(t,s)g(s)z(s)ds ,$$

$$L_g^{(n)} z(t) = \int_{\frac{1}{n}}^{1-\frac{1}{n}} G(t,s)g(s)z(s)ds ,$$

where $\frac{1}{n} \leq a_0 < b_0 \leq 1 - \frac{1}{n}$, $a_0$ and $b_0$ are as in lemma 2.2.

It is easy to know that $L_g$ and $L_g^{(n)}$ are compact in $C[0,1]$ and map $K$ into $K$. Let $r(L_g)$ and $r(L_g^{(n)})$ stand for the radius of the spectrum of $L_g$ and $L_g^{(n)}$, respectively. If we denote that $\mu_1(L_g) = \frac{1}{r(L_g)}$ and

$\mu_1(L_g^{(n)}) = \dfrac{1}{r(L_g^{(n)})}$, then $0 < \mu_1(L_g), \mu_1(L_g^{(n)}) < \infty$ by $0 < r(L_g), r(L_g^{(n)}) < \infty$. Specially, if $g \equiv 1$, $\mu_1(L_g)$ is written usually as $\mu_1$.

**Lemma 2.4** *([10], Lemma 2.2) For any* $\varepsilon > 0$, *there exists* $n_0 > 0$ *such that* $\mu_1(L_g^{(n)}) \le \mu_1(L_g) + \varepsilon$ *for* $n \ge n_0$.

We shall use the following known result(for example [1]), which can be proved by using the Leray-Schauder degree theory for compact maps in Banach space.

**Lemma 2.5** *Let* $E$ *be a Banach space,* $\Omega_1$ *and* $\Omega_2$ *be two bounded open sets of* $E$, *and* $\theta \in \Omega_1 \subset \Omega_2$, *where* $\theta$ *is zero element of* $E$. *If* $F : \overline{\Omega_2 \setminus \Omega_1} \to E$ *is compact and satisfies*

*(i)* $x \ne \lambda Fx$ *for all* $x \in \partial\Omega_1$ *and* $0 < \lambda \le 1$.

*(ii) There exists* $x_0 \in E \setminus \{\theta\}$ *such that* $x \ne Fx + \lambda x_0$ *for all* $x \in \partial\Omega_2$ *and* $\lambda \ge 0$.

*Then* $F$ *has a fixed point in* $\Omega_2 \setminus \overline{\Omega_1}$.

## EXISTENCE OF POSITIVE SOLUTIONS OF (1.1)-(1.2)

In this section, we shall use Theorem 2.1 and lemma 2.1 to prove the existence of positive solutions of (1.1)-(1.2).

**Theorem 3.1** *Suppose that* $(A_1)-(A_3)$ *and the following conditions hold.*

$(C_1)$ *There exist* $r_0 > \dfrac{2\|W\|}{p_{\min}}$, $\varphi \in L_+[0,1]$ *with* $\int_0^1 \varphi(s)ds > 0$ *and* $\varepsilon_1 \in (0, \mu_1(L_\varphi))$ *such that*

$$f(t,z) \le (\mu_1(L_\varphi) - \varepsilon_1)\varphi(t)z \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [0, r_0]. \tag{3.1}$$

$(C_2)$ *There exist* $\rho_0 > 0$, $\psi \in L_+[0,1]$ *with* $\int_0^1 \psi(s)ds > 0$ *and* $\varepsilon_2 > 0$ *such that*

$$f(t,z) \ge (\mu_1(L_\psi) + \varepsilon_2)\psi(t)z \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [\rho_0, \infty). \tag{3.2}$$

*Then (1.1)-(1.2) has a positive solution.*

***Proof***

The proof is divided into three steps.

*Step1*. Let $\Omega_1 = \{z \in C[0,1], \|z\| < r_0\}$, we prove that

$$z \ne \lambda A^* z \quad \text{for } z \in \partial\Omega_1 \text{ and } 0 < \lambda \le 1. \tag{3.3}$$

In fact, if there exist $z_0 \in \partial\Omega_1$ and $0 < \lambda_0 \le 1$ such that $z_0 = \lambda_0 A^* z_0$. Let $w_0(t) \equiv 0$ on $[0,1]$. Since $\|z_0\| = r_0 > \dfrac{2\|W\|}{p_{\min}}$, by Theorem 2.1, we know that $z_0(t) > 0$ on $(0,1)$. Similar to the poof of Theorem 3.1 step1 in [10], then we have (3.3) holds.

*Step2*. By $(A_1)$, there exists $g_{\rho_0} \in L_+[0,1]$ such that $|f(t,z)| \le g_{\rho_0}(t)$ for a.e. $t \in [0,1]$ and all $z \in [0, \rho_0]$, then $f(t,z) + g_{\rho_0}(t) \ge 0$ for a.e. $t \in [0,1]$ and all $z \in [0, \rho_0]$. And from $(C_2)$, we know $f(t,z) + g_{\rho_0}(t) \ge 0$ for a.e. $t \in [0,1]$ and all $z \in [\rho_0, \infty]$. Let $g_0(t) = g_{\rho_0}(t)$ in (2.9), then it is clear that $f$ satisfies (2.9).

In Lemma 2.2, we set $g(t) = g_0(t)$ and $h(t) = \dfrac{\varepsilon_2}{2}\rho_0\psi(t)$, then there exist $0 < a_0 < b_0 < 1$ such that

$$\frac{\varepsilon_2}{2}\rho_0\int_a^b G(t,s)\psi(s)ds \ge \int_0^a G(t,s)g_0(s)ds + \int_b^1 G(t,s)g_0(s)ds \quad \text{on } [0,1],$$

for all $0 < a \le a_0$ and $b_0 \le b < 1$.

By Lemma 2.4, there exists $n_0 > 0$ such that $\mu_1(L_\psi^{(n_0)}) \le \mu_1(L_\psi) + \frac{\varepsilon_2}{2}$ and $\frac{1}{n_0} \le a_0 < b_0 \le 1 - \frac{1}{n_0}$. And we take $n_0$ big enough to satisfy $\left(\frac{n_0 p_{\max}}{p_{\min}} + 1\right)(\rho_0 + \|g_*\|) > r_0 + \|g_*\|$. Thus, from the information mentioned above, we obtain that

$$\frac{\varepsilon_2}{2}\rho_0\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)ds \ge \int_0^{\frac{1}{n_0}} G(t,s)g_0(s)ds + \int_{1-\frac{1}{n_0}}^1 G(t,s)g_0(s)ds \quad \text{on } [0,1]. \tag{3.4}$$

Let $R = \left(\frac{n_0 p_{\max}}{p_{\min}} + 1\right)(\rho_0 + \|g_*\|)$ and $\Omega_2 = \{z \in C[0,1], \|z + g_*\| < R\}$, then it is clear that $\theta \in \Omega_1 \subset \Omega_2$.

Without the loss of generality, we may assume that $A^*$ has no fixed point in $\partial\Omega_2$ (in fact, if $A^*$ has a fixed point $z$ in $\partial\Omega_2$, then by Theorem 2.1, we know that $z(t) > 0$ on $(0,1)$ and $z = A^*z = Az$, the result is already proved). Letting $Bz = \mu_1(L_\psi^{(n_0)})L_\psi^{(n_0)}z$, and then $B(K) \subset K$ and $r(B) = 1$. The well-known Krein-Rutman theorem ([1], Theorem 19.2) shows that there exists $z^* \in K \setminus \{\theta\}$ such that $Bz^* = z^*$. And we can obtain that

$$z^*(t) = \mu_1(L_{n_0})L_{n_0}z^*(t) = \mu_1(L_{n_0})\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)z^*(s)ds. \tag{3.5}$$

Next, we prove that

$$z \ne A^*z + \mu z^* \quad \text{for } z \in \partial\Omega_2 \text{ and } \mu \ge 0. \tag{3.6}$$

In fact, if there exist $z_0 \in \partial\Omega_2$ and $\mu_0 \ge 0$ such that $z_0 = A^*z_0 + \mu_0 z^*$, then $\mu_0 > 0$ since $A^*$ has no fixed point in $\partial\Omega_2$. Together with (2.10) and (3.5), we have that

$$\|\alpha\| = \|z_0 + g_*\| = R = \left(\frac{n_0 p_{\max}}{p_{\min}} + 1\right)(\rho_0 + \|g_*\|) > \left(\frac{p_{\max}}{p_{\min}} + 1\right)(\rho_0 + \|g_*\|). \tag{3.7}$$

Lemma 2.3 implies that there exist $0 \le a_1 < b_1 \le 1$ such that $z_0(t) \ge \rho_0$ on $[a_1, b_1]$ and

$$a_1 \le \frac{p_{\max}(\rho_0 + \|g_*\|)}{p_{\min}(\|\alpha\| - \rho_0 - \|g_*\|)}, b_1 \ge 1 - \frac{p_{\max}(\rho_0 + \|g_*\|)}{p_{\min}(\|\alpha\| - \rho_0 - \|g_*\|)}.$$

From (3.7), we know that $\frac{1}{n_0} = \frac{p_{\max}(\rho_0 + \|g_*\|)}{p_{\min}(\|\alpha\| - \rho_0 - \|g_*\|)}$. Hence, we have $0 \le a_1 \le \frac{1}{n_0} \le a_0 < b_0 \le 1 - \frac{1}{n_0} \le b_1 \le 1$, and then

$$z_0(t) \ge \rho_0 \quad \text{on } \left[\frac{1}{n_0}, 1 - \frac{1}{n_0}\right].$$

By Theorem 2.1, putting $\lambda = 1$, then we know $z_0(t) > 0$ on $(0,1)$. Similar to the poof of Theorem 3.1 Step 2 in [10], then we have (3.6) holds.

*Step3.* Since the condition $(A_1)$ guarantees that $A^*$ is compact from $C[0,1]$ to $C[0,1]$. Through the above discussion and utilize Lemma 2.5, we obtain that $A^*$ has a fixed point $z$ in $\Omega_2 \setminus \overline{\Omega_1}$ and it is clear that $\|z\| > \frac{2\|W\|}{p_{\min}}$ holds. Then by Theorem 2.1, we know $z(t) > 0$ on $(0,1)$, and then $z = A^* z = Az$. This shows that $z$ is a positive solution of (1.1)-(1.2).

This completes the proof. □

**Example 3.1** We consider (1.1)-(1.2) for $p(t) = 1$ and $f(t,z) = c\max\{z-1,0\} - t^2$, where $c > \pi^2$ is a constant. Let $w(t) = -t^2$, then it is easy to know that $f$ satisfies $(A_1)$ and $(A_2)$, and $\|W\| = \frac{1}{3}$. Let $\varphi(t) = \psi(t) \equiv 1$, $r_0 = 1$ and $\rho_0 = \frac{2(c+1)}{c-\pi^2}$, it is obvious that $r_0 > \frac{2}{3} = \frac{2\|W\|}{p_{\min}}$. Notice that $\mu_1 = \pi^2$ [9], then for $\varepsilon_1 = \frac{\mu_1}{2}$ and

$\varepsilon_2 = \frac{c-\pi^2}{2}$, we have

$$f(t,z) = -t^2 \le (\mu_1 - \varepsilon_1)z \quad \text{for } t \in [0,1] \text{ and all } z \in [0, r_0],$$

$$f(t,z) \ge (\mu_1 + \varepsilon_2)z \quad \text{for } t \in [0,1] \text{ and all } z \in [\rho_0, \infty).$$

Then by Theorem 3.1, (1.1)-(1.2) has a positive solution $z$ in $C[0,1]$.

**Remark 3.1** Since $f(t,z)$ in Example 3.1 does not satisfy $f(t,0) \ge 0$ [9, 10], $\lim_{z\to\infty} \min_{a\le t\le b} \frac{f(t,z)}{z} = c < \infty$ [2], and $\int_a^b \liminf_{z\to\infty} \frac{f(t,z)}{z} dt = (b-a)c < \infty$ [6] for all $0 < a < b < 1$. Then the existing results can not be used to treat it. Hence Theorem 3.1 is a new result.

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