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Multiple server queues with vacations according to two standards of system performance (average waiting time in row Wq, average row length Lq)

Shikhah Ibrahim Abdullah ALothman
Lecturer, Prince Sattam Bin Abdulaziz University, Kharj, Kingdom of Saudi Arabia

***Corresponding author**
Shikhah Ibrahim Abdullah ALothman

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Abstract: This study aims at introducing a multi-server queues with vacations according to two system performance metrics (average waiting time in row Wq, average row length Lq). We examine how such a dynamic impacts common system measures such as stability, expected number of customers in the system, probability of waiting and expected waiting time. We have designed a method to calculate performance metrics for the previous queue system defined as average system uptime and average vacation time. We designed a method to calculate performance metrics for the previous queue system defined as the average waiting time in the row and the average grade length. We changed the different parameters of access rates, service rates, and number of servers to achieve the best results and try to improve and develop the system, making it easier for servers to perform their service To obtain the appropriate vacations for them without being affected by the customer service and the system's interest is disrupted and we obtained the following results where the results related to average waiting time in row Wq, average row length Lq were found.
Keywords: Multi-Server, Queues, vacations, Performance Metrics, waiting Time, row length

INTRODUCTION

Queuing Systems

Queuing theory is a mathematical study of *Queuing* classes or *Queuing* lines. The *Queuing* system consists of customers or units that require a type of service and the *Queuing* lines that customers enter when the service is not available immediately and there are cases where customers leave the system without interest in the *Queuing* and the service center that provides service to the customers.

The year 1910 was the real birth of the theory of *Queuing* by the Danish scientist Erlang, where he began that year to conduct experiments related to the problem of congestion in the center of telephone exchange through the workers in the telephone call center as it was found that callers are often subjected to some delays during the periods In which telephone calls are frequent because of the inability of staff to meet orders in a synchronous manner with the speed with which they occur.

The Erling scientist calculated the duration of the delay for one worker in the division and then circulated his study and the results to a number of workers. The development of telephone traffic continued on the bases set by Erling and by the end of the Second World War, the use of this method was expanded and included a number of general cases with queues [1].

Kendall [39] was one of the individuals who want to view the theory of waiting rows from the perspective of random processes.

The last century has witnessed a tremendous development in queue theory, which has been highly regarded as a research area in both operational research and operational probabilities.

The concept of the *Queuing* model is that when the customer reaches the server, he enters the waiting line, and then the service provider serves the customer from this class. When the service is completed, the service is repeated to the new customer from the queue.

It is assumed that there is no time lag between the completion of the service to a particular customer and the commencement of service to the next customer in the queue if the service provider does not need time to prepare to provide service to a new customer.

The key elements of the *Queuing* model are the customer and the service provider. In waiting queue models, we are concerned with the interaction between the customer and the service provider that affects the time required by the customer to obtain the service.

Queuing systems are generally expressed in Fig. 1, which shows that customers arrive at the waiting row system expressed in a dashed rectangle and then stand in the queue waiting for their turn to get the service and then move to the service centers (service center) Customers system after obtaining the required service [2].

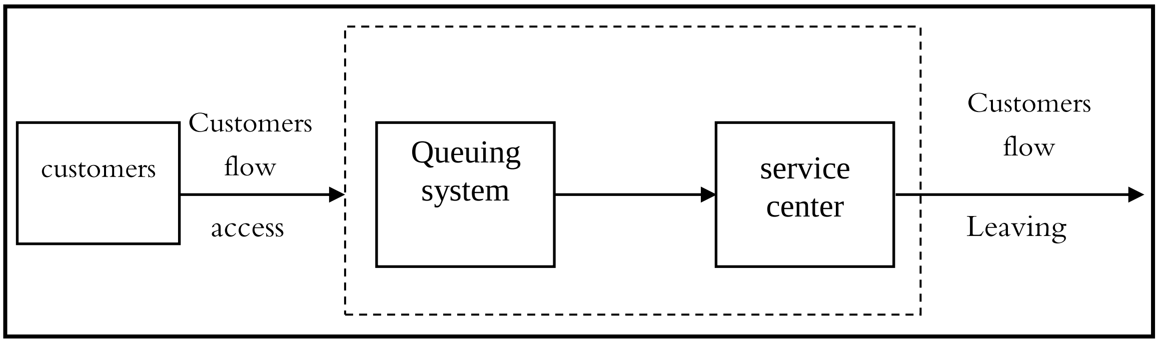

Fig-1: Queuing system

The queue consists of a group of customers (individuals or objects) who are waiting for their turn to receive the service.

The service station is where customers go to get the service they need. If we combine service centers with queues, this is known as the system and system management (the system here is the channel or service channels plus rows) [3].

The Properties of Queuing System

Queuing systems have the following characteristics:

Arrival Process

The *Arrival Process* is a description of how customer service seekers enter service stations [4].

Queue Capacity

It is the largest number of persons allowed in the system [4]

Service Discipline

The *Service Discipline* is the way in which the service is provided to customers. The most popular customer service principles are:

A. a. "Service in order of arrival" means that it first arrives, it will be first served or it first arrives first, it will first go out (First -Out- In- (First or FIFO), and this principle is the principle in most grade systems.
B. B. "Service by Reverse Access Order" or who finally arrives serves first (Last-In-First-Out) or LIFO (abbreviation). This principle is used in most storage systems where the goods that are finally stored are consumed because they are within reach.
C. C. "Random service" where items are randomly selected to serve them regardless of their arrival time (Random - In - Selection or SIRO) as in some data entry operations.
D. Service according to "Priority" or "Priority" (PRI), where preference is given to some clients such as patients arriving at hospitals or clinics in serious condition and cases of important military messages. Or be a preference according to a particular order or scale.

The percentage of time the server is busy providing service to the customer is given by the following formula: $a = \frac{\lambda}{\mu}$ Where a is the ratio between the rate of access, λ is the service rate μ is a dimensionless whose numerical value gives a measure of the service demand of the system and is known as the Erling units in honor of the Erling scientist.

The quantity $\rho = \frac{a}{c}$ is called the usage parameter where c is the number of servers where c ≥ 1 and where there is one server, so $\rho = a$ [2].

Service Time

The period of time spent in serving the client [2] (Service Stages and Number of Service Channels)

Queuing queues models in terms of the number of channels and stages of service are classified into the following models [38]:

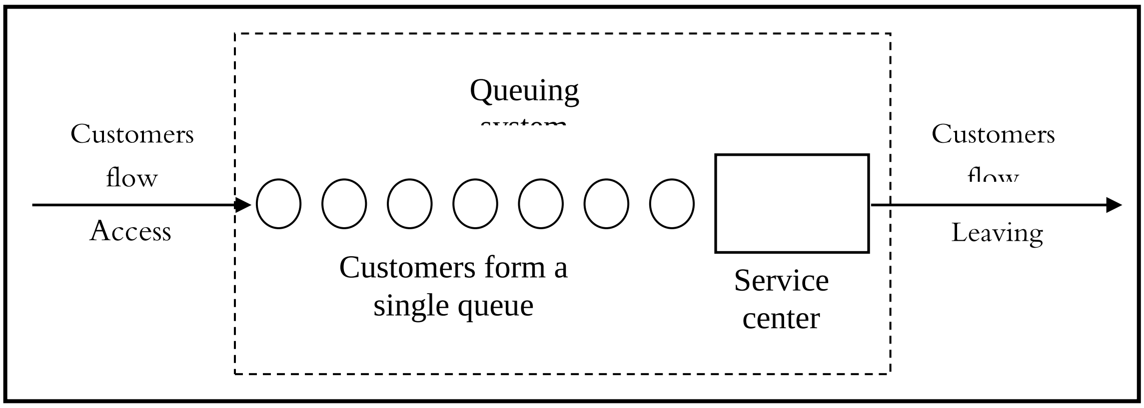

Fig-2: A one-channel and one-phase waiting queue model

For example machines in front of a single maintenance unit.

B. Multiple Stages – Single Channel Model

In this case, the customers constitute more than one waiting line on more than one stage. Each stage provides a different type of service than the other stages, as in Figure (3) below:

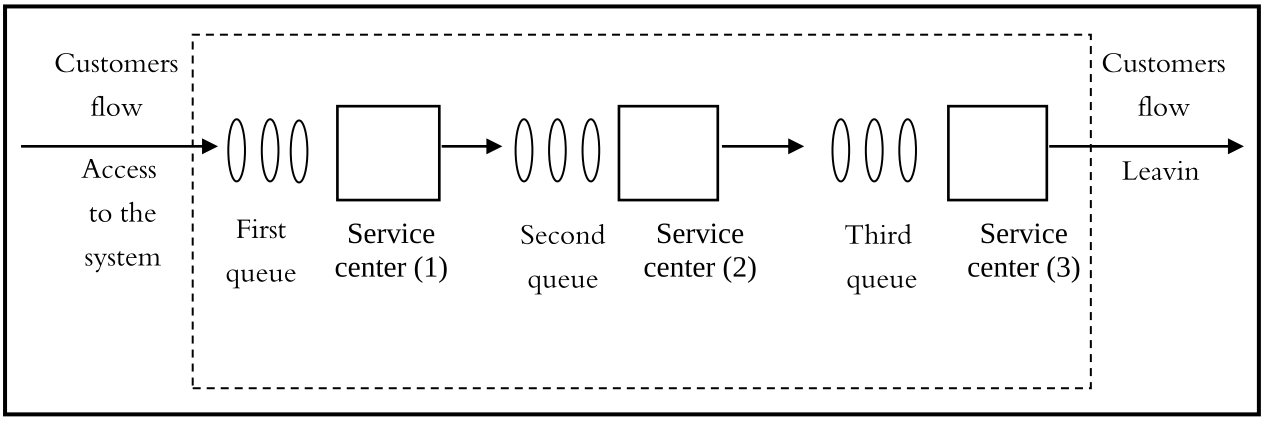

Fig-3: Multiple Stages – Single Channel Model

For example patients who move from one clinic to another for a comprehensive medical examination.

C. Single Stage - Multiple Channel Model

In this case, the service centers are multiple with one waiting queue

Queueing Applications

The theory of *Queueing* has taken the form of progress, development and prosperity both in the field of methodological theory and in practical applications.

In the field of methodological theories, we find that there are a large number of those interested in studying the models of *Queueing* and this number are university students in various stages of advanced university education and graduate students in different related disciplines.

Queuing theory has become a target for practical applications. For example, the CPU is the server and the programs that are waiting to be executed are the clients.

In the area of air traffic planning, waiting for aircraft in the air taken into consideration waiting for the existence of an empty corridor that the aircraft is expected to use in landing operations. More complex wait queues have been developed to obtain the design of traffic lights at the crossroads. queues models are also widely used in industrial fields to overcome the waiting problems that accompany some of the work. These models are also applied to determine the optimal number of berths that receive vessels at ports in order to reduce total costs [5].

Birth -and -Death Process

There are many systems that are interesting in their study that evolve over time. Their study focuses on the dynamic behavior of these systems. These systems always include randomness in some way such as [6]:

- Study of queues systems
- The number of successful students in a course over different years.
- The temperature of the air in a particular city
- Number of data packets in an information network.

The random process x_t or x (t) is defined as a sequence of random variables by index (t), which is often the time. Thus, the random process is plotted from the sample S space to a function in time t so that the x_t (j) for each element jsS can say that:

- For each value j, $x_t(j)$ becomes a function in time.
- For each value t, $x_t(j)$ becomes a random variable

The $x_t(j)$ called the function of a given value is called realization of the random operation and defines the set of possible x_t values in the state space. The set of t values is defined as the parameter space.

Random processes are classified according to the fact that the vacuum of the case or the parameter space is connected or sporadic. For example, it is possible to talk about a random process that is time-related or time-spaced according to the parameter space.

The study of random processes, for example, is concerned [7]:

A. *Time Dependent Distribution*

It is possible thatx_t takes a value in a subset of S at a known moment t.

B. *Stationary Distribution*

It is possible that x_t takes a value in a subset of S when $\infty \rightarrow t$ ((assuming the existence of end).

C. *Covariance*($\boldsymbol{x}_{\mathrm{j}}$, $\boldsymbol{x}_t$)

The heterogeneity or correlation between x_j, x_t is determined at two different times t, j.

D. *Hitting Probability*

The possibility of returning to a state of S

E. *(First Passage Time)*

Is the moment when the random process enters for the first time a case or set of cases when it starts from a certain initial state.

Queueing Systems Notations

As noted above, access and departure have their own allocations and that there are various service principles. We have also indicated that customers may be born from an end-to-end source as we have indicated that system capacity may be limited (limited to a certain number of customers) and may be unlimited. And that the system may contain one channel or several parallel channels. The scientist Kendall [39] suggested a symbolic way to denote all these things as follows:

A/B/C/X/Y/Z

Where the characters in this code mean the following:

A: It represents the distribution of the number of customers accessing the system (or for the distribution of time between two consecutive accessions). The M code is usually used to indicate that the number of clients follows the Poisson

distribution or to indicate equivalently that the time between two consecutive hits follows the exponential distribution. This indicates Markov's property (relative to the scientist of Markov, whose name is associated with random processes).
B: is the distribution of the number of customers who leave the system (or to divide the service time of a customer). The M code is usually used to indicate that the distribution of the service time of a customer follows the exponential distribution
C. Denotes the number of parallel service channels (C= 1,2,3……,∞)

X: The service principle we have seen is LIFO FIFO, SIRO, PRI. The GD symbol is used to denote a general discipline that can be any of the above principles or any other principle.
Y: The capacity of the system is the number of customers that can be accommodated by the system (in row or waiting rows and in the channel or service channels) so that, (Y= 1,2,.....,∞)
Z: Source energy is the number of source elements from which customers are generated. This energy may be terminated or incomplete (Z= 1,2,…∞)
Below are some of the known queue systems:
The symbol (M/M/1/FIFO/∞/∞)indicates a queue system characterized by:
The time interval between two consecutive access times and the time of a customer service following the exponential distribution and the system has one service channel and the service principle is "service in order of access" and the system power is unlimited and the source power is unlimited.
When the last three symbols are deleted, they indicate FIFO /∞/∞. For example, the M / M / 1 symbol denote the same concept as the previous code.

The symbol (M / M / 1 / FIFO / K */∞)* has the same previous concept except that the system energy is defined by the number K elements.

The symbol (M ^ X / M / S / N / FIFO) means that customers arrive in the system in the form of payments followed by the Poisson process and the service time follows the exponential distribution and the number of servers S server and the maximum number allowed to exist within system N, First serves the first, and X batch size which can be fixed or follows a given distribution.

The symbol (M ^ X / E_K / S / N / FIFO) means that the time between the arrival of customer payments follows Exponential distribution. The service time is followed by the K-type distribution, the number of S servers and the maximum number allowed existing within System N, the next service principle first serves first, and X the batch size is fixed or follows a certain distribution.

Now if we have a row system we mean that this system is a system of birth and death and in this case we will use the following symbols:
N
N (t): represents the random variable that represents the number of clients in the system up to the moment t where $t \geq 0$. We mean the number of customers in the Queueing system

Number of customers in a row system = number of customers in row or queue rows + number of customers served in a channel or service channel.

$P_n(t)$: Is the probability distribution of the number of customers in the system at time t

$$P_n(t) = P(N(t) = n) \qquad (1.5.1)$$

As we define the parameters $P_n(t)$ and N (t), the study of queuing systems is generally dependent on time t. However, many queue systems depend on the beginning of their work over time, but over time they reach stability.

In the stability case, the random variable N (t) and its probability distribution function $P_n(t)$ become independent of time and are denoted by N and P_n where:

$$P_n = P(n) = P(N = n) \qquad (1.5.2)$$

Performance Measures

The main objective of the study of queue systems is to achieve an appropriate level of service with a reasonable level of costs. To achieve this goal, we need to find measures called efficiency or performance measures by which we can study and analyze queue systems and then re-adjust or reconstruct these systems in such a way that we reach the desired goals.

Such as determining the optimal number of service channels and the optimal speed of customer service [2]

There are some concepts that are used in performance measures such as:
a. Arrival Time
Is the time when the customer reaches the class and is symbolized by AT.
B. Departure Time
This is the time when the customer completes the service and leaves the system and is symbolized by DT.
C. Time of Beginning The Service
Is the time when the customer leaves the class to start the service and is symbolized by the TSB
D. Time in Queue
Is the time in which the customer is waiting in line to receive the service and is symbolized by the symbol TiQ.
E. Time to stay in the system (Sojourn Time)
Is the time in which the client stays in the system until he takes the service and leaves the system and is symbolized by the symbol SJ.

Customer Performance Measures

Are measures that measure the effectiveness of the client; for Example:

a. Time in a row: while the time in the row is a little, it is better for the customer.
b. B. Service time whenever service time is short, it is better for the customer.
c. C. Waiting costs If the waiting costs are few, it is the better the customer.

Server Performance Measures

These measures measure the effectiveness of the server. Examples include:
a. Service time the faster the service time, it is the better.
B. Server downtime if few is better.
C. The rate of leaving customers from the server if large is better.

Sultan [35] calculates some of the following criteria for the waiting queue system:
1. Average customer waiting time per row. The average time between joining a customer's grade and joining the service is the following:

$$W_Q = \frac{\text{Total customer waiting times row}}{\text{The total number of customers in row}} \qquad (1.6.1)$$

2. Average customer waiting time for the system. The average time between client joining the system and completing the service is the following:

The average number of customers by grade is represented in the following relationship:

$$W_S = \frac{\text{Total customer waiting times in the system}}{\text{Total number of customers in the system}} \qquad (1.6.2)$$

$$L_Q = \frac{\text{Total number of clients waiting in row}}{\text{Total Customer time in the system}} \qquad (1.6.3)$$

4. The average number of customers in the system is represented in the following relationship:

$$L_S = \frac{\text{Total number of customers waiting within the system}}{\text{Total customer time in the system}} \qquad (1.6.4)$$

5. Service Centers may be busy while a client is accessing the system. Of the following relationship:

$$P_B = \frac{\text{Average number of busy servers}}{\text{The total number of servers within the system}} \qquad (1.6.5)$$

Little's Law

Assume that λ denotes the access rate in the unit of time, W the average waiting time in the system, and L the average number of clients in the system. The Little formula is a useful result in waiting queue systems and this formula gives the relationship between L and W as follows [8]:

$$L = \lambda W \qquad (1.7.1)$$

The relationship between the mean length of the waiting row L_Q and the wait time W_Q can be deduced:

$$L_Q = \lambda W_Q \qquad (1.7.2)$$

The relationship between the average waiting time for the client in W_S and the average number of customers in the L_S system can also be *deduced*:
:

$$L_S = \lambda W_S \qquad (1.7.3)$$

The significance of Little's formula is that the average waiting time can be calculated from the average grade length and vice versa. Thus, for example, it is enough to measure the length of the row and thus we can obtain the average waiting time so that:

$$(1.7.4) \qquad W_Q = \frac{L_Q}{\lambda}$$

Since we are going to study the M / M / c queue model, we will address the potential distributions used for this type.

Probability distributions used in the stud

Poisson distribution

The Poisson process of potential probability distributions is common in many applications, and the Poisson process of discrete probability distributions is named after Poisson. The Poisson experiment is a process that produces numerical values for a random variable X representing the number of times of an accident (or phenomenon) in a time period symbolized by the symbol t. The exponential distribution is characterized by loss of memory meaning that if we had a system, the time required for the holidays available to the servers does not depend on the previous holiday or the following in the sense that they are independent.

Characteristics of the Poisson process:

a. The probability mass function is given by relationship

$$f(x) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \quad , x = 0,1, \dots .. \qquad (1.8.1)$$

b. Distributing to a separate random variable takes non-negative values.

c. the Poisson process is expected given from the relationship:

$$\mathrm{E}[x] = \lambda t \qquad (1.8.2)$$

d. The variation of the Poisson process is given from the relationship [2]:

$$\mathrm{V}[x] = \lambda t \qquad (1.8.3)$$

The Exponential Distribution

The exponential distribution is one of the important distributions because it has many applications, most notably the waiting lines. The time period between the arrival of a client and another of the service center follows the exponential distribution. The reason for this designation is that this distribution is based on an exponential mathematical equation and this equation is $f(t) = \lambda e^{-\lambda t}$, $\lambda > 0$, $t \geq 0$ where t is the time period between the arrival of another client and λ Time unit.

The exponential distribution has to do with the distribution of Poisson. If events occur following Poisson distribution, the period between two events follows the exponential distribution. For example, if customer access to a service center follows the Poisson distribution, the time between arrivals follows the exponential distribution.

Characteristics of exponential distribution

a. Continuous distribution and identification of all non-negative real numbers.
b. B. The probability distribution function is:

$$F(x) = \int_{-\infty}^{x} \lambda\, e^{-\lambda t}\, dt \; = 1 - e^{-\lambda x} \quad , x \geq 0 \qquad (1.8.4)$$

C. Expect exponential distribution given by the relationship:

$$E(x) = \frac{1}{\lambda} \qquad (1.8.5)$$

D. The exponential distribution variation is given by the relationship [2]:

$$V(x) = \frac{1}{\lambda^2} \qquad (1.8.6)$$

Phase Type Distribution

PH is a probability distribution that results in the probability of one or more Poisson operations, a distribution that can generate random variables for time. Markov is able to achieve a Markov property for each phase. All random variables generated by this distribution are positive values.

If the distribution of PH is one phase, it can be considered an exponential distribution, but if the distribution is more than one stage, it can be considered a generalization of the Erling process.

Since we will study the multi-server wait queues with holidays, the holiday time follows PH distribution and this distribution was chosen because it is a mixture of continuous and intermittent distribution [9].

Below is the distribution of PH of grade m

Is a Markovian process with a number of (m + 1) case, which has the generation matrix as follows:

$$Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$$

Where:

- T is a square matrix of m, denoted by elements T_{ij} , $i, j = 1, \dots, m$ Check:
  1. $T_{kk} < 0$ for each values $k \leq m$ (where T_{kk} is the main diameter element).
  2. $T_{kl} \geq 0$ for each values $k \neq l$ These values give transfer rates from case k to case l.
  - T^0 is a $m \times 1$ column vector that gives rates of transition to absorbent cases (absorbing states), those cases that cannot be transferred to any other case
  - As long as Q is the smallest generator of the transition potential (we will recognize it in Part 3), where 0 is the zero vector and e is its vector.
  - Vacation Time is always defined.

The following is an example of PH class 2 distribution, the following is the distribution:

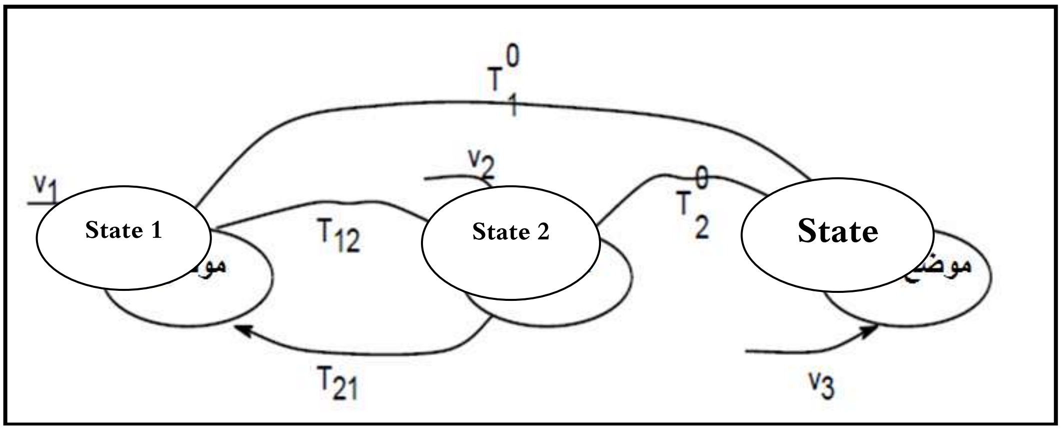

Fig-4: PH distribution of rank 2

The previous figure describes the vacations for our servers, where m = 2, ie, the number of cases, the number of cases 3, the transmission and the case of one, the system takes a holiday as long as it moves between cases 1, 2, and once it moves to the pipette with the possibility of v_3 it returns to service and when the server leaves The server switches to one of the two cases 1 or 2 with the probability of v_1 ، $v_2$respectively, which means that the server is in a vacation state [10, 11].

Queueing Previous Studies of the Systems

Many researchers have analyzed multi-server wait queues (multi-server) with vacations.

Levi et al. [12] studies queues System from the kind of with M/M/C with vacation. The aim of this study is to analyze the M / M / C waiting queue system where it is assumed that the times between arrival and service times follow the exponential distribution and the number of service operators C and the principle of service is in order of arrival (FIFO) and the system and source power is unlimited. Vacation times follow exponential distribution. The researchers obtained the distribution of the number of servers occupied and the expected number of customers in the system.

Fischer [13] also applied the function of the generated function. This study aimed to obtain the efficiency measures of a multi-channel control system and gave an analysis of the state of the system in stability mode.

Neuts [9] developed an analytical matrix (MAM) method that provides a powerful and effective tool for studying complex random systems.

Gross and Miller [14] used both random and random variables for computational analysis of Markov's operations in both sporadic and continuous cases. The researchers aimed to obtain efficiency measures for some grade systems. The researchers calculated the server occupancy times and the average number of events occurring in the system During specific time periods as they discussed many of the applied random processes.

Because of the complexity of multi-server holiday systems, birth and death are treated as inappropriate or inappropriate. The analytical matrix method was developed in the 1980s and was more suitable for analysis of the multi-holiday system, which can be described as quasi-birth-and-death-process (QBD)

Vinod [15] was the first to use the QBD process to study the exponential holiday M / M / C column and suggested using the numerical method to find constant distributions of row length and waiting time using the analytical matrix method.

Igaki [16] studied the M / M / 2 queue system, This study aims to analyze the waiting queue of type M / M / 2, where it was assumed that the times between arrival and service times follow the exponential distribution and the number of service providers and allowed only one server to take the holiday. The researcher explained the characteristics of analysis of random systems and concluded the efficiency and effectiveness of measurement and performance when there are busy servers.

Tian and Zhang [17] studied the M / M / C waiting line; it aimed at finding a clear algebraic expression of the relative rate matrix. The researchers obtained the longitudinal distribution of queues and the waiting time in the various holiday waiting systems. They established the characteristics of the analysis of the multi-server conditional random systems for both line length and waiting time and it resembles the Random systems of single-server non-conditional time-out.

Chaudhry [18] studied the M / G / 1 model. The researcher's goal was to study the model taking into account the holiday times of the service providers following the general distribution and the arrival time of payments followed by the exponential distribution and the service provider was only one.

The study of the researcher in 2003 was based on an analysis of the same model under the individual holiday policy, in which the service provider takes one holiday between two consecutive working periods. The potential distribution of the customers is queued at various points and in the case of stability.

Madan et al. [19] [M & M / 2] conducted the study of the waiting line of M / M / 2 aimed at analyzing the model of vacation servers Bernoulli plan and a single vacation, and researchers have been able to generate stable state functions of the system size of the different state of servers.

Ke J.C [20] studied the policy of N for the M / G / 1 waiting line model, where the service-based employee is disconnected from work and takes a consecutive break when the number of clients is less than N and if the number of customers reaches N, In the service delivery and also some breakdowns and repair of the service-based periods of service The researcher analyzed this model and measured the total cost expected to perform the service and got better and less expensive.

Wang [21] studied the M / G / 1 waiting row model where the service time is followed by a general distribution. Through this model, the customer service is essential and the service is optional. The researcher has obtained some measures of efficiency in the transition situation and also in the case of stability.

Wang et al. [22] has studied the M / G / 1 waiting line model and has taken care to remove the service provider when it has a temporary disability to serve all clients in each policy. The researchers used the available energy theory and reached a rough equation for distribution Potential customers in the system are in stable condition.

Kawanishi [23] conducted a study entiteled the M / M / C queue patiently or patiently; this study aimed analyzing the queuing models of the communications center of any multi-server system in a practical way, so that each server can go to one holiday. It is assumed that the access numbers follow Poisson distribution and that the times of service and holiday follow the exponential distribution. It was also assumed that the client may leave the system due to impatience and that the clients' patience times are independent and has the same potential distribution. The QBD process was used to derive the appropriate waiting row model for this case. The case of forcing PH distribution as a basic distribution of customer impatience is not appropriate, as it cannot keep pace with the growth of the size of matrices that depend on different levels, while proving that the imposition of the case of those who abstained from service and finally left before the termination of the case of practical calculation to determine the efficiency of the system.

In this study, we will analyze the multi-server M / M / c waiting queues with holidays. We describe the multi-server wait queue system, how customers access service centers and then how to leave the system immediately after receiving this service. In the systems under study, the servers or the server take a holiday independently of each other in the case of the holiday inactivity and this holiday may be multiple or single or have time to prepare if the holiday multiple, the servers repeat this holiday until the advent of a client in the system, Or servers return to the system after the end of the holiday even if there is no client in the system, or there is a time to set up when the first client arrives for the system.

In the case of simultaneous holiday, the servers take a holiday together and this holiday may be multiple or single or have time to prepare if this holiday multiple, the servers repeat this holiday until the advent of a client in the system, but if the holiday alone, the servers return to the system after the end of this Holiday even if there is no client in the system, or there is time to set up when the first client arrives for the system.

Simlution of Queueing Systems

Sometimes, in practice, there are many complex problems that are difficult to put into mathematical models or mathematical models are difficult to solve analytically. In such cases, we have to search for optimal solutions and methods that succeed in finding such optimal approximation solutions is simulation method

There are cases where simulation is recommended:

- Difficulty obtaining data about the real problem or the high cost of obtaining such data.
- The difficulty of constructing a mathematical formulation that can be used to find an analytical solution to the problem.
- There is no direct analytical way to solve the problems in question.
- The difficulty of verifying results in some experiments

Method of Randomized Trials

The randomized method of experimentation expresses the simulation method using the sample. These samples are taken from a theoretical society, where the probability distribution of the variable we are studying is determined. We then generate random variables representing these samples. Is a simulation of the random processes that arise in the system, which leads to the development of an algorithm to formulate the functions of the system studied in doing various work, in addition to monitoring and analysis of random processes that arise in the system studied and the preparation and dissemination of monitoring results [24].

Methods of Generating Random Variables

Several methods have been designed to generate random variables, including:

Linear Congruential Method

The linear Congruential method was used by lehmer in 1951, where it presents a series of integers where we find a set of random numbers that follow the regular distribution U[0,1]. This method is based on generating a set of numbers between 0 and m-1. The same sequence of random numbers when required by the following repetitive relationship:

$$X_{i+1} = (a\, x_i + c) mod\, m \quad , \quad i = 0,1,2, \ldots. \qquad (2.11.1)$$

Inverse Transformation Method

If the probability density function f(x) and the Inverse distribution function F(x) have an inverse distribution function$0 \le F(x) \le 1$, it is known that the Inverse function field F(x) Period F^(−1) (x) If y is a random number in the period [1.0], it will be:

$$F(x) = Y \qquad (2.11.2)$$

Thus,

$$x = F^{-1}(Y) \qquad (2.11.3)$$

where $F^{-1}(Y)$ Inverse function of $F(x)$ and $0 \le F(Y) \le 1$

If we want to generate random variables that follow the exponential distribution of the parameter and whose probability density p.d.f is given as follows:

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x} & ; \ x > 0 \\ 0 & ; \ otherwise \end{cases} \qquad (2.11.4)$$

The cumulative distribution function is:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx = 1 - e^{-\lambda x} \quad , x \ge 0 \qquad (2.11.5)$$

In order to generate random variables from exponential distribution using the reverse conversion method, we find the cumulative distribution function of the random variable, as in (2.11.5), and then we put $F(x) = R$ on the x field and thus the equation become:

By taking the logarithm of the parties

where $$R \sim U(0,1) \qquad (2.11.6)$$

and thus

$$1 - R \sim U(0,1)$$

$$\therefore \ x = \frac{-1}{\lambda} \ln R$$

We generate the required number of random numbers and then calculate the views of the variable [25].

(2-11-3) Acceptance Rejection Method

The method of acceptance and rejection is used to generate random variables from different distributions. The basis of this method is initially based on the generation of random numbers that follow the regular distribution in a repetitive manner and only accept them to meet certain conditions. These conditions are designed to accept the random variables that follow the required distribution Generate random numbers from it.

Convolution Method

The method of Convolution is a method for variables that consist of two or more independent random variables. If we assume that we have X, a random variable is the sum $Y_1 + Y_2 + \cdots + Y_m$ where Y_j , $j = 1, .. m$ Fixed number constant $Y_1 + Y_2 + \cdots + Y_m$ is called a wrap of variable m [25].

Composition Method

This method is a direct method for generating random variables. This method is used when the function consists of a sum of m of probability distributions as follows:

$$f(x) = \sum_{i=1}^{m} p_i f_i(x) \quad , \quad 0 \le p_i \le 1 \qquad (2.11.9)$$

The method is to generate a separate random variable K that follows the distribution:

$$P[K = i] = p_i \quad , \quad i = 1, 2, \ldots, n \qquad (2.11.10)$$

Then we generate $Y_i \sim F(x)$ the cumulative distribution function of the function $f(x)$ and then put $X(u) = Y_i$ and then calculate the value u [36].

Midsquare Method

This method was introduced by Von-Neumann in the mid-1940s, where this method starts with an initial number and is then squared. The random number is then selected after excluding the fractions and then quadrature this number to in turn generate the second random number and so on [26].

Costant Multiplier Technique

In this method, two points are chosen (X_0, X_1) so that the number X_0 is fixed and multiplied by the number X_1 and then exclude the fractions to get the random number R_i and continue as follows:

Assuming that $X_1 = 7223$ and $X_0 = 3987$ and for getting random numbers we follow the following:

$V_0 = X_0 X_1 = (3987)(7223) = 28798101 \Rightarrow X_2 = 7981$

$R_1 = 0.7981$

$V_1 = X_0 X_2 = (3987)(7981) = 31820247 \Rightarrow X_3 = 8202$

$R_2 = 0.8202$

The success of this technique depends on the choice of the constant [26].

Methods Terminating Trile

There are several criteria by which the simulation of a system under study can be terminated [27].

1. Determine a certain number of events e.g 100 customer service.
2. Determine the operating time e.g ended after 3 hours on the computer i.e 3 hours simulation.
3. According to real time such as the start of the Hajj season.
4. Internal event within the system such as malfunction of all devices.
5. An external event such as that the user decides to change the experiment.

M/M/c Queue with Asynchronous Vacations

There are two holiday mechanisms [8]:

- *Holiday service station.*
- *Holiday server.*

Chao and Zhao [37] studied the previous classification of the holiday. The study aimed at finding the stable probability of the different cases. The algorithms were created to arrive at numerical solutions. Both types of analysis were analyzed and compared by numerical observations of both systems, below we present both types as follows:

Service Station Vaction

All servers in the station get a holiday at the same time, and this happens whenever the system is empty, and then all system servers leave the system for a holiday and return to the system at one time when the holiday is over.

Server Vaction

It is the second mechanism of vacation and is often found in practice, which is the most common. In this case each server unit is independent and can take its own holiday when the customer service is finished and can be used in any other secondary work.

This phenomenon occurs, for example, in post offices where and when there is a window without a queue of customers. Here the writer goes to other types of work (sorting, distribution, etc.).

Note that if c = 1, where c is the number of servers, ie, one server system, the two holiday types match, Tian el al. [28] study that situation.

They analyzed the queue of type GI / M / 1. The objective was to study the model taking into account the holiday time of the service providers followed by the exponential distribution and the arrival time of the customer follows the general distribution and the service provider was one

If we compare multi-server system vacation models with single-server vacation models, we find that multi-server vacation models are more complex and have different dynamics than single-server vacation models [8].

Policies in Multi- Server Models (Vacation)

Here are several holiday policies for wait queue servers for many practical applications. We will review four different policies here [29]:

A. Asynchronous All Server Vacation Policy

The definition of *Asynchronous Vacation is that all servers in the M / M / c queue must start independently and randomly, where the server can take the holiday when it ends up serving one of the clients with another customer offering the service. At the moment there are servers that provide the service for other customers or on vacation or idle do not work (In the case of the only vacation).*

This holiday may take the so-called multiple vacation as the server or servers take another vacation if there is no customer or clients in the queue waiting for the completion of the previous vacation, and repeat the vacation until a customer appears in the waiting queue within the system and symbolizes this type (Asynchronus Multiple Vacation) (AS, MV).

In the case of single-server *Vacation*, the server or servers take only one *Vacation* when the queue is empty. After completing this *Vacation*, either the server or the server will remain idle or serve up to customers, and this code is marked asynchronus single vacation (AS, SV)

The difference between a multiple *Vacation* and one *Vacation* is that the server or servers on multiple *Vacation*s repeat the *Vacation* in the absence of a client or clients on the waiting queue. The only *Vacation* after the *Vacation* ends, the server or servers will be idle if there is no customer or clients on the queue.

However, if the server or servers took time to prepare before the client or customer service, this type is called the asynchronous setup time format and is abbreviated as ASynchronus Setup (AS, SU).

It is noted that the ASynchronus vacation is not comprehensive, as the servers take a single vacation on a regular basis, as long as the waiting row is zero, and the concept of the ASynchronus vacation is not the same as the concept of vacation server

B. Synchronous All-Server Vacation Policy

First, I will refer to the definition of a recursive vacation. All the servers in the M / M / C server queue start simultaneously at the same time, for example in the multi-user terminal system, where the super computer is shut down or weak Power supply can be considered a synchronized vacation.

This vacation may take the so-called multi- vacation, where all servers take another vacation together if the system is not busy for the completion of the previous vacation, and the vacation process is repeated until the customer appears in the waiting queue within the system where you resume customer service and continue servers c in the customer service, And is synonymous with Multiple Synchronization (SY, MV).

In the case of unicast servers, all c-servers take only one vacation and together when the system becomes empty once the last customer service is completed in the system. After completing this vacation either the servers will remain idle or serve up from the customers, this type is Synchronous Single Vacation (SY, SV).

Another type of synchronous vacancy is that the synchronous form is timed and synchronized Setup (SY, SU) and is called the synchronous setup time. In such a system, the system becomes empty once the last client service is completed in the system. In this case, All servers are out of service or out of business. When one or more clients arrive, c servers are going to open or run and there is time to set up before starting to work in the customer or customer service.

It is noted that the vacation is inclusive, meaning that all servers are included in the vacation.

The concept of synchronous vacation is like the concept of service station vacation.

C. Some-Server Vacation Policy

Is the number of servers allowed to take the vacation. The maximum number of servers on the vacation should not exceed the number d where $d \leq c < 0$ where c is the number of servers. For example, doctors in the hospital must specify the number of doctors allowed to take the vacation. Doctors take the vacation so there is no harm to patients so we need to determine the number of doctors who take the vacation.

D. Threshold Vacation Policy

All servers start a vacation at the moment of service completion when the system becomes busy and the servers continue to take the concurrent holiday until the number of customers in the system is at least N, ie the system starts to serve customers when the number of customers in the system N number where N = 1,2, ...

To be able to identify vacation waiting systems, we need to explain almost the process of birth and death.

Quasi-Birth-and-Death Process

Most studies of multi-server holiday models have focused on M / M / C systems, which use Markov series for queues, the most commonly used mathematical tool that changes in time. The use of Markov's method is the advantage of this method in terms of ease of use and simplicity. Since the multi-server system serving a number of customers is a system in which the number of clients changes in time, Markov's method is ideal for designing a system that demonstrates how the system handles customers.

The process of *Quasi* birth and death (QBD) is similar to the process of birth and death described previously. The difference between them is that the process of birth and death in one dimension, so it is one-dimensional, while *Quasi* birth and death is two-dimensional. In the process of birth and death, this dimension represents the number of clients in the *Quasi* birth and death in addition to the number of customers there is a second dimension which is the case of servers, and we need to add this dimension until the system is described.

The infinitesimal generator (the smallest matrix with all probability) is a QBD represented by a three-tiered matrix whose elements are matrices within the original matrix.

The process $\{X(t), J(t), t \geq 0\}$ can be defined on the vacuum of the given state Ω through:

$$\Omega = \{(k, j) : k = 0,1,2,3,\ldots, \ j = 1,2,3,\ldots m\}$$

Where k stands for the number of clients, and j represents the state of the server in terms of the presence of a holiday or its absence and m represents the maximum case.

The transition to k is limited only to the adjacent cases, but is not restricted in the case of case j, and in more detail where the system is in the case $(k, j) \in \Omega$ where Ω represents the state space, it can move to one of the following situations:

) }k, j) ، (k-1, j) ، (k+1, j { (

While you cannot move to the position with the following shape:

$\{(k \pm n, j) \in \Omega : n \geq 2\}$.

When the QBD transition method does not depend on k, it is called a homogenous QBD process or a semi-spontaneous process of birth and death.

If the transition process is dependent on k, we call it a heterogeneous QBD process, or a keratogenesis and k-dependent process.

The operation $\{X(t), J(t), t \geq 0\}$ is called QBD if the finite generator of this process is given

$$Q = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & B_3 & A_3 & C_3 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \qquad (3.3.1)$$

As all the partial arrays of matrix Q are a square matrix of class c where c is the total number of servers [30, 31].

We note that the matrix A_k where $k \geq 0$ represents a matrix in which the arrival and departure rates and the country elements in this matrix are negative numbers and the non-Qatari elements are positive numbers.

The matrix B_k where $k \geq 0$ represents a matrix with only the departure rates and all its elements positive numbers.

The C_k matrix represents a matrix with only access rates and all its elements positive numbers.

The matrix Q is required to be

$$(A_0 + C_0)e = (B_k + A_k + C_k)e = 0, k \geq 1$$

Where e is the vector of all its elements equals the correct one [Neuts (1981)

The group {0,1}, ..., (0, c)} expresses the absence of clients, called boundary levels

The group*{(k,1),.....,(k,c)}* expresses the existence of a number of clients, k where k = 1,2,3 ..., called non-boundary levels, calledA_0, C_0, $B_{1,}$ border matrices.

In some applications, if the micro-generator (infinite) matrices are not defined and independent and do not depend on k, the matrix Q is written on the image [29]:

$$Q = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A & C & & & \\ & B & A & C & & \\ & & B & A & C & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \qquad (3.3.2)$$

where
A_0 Matrix of rank $m_1 \times m_1$
C_0 Matrix of rank$m_1 \times c$
B_1 Matrix of rank $c \times m_1$
m_1 represents the number of cases at the level *k=1*

Matrix $A_0, C_0, B_{1,}A, B$, can be explained by the following QBD model:

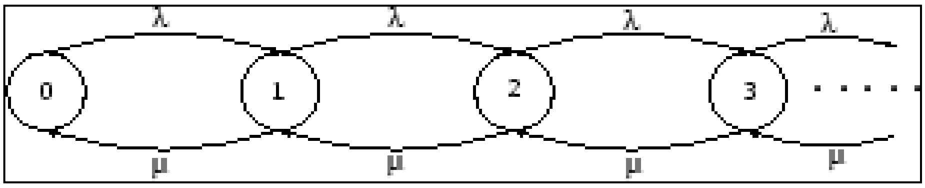

Fig-(3-1): shows the status transition diagram of the M / M / 1 waiting queue model

Figure (3-1) Example of Operation (QBD (In this case, the matrix Q is on the image:

$$Q = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A & C & & & \\ & B & A & C & & \\ & & B & A & C & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$A_0 = (-\lambda) \quad , \quad A = \begin{pmatrix} -(\mu+\lambda) & \mu \\ 0 & -(\mu+\lambda) \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \mu \\ 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$$

$$C_0 = (\lambda \quad 0) \quad , \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

The following can be analyzed (QBD) in standard or legal status (canonical form):

(QBD) is a positive recurrence process, that is, the same state can be accessed more than once through a process that is controlled by Markov chains. The number of system accessers follows Poisson distribution at λ, and the number of servers in the system also follows the distribution and the time between each successive access follows the exponential distribution at λ. The wait time in the queue follows the exponential distribution with an average of μt, and the probability of moving from case i to state j in n is a step given by the relationship

$$\pi_{ij}^{(n)} = (X_n = j \,/\, X_0 = i) \quad , \quad n \geq 0 \, , i,j = 0,1,2,\ldots$$

WhereX_0represents case i and is a stand-alone random variable, X_{n} represents case , which is a stand-alone random variable.

If $X(t)$represents the number of clients at the moment t and $\mathrm{J(t)}$ represents the number of servers occupied.
X = t = = X and that: (Jt) = J (t) = J Therefore the stability probabilities can be defined as follows:
Assume that $\underset{t\to\infty}{Lim}\ X(t) = X$ and $\underset{t\to\infty}{Lim}\ \mathrm{J(t)} = \mathrm{J}$, so Stability probabilities can be defined as follows:

$$\pi_{ij} = P(X = i, J = j) = \lim_{t\to\infty}(X(t) = i, J(t) = j)$$

Where π_{ij} is the probability of being in case j of the previous case I and $\pi_{\mathrm{i}} = (\pi_{i1}\pi_{i2} \ldots \pi_{ic})$, is the row i in the transition matrix where $(i,j) \in \Omega$ is the vacuum of cases Ω [9].

Below we will present some of the special theories of the process (QBD) without proof [29]:

Theory 1
The *irreducible* birth and death (QBD) is recognized as where any case can be accessed from any other case that is a recurring positive means that we can reach the same situation more than once during Markov's process, if and only if:

1. Array equation: $R^2B + RA + C = 0$ has the least non-negative (R or R), the matrix rate and the A, B and C matrices previously mentioned (3.3.2).
2. The set of homogeneous linear equations on the image has a positive solution π_0 and satisfies the normal condition: $\pi_0(A_0 + RB_1) = 0 \quad \pi_0(I-R)^{-1}$, where e is the vector of all its elements equal to the correct one.
3. Stable probability distribution can be expressed in a sequence of geometric matrices As follows: $\pi_l = \pi_{l-1}R \quad , \quad l = 1,2,3 \ldots$

Thus, the final formula is as follows: $\pi_l = \pi_0 R^l \quad , \qquad l \geq 0$

In many practical applications, the properties and variables defined in the QBD process are used in the standard image (QBD) in the non-canonical image called QBD with complex boundary behavior Which does not depend on the fact that the $l = 0$ level is only the boundary, but the value l changes and takes the values $l = 0,1,2, \ldots$ c, so that the limits become variable levels and we note the following:

The infinite probability generation matrix Q has the same pattern as the previous:

$$Q = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A & C & & & \\ & B & A & C & & \\ & & B & A & C & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

A_0 Matrix of rank $m_1 \times m_1$

B_1 Non-square matrix $c \times m_1$

C_0 Non-square matrix $m_1 \times c$

Where $m_1$represents the number of cases at the level k = 1

The following is the theory of non-standard image:

Theory 2

(QBD) is irreducible (Positive recurrent) if and only if:

1. Equation of the following matrices: $R^2B + RA + C = 0$ has the least non-negative solution R, since all matrices in the previous equation were defined by theory (3-3-1).
2. The homogeneous linear equations on the image $(\pi_0, \pi_1)B[R] = 0$ have a positive solution, where:

$$B[R] = \begin{bmatrix} A_0 & C_0 \\ B_1 & A + RB \end{bmatrix}_{(c+m_1)\times(c+m_1)}$$ Square matrix

(π_0, π_1) Achieves the normal condition $\pi_0 e + \pi_1 (I - R)^{-1} e = 1$

3. Stable probability distribution can be expressed in a sequence of geometric matrices where:

$$\pi_l = \pi_{l-1} R \quad , \qquad l = 2,3, \ldots$$

The final version is therefore as follows: $\pi_l = \pi_1 R^{l-1} \quad , \qquad l = 2,3, \ldots.$

4. The probabilistic distribution of the previous case π_l where $l \geq 0$ is called the modified matrix distribution.

In this case, the probability generation matrix Q differs from that in the case of the standard process only in the probability of transition from the border situations.

When parsing multiple server wait queues with M / M / c vacation we will use QBD in a more complex and variable way. There is no single boundary level $l = 0$, but the limits are multiple to $l = 1, 2, 3, \ldots, c$, and those levels contain a different number of cases, and the number of cases at level l will be denoted by m_l where: $0 \leq l \leq c - 1$

The infinite micro generator gives the image:

$$Q = \begin{bmatrix} A_0 & C_0 & & & & & & \\ B_1 & A_1 & C_1 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & B_{C-1} & A_{C-1} & C_{C-1} & & & \\ & & & B_C & A & C & & \\ & & & & B & A & C & \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (3.3.3)$$

Where
A_k Matrix of rank $m_k \times m_k$, $0 \le k \le c-1$
$\mathrm{B_k}$ Matrix of rank $m_k \times m_{k-1}$, $0 \le k \le c-1$
$\mathrm{C_k}$ Matrix of rank $m_k \times m_{k+1}$, $0 \le k \le c-2$
C_{c-1} Matrix of rank $m_{c-1} \times c$
$\mathrm{B_C}$ Matrix of rank $c \times m_{C-1}$
A,B,C Square matrices of rank $c \times c$

The matrix Q in (3.3 3) can be divided as follows:

$$(3.3.4) \quad Q = \left[\begin{array}{cc|cccc} \mathcal{A}_0 & \mathcal{C}_0 & & & & \\ \hline \mathcal{B}_1 & A & C & & & \\ & B & A & C & & \\ & & B & A & C & \\ & & \ddots & \ddots & \ddots & \end{array}\right]$$

where $\mathcal{A}_0$ *Square matrices of* rank $m^* = m_0 + \dots + m_{c-1}$

$$\mathcal{B}_1 = (0, B_k, k = C) \text{ ، } \mathcal{C}_0 = \begin{bmatrix} 0 \\ \mathrm{C}_{c-1} \end{bmatrix}$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{C-2} & A_{C-2} & C_{C-2} \\ & & & & B_{C-1} & A_{C-1} \end{bmatrix}$$

where:
$\mathcal{B}_1$ Non - square matrix of rank $c \times m^*$
$\mathcal{C}_0$ Non - square matrix of rank $\mathrm{m}^* \times c$
According to the above, the previous theory can be reformulated as follows:

Theory 3
Non-reducing *QBD be positive and repeated if and only if:*

1. Equation of the following matrices: $R^2 B + RA + C = 0$ has the least non-negative solution R.
2. Homogeneous linear equations is on the image

$$(0 = \pi_0, \pi_1, \dots, \pi_C)B[R]$$

It has a positive solution $((\pi_0, \pi_1, \dots, \pi_C)$, where:

$$B[R] = \begin{bmatrix} A_0 & C_0 & & & & & \\ B_1 & A_1 & C_1 & & & & \\ & B_2 & A_2 & C_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B_{C-1} & A_{C-1} & C_{C-1} & \\ & & & & B_C & A+RB & \end{bmatrix}$$

The positive solution $((\pi_0, \pi_1, \ldots, \pi_c)$ satisfies the standard requirement, where

$$\sum_{l=0}^{c-1} (\pi_l e + \pi_c [I-R]^{-1} e) = 1$$

3. In addition, the stable probability distribution of the M / M / c queues model with holidays can be expressed mathematically as a geometric matrix expression as follows:

$$\pi_l = \pi_c R^{l-c}, l \geq c$$

Conditional Stochastic Decomposition

Assuming that the two-dimensional non-negative vector (X, J) has a common probability distribution:

$$\pi_{kj} = P(X = k, J = j), k \geq 0, 0 \leq j \leq c$$

Assuming that: $\pi_k = (\pi_{k0}, \pi_{k1}, \ldots, \pi_{kc}), k \geq 0$

Also assuming that: (X, J) follows the geometric matrix distribution (a matrix of matrices follows the geometric distribution), and there is a non-negative matrix of grade c + 1 and therefore:

$$\pi_k = \beta R^k, \quad k \geq 0 \;\; ; \quad \beta (I-R)^{-1} e = 1$$

When k = 0, $\beta = \pi_0 = (\beta_0, \beta_1, \ldots, \beta_c)$

If R is a top trigonometric matrix (all elements below the main diameter are equal to zero) it can be divided as follows:

$R = \begin{bmatrix} H & \eta \\ 0 & r \end{bmatrix}$ where H is a square matrix, cxc, and η is a cx1 column vector, and r is a real number.

We know the following conditional random variable $X^{(c)} = \{X | J = c\}$, that is, $X^{(c)}$ is a random variable equal to X provided that J is equal to c.

The variable $X^{(c)}$ is the sum of two independent random variables: $X^{(c)} = X_0 + X_d$

Where X_0 follows the engineering distribution with parameter r, and X_d follows the distribution of PH of class c and has the function generating the following probabilities:

$$X_d(z) = \frac{1}{\sigma} \{\beta_c + z(\beta_0, \beta_1, \ldots, \beta_{c-1})(I - zH)^{-1} \eta\}$$

where

(3.4.1) $$\sigma = \beta_c + (\beta_0, \beta_1, \ldots \beta_{c-1})(I - H)^{-1} \eta$$

If R is a trigonometric matrix, $R = \begin{bmatrix} r & 0 \\ \xi & H \end{bmatrix}$ (all elements above the main diameter are zero), where H is a square matrix $c \times c$ and ζ is a cx1 vector and r is a real number located in category *(0 , 1)*.

Thus we know the conditional random variable $X^{(c)}$ as follows:

$X^{(c)} = \{ X | J = 0$, that is, $X^{(c)}$ is a random variable equal to X provided that J Equals 0.

The variable$X^{(c)}$ is the sum of two independent random variables: $X^{(c)} = X_0 + X_d$

Where X_0 follows the engineering distribution with parameter r, and X_d follows the distribution of PH of class c and has the function generating the following probabilities:

where

$$X_d(z) = \frac{1}{\sigma}\{\beta_c + z(\beta_0, \beta_1, \dots, \beta_{c-1})(I - zH)^{-1}\zeta\}$$

$$\sigma = \beta_c + (\beta_0, \beta_1, \dots \beta_{c-1})(I - H)^{-1}\zeta \quad [32] \qquad (3.4.2)$$

We will also discuss the different models of the non-stop waiting queue systems, which include:

- Multiple vacation models.
- Single vacation models and setup time

(Multiple Vacation Model)

In this case, we find that in a multi-service system M / M / c at the λ access rate and μ service rate, any server starts the vacation once a customer has finished serving and no client is waiting in the system.

At the end of the vacation for a server, if there is no client waiting in the queue, the server starts on another vacation, but if there are clients waiting in the queue, the server starts to serve them immediately.

Since servers start their own vacations independently of each other, this system is called the inertial vacation model and is symbolized by the symbol M / M / c (AS, MV).
We will assume the following:

- Vacation time follows
- Non-square matrix $\mathcal{B}_1 = (0 \quad B_C)$, $(c+1) \times c^*$
- $\mathcal{C}_0$ Non-square matrix $\mathcal{C}_0 = \begin{pmatrix} 0 \\ C_{C-1} \end{pmatrix}$, $c^* \times (c+1)$
- A_k square matrix of rank $(k+1)$, $0 \le k \le c-1$

- Square matrix of rank known as the following:

$$A_k = \begin{pmatrix} -h_0 & c\theta & & & \\ & -h_1 & (c-1)\theta & & \\ & & \ddots & \ddots & \\ & & & -h_{k-1} & (c-k-1) \\ & & & & -(\lambda + k\mu) \end{pmatrix}$$

where h_k Is a function in λ ،μ ،θ defined as follows

$h_k = h_k(\lambda, \mu, \theta) = \lambda + k\mu + (c-k)\theta,\quad 0 \le k \le c$

- $A_0 = -\lambda$ ،$B_1 = \begin{pmatrix} 0 \\ \mu \end{pmatrix}$ ،$C_0 = (\lambda \quad 0)$
- B_k Non - square matrix of rank $(k+1) \times k$, $0 \le k \le c-1$,

$$B_k = \begin{pmatrix} 0 & & & & \\ 0 & \mu & & & \\ \vdots & & 2\mu & & \\ & & & \ddots & \\ & & & & (k-1)\mu \\ 0 & & & & k\mu \end{pmatrix}_{(k+1)\times k}$$

- C_k Non - square matrix of rank $(k+1)\times(k+2), 0 \le k \le c-1$

$$C_k = \begin{pmatrix} \lambda & & & & 0 \\ & \lambda & & & \vdots \\ & & \ddots & & \\ & & & & \\ & & & \lambda & 0 \end{pmatrix}_{(k+1)\times(k+2)}$$

$$A = \begin{pmatrix} -h_0 & c\theta & & & \\ & -h_0 & (c-1)\theta & & \\ & & \ddots & \ddots & \\ & & & -h_{c-1} & \theta \\ & & & & -h_c \end{pmatrix}_{(c+1)\times(c+1)}$$

- *B A square diagonal matrix of the rank c + 1*

$$B = diag(0, \mu, 2\mu, \ldots c\mu) = \begin{pmatrix} 0 & & & & \\ & \mu & & & \\ & & 2\mu & & \\ & & & \ddots & \\ & & & & c\mu \end{pmatrix}_{(c+1)\times(c+1)}$$

- C is a square diagonal matrix of c + 1

$$C = \lambda I = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix}_{(c+1)\times(c+1)}$$

(Levy and Yechiali [12], Vinod [15], Tian and Li [33])

The analysis of the system of any queue to access the system stability condition and the condition of the system stability in the model used is that the access rate is lower than the departure rate. This is achieved if: $\lambda < c\mu$, the split of the two extremes on cμ results in: $\rho = \lambda/_{c\mu} < 1$ stability condition, R is the solution of the equation of the famous matrices:

B + RA + C = 0 R^2, *and in order to get to them the following theories must be used* [29]:

Theory 1

If $\rho = \lambda / c\mu < 1$, the following quadratic equation has two different solutions:

$k\mu z^2 - [\lambda + k\mu + (c-k)\theta]z + \lambda = 0 \,; 1 \le k \le c$, and the two solutions $r_k < r_k^*$ achieve the following condition: $0 < r_k < 1, r_k^* \ge 1$

Theory 2

If ρ <1 is the least negative solution R (where R is defined as in theory 1 is given by the following square *(c+1)* matrix:

$$R = \begin{pmatrix} r_0 & r_{01} & & r_{0c} \\ & r_1 & \cdots & r_{1c} \\ & & \ddots & \vdots \\ & & & r_c \end{pmatrix} \quad (3.5.2)$$

where

$0 < r_0 = \lambda / (\lambda + c\theta) < 1$, $0 < r_k < 1, 1 \le k \le c-1$ are roots that were previously defined in Theory 1.

- $r_c = \rho < 1$.
- *Non- Diameter elements (ie, not located on the main country) achieve the following frequency relationship:*

(3.5.3) $j\mu = \sum_{i=k}^{j} r_{ki} r_{ij} + (c-j+1)\theta r_{k,(j-1)} - [\lambda + j\mu + (c-j)\theta] r_{kj} = 0$

A relationship that gives the value of the element r_{kj} in terms of the next element in the same column directly $r_{k(j+1)}$ which implies that j> k, so that:

$$0 \le k \le c-1 \,, k+1 \le j \le c$$

It is clear that the calculation of $r_{kk}= \; r_k$ values can be computed directly from the square root value in theory (3.5.1) as follows:

$$r_k = [\lambda + k\mu + (c-k)\theta - \sqrt{(\lambda + k\mu + (c-k)\theta)^2 - 4\lambda k\mu}] \; / [2k\mu]$$

However, the calculation of non- *Diameter* values r_{kj} is very difficult because it is calculated sequentially by calculating its previous values and the following figure (3-2) shows an example of the case of c = 4:

$$\boxed{\begin{array}{ccccccccc} r_0 & \rightarrow & r_{01} & \rightarrow & r_{02} & \rightarrow & r_{03} & \rightarrow & r_{04} \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & r_1 & \rightarrow & r_{12} & \rightarrow & r_{13} & \rightarrow & r_{14} \\ & & & & \uparrow & & \uparrow & & \uparrow \\ & & & & r_2 & \rightarrow & r_{23} & \rightarrow & r_{24} \\ & & & & & & \uparrow & & \uparrow \\ & & & & & & r_3 & \rightarrow & r_{34} \\ & & & & & & & & \uparrow \\ & & & & & & & & r_4 \end{array}}$$

Fig-(3-2): Sequential calculation of the non-national values of matrix R in the case of c = 4

After the previously mentioned, we assume that:

The stable end of the random operation: $\{[L_v(t), J(t)], t \geq 0\}$ specified to the system *M/M/c (AS, MV)* is (Lv, J) when t becomes ∞ .
The stable probability of transition from case k to state *j* jis defined as follows:

$$\pi_{kj} = P\{L_V = k, J = j\} = \lim_{t\to\infty} P\{L_V(t) = k, J(t) = j\}, (k, j) \in \Omega$$

where Ω is Vacuum cases
Below is the vector form of the transition probability:

$$\pi_0 = \pi_{00}, \pi_1 = (\pi_{10}, \pi_{11}), \dots \pi_k = (\pi_{k0}, \pi_{k1}, \dots, \pi_{kk}), 0 \leq k \leq c$$

and in case k > c All probability vectors are equal:

$$\pi_k = (\pi_{k0}, \pi_{k1}, \dots, \pi_{kc})$$

The following theory illustrates the serial relationship of probability vectors:

Theory 3
If ρ <1, stable probability vectors investigate the relationship:

$$\lambda\, \pi_k e = \pi_{k+1} T_{k+1}^0$$

where

$$T_k^0 = (0, \mu, 2\mu, \dots, k\mu)^T, \qquad 0 \leq k \leq c-1$$

$$T_k^0 = (0, \mu, 2\mu, \dots, c\mu)^T, \qquad k \geq c$$

it is the relation Which gives the vector π_{k+1} *in vector* π_k.

For the comparison of M / M / c (AS, MV) and traditional M / M / c we do:

First, we identify the probability of existing a number of clients in the system M / M / c (AS, MV) (Lv - c) provided that all the servers are busy and in the form:

$$L_v^{(c)} = P\{L_v - c / J = c\}$$

Second, we know β_k: is the positive solution to equation arrays:

(3.5.4) $\quad (\pi_0, \pi_1, \dots, \pi_c) B[R] = 0$

Where matrix B [R] is a square matrix of rank $\frac{1}{2}(c+1)(c+2)$ on the image

$$B[R] = \begin{pmatrix} \mathcal{A}_0 & \mathcal{C}_0 \\ \mathcal{B}_1 & A + RB \end{pmatrix} = \begin{pmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{C-1} & A_{C-1} & C_{C-1} \\ & & & & B_C & A + RB \end{pmatrix} \qquad (3.5.5)$$

Third, we rephrase βc, B [R] as follows:

$$\beta_c = (\beta_{c0}, \beta_{c1}, ..., \beta_{cc}) = (\delta, \beta_{cc}), \delta = (\beta_{c0}, \beta_{c1}, ..., \beta_{c(c-1)})$$

(3.5.6) $$R = \begin{pmatrix} H & \eta \\ 0 & \rho \end{pmatrix}, H = \begin{pmatrix} r_0 & r_{01} & \cdots & r_{0(c-1)} \\ & r_1 & \cdots & r_{1(c-1)} \\ & & \ddots & \vdots \\ & & & r_{c-1} \end{pmatrix}_{c\times c}, \eta = \begin{pmatrix} r_{0c} \\ r_{1c} \\ \vdots \\ r_{(c-1)c} \end{pmatrix}$$

The following theories illustrate the relationship between M / M / c (AS, MV) and the traditional system M / M / c:

Theory 4

If ρ <1, the segment $L_v^{(c)}$ can be divided into two parts: $L_v^{(c)} = L_0^{(c)} + L_d$, where:

$L_0^{(c)}$ Is the random variable that corresponds to the corresponding variable in M / M / c, the number of clients waiting in queue, provided that all servers are busy and follows the engineering distribution.

and L_d is The length of the extra line resulting from the vacation, follows the PH distribution of the class *c:*
where

$$P\{L_d = k\} = \begin{cases} \frac{1}{\sigma}\beta_{cc} & , \quad k = 0 \\ \frac{1}{\sigma}\delta H^{k-1}\eta, & k \geq 1 \end{cases} \qquad (3.5.7)$$

where

$$\sigma = \beta_{cc} + \delta(I - H)^{-1}\eta$$

Sengupta [32] has demonstrated that the distribution of the private PH in L_d must be a sporadic PH distribution of class c and that it has derived its distribution function.
It was also concluded that predicting the conditional random variable $L_v^{(c)}$

$$E[L_v^{(c)}] = \frac{1}{1-\rho} + \frac{1}{\sigma}\delta(I - H)^{-2}\eta$$

We also define the conditional waiting time: $W_V^{(c)} = (W_V | J = c)$. The following is the theory of conditional waiting time:

Theory 5

If ρ <1, the segment $W_v^{(c)}$ can be divided into two parts: $W_v^{(c)} = W_0^{(c)} + W_d$, where:

$W_0^{(c)}$ Is the random variable that corresponds to the corresponding variable in M / M / c Any waiting time in the queue without vacation provided that all servers are busy, which follows the exponential distribution with the parameter cμ (1-ρ).

W_d Is the delay time of the return of the vacation and follows the distribution of exponential matrices where:

(3.5.8) $$P\{W_d \leq x\} = 1 - \frac{1}{\sigma}\delta.\exp\{-c\mu(I - H)x\}(I - H)^{-1}\eta$$

It was also possible to infer the expectation of conditional waiting time:

$$E[W_v^{(c)}] = \frac{1}{c\mu(1-\rho)} + \frac{1}{c\mu\sigma}\delta(I - H)^{-2}\eta = \frac{1}{c\mu}E[L_v^{(c)}]$$

(Sengupta [32]*)*

(Single Vacation and Setup Time Models)

Assuming that we are studying the M / M / c (AS, SV) system model as a single vacation:
In this system, every server gets a single vacation with a fixed time at the moment it is terminated for the service of a customer, and then returns to the service of a customer if someone is in the queue, or remains idle.

Thus there are only three cases per server:

1. To serve a customer.

2. Have a vacation.
3. To be without work.

M/M/c Queue with Synchronous Vacations

We have mentioned earlier that there are multiple vacation policies for queue servers that are suitable for many applications and we have introduced one of these policies, which is the holiday policy of all the servers that are not synchronized, and here we will be interested in studying another policy which is the vacation policy of all servers synchronized whether this holiday multiple or single or have time Prepare and then build a system to evaluate the performance of this type of recess.

M/M/c Queue with Synchronous Vacations

In this section we will learn about the different models of intercontinental waiting line systems that include:
• Multiple *Vacation* model
• Single *Vacation* models and setup time

Multiple Vacation model

The M / M / c multi-server queue system has the λ access rate, the μ service rate, the FIFO service provisioning system, and the c server system. The stability requirement for this system is [40]:

$$\rho = \lambda(c\mu)^{-1} < 1$$

In the system M / M / c (SY, MV):

The number of customers at the moment *t* in this system is indicated by $L_v\ (t)$

Then we identify J(t): the function that indicates the existence of a vacation for servers or lack thereof and is defined as follows:

$$\mathrm{J(t)} = \begin{cases} 0 & \textit{if servers are not on vacation,} \\ \mathrm{j} & \textit{if} \text{ servers are on vacation at phase j, j} = 1,2..\mathrm{c} \end{cases}$$

Because vacations are synchronized, at least one server is busy during the non-vacation period, and some servers may not be working.

In the single synchronous holiday system, which is code M / M / c (SY, SV), all unoccupied servers take a single vacation together at the same time as the completion of the service for all customers, at which time the system becomes empty and at the end of the vacation either Servers remain idle, or serve customers if a customer (or clients) appears to request service from the system [34]. (Computational Analysis of Synchronous Vacations Systems)

Analysis of *Synchronous Vacations Systems* with simultaneous vacations based on:
1- Structural structure of the system under study.
2. System operation policies.

(System Structure)

System Structure is the basic rules on which the system is based:
1. *The system consists of a set of servers and their number s where s = 1,2 ....*
2. *The system allows an infinite number of clients with i where i = 1,2 ....*
3. *Customers arrive at the system in a random way.*
4. *Queue capacity is not finished*
5. *Customers queue up to order service from multiple stations or service centers.*
6. *Service time is random.*
7. *Generate access times for customers from exponential distribution at λ.*
8. *Generation of service times according to exponential distribution at* μ_s
9. *Randomly generate vacation times from exponential distribution at a rate of* μ_V

Operating Policies

In order to run this system, we are working on a model to simulate this system and running this system through the following steps:
1. Assume that the number of customers requesting service is i where i = 1,2 ... and to generate the arrival time for customers from exponential distribution at λ according to Banks [27].

AT(i+1)=AT(i)+IAT(i), i=1,2,… (4.3.1)

Where AT(i) is the time of client access *i* , IAT(i) time between client *i* and *i+1 AT(i+1)* client arrival time.

2. Service time varies from one customer to another depending on the status of the service provider where the customer can receive the service directly if the service provider is idle or waiting in line until the service provider finishes providing service to the previous customer.
3. The service principle that will be used from first comes first served (FIFO).
4. Randomly generate vacation times from exponential distribution at a rate of μ_V
5. Customers are serviced from customer 1 to client S where S is the number of servers on the system without waiting to assume that the system receives the first S customer without giving a holiday to the system when the system starts
6. As for the rest of the customer from customer S + 1 to the last client can indicate the timing of the completion of the provision of the server i for the client to whom this server provides the required service With Dts (i) where i = 1,2, ... S. Therefore, we have a vector of S elements, DTs, which represents each of its elements. When each S server can provide the service to a new customer after its end to provide its current customer service.

RESULTS AND DISCUSSION

This section shows the results and evaluation of the performance of the wait-and-wait system. We have sought to obtain the best results for the calculation of performance measures. We have worked to change the λ access rate, which varies from 5client to 20 client., 10 The number of servers c takes values. We also imposed different values for exit rates that take μ = 1,3,5,10 and the number of servers c takes values $c = 1,2,3,4,5$

We designed a method to calculate performance metrics for the previous queue system defined as the average waiting time in the row and the average grade length. We changed the different parameters of access rates, service rates, and number of servers to achieve the best results and try to improve and develop the system, making it easier for servers to perform their service To obtain the appropriate vacations for them without being affected by the customer service and the system's interest is disrupted and we obtained the following results:

Standard 1: Average waiting time in row

We found that the average waiting time in the row was clearly affected by the number of servers, the λ access rate and the service rate μ, which resulted in a change in the value of $\rho = \frac{\lambda}{\mu}$, where we found the following:

A. The higher the access rate of λ for the same number of servers, the average waiting time in the class is clearly increased, ie, there is a direct correlation between them. The increase in one of them results in an increase in the other and we can observe this increase from Table (1).

B. The higher the rate of service μ for the same value of the access rate and the number of servers itself, the lower the waiting time in the row, ie, there is an inverse relationship between them and we notice this decrease from Table (1).

C. As for the number of servers, the greater the number of servers and the values of both the access and service rates, the lower the average waiting time in the row, and we notice this decrease from Table (1).

In general, the lower the value of ρ, the more stable the system, and the values converge to a fixed value, which shows the average waiting time in the row as in the following table:

Table–1: Average waiting time in row

	Λ	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	c = 1	1.979	16.668	27.293	39.791	46.079	51.876	56.372	60.859	63.871	66.629	69.235	71.559	73.119	74.679	76.239	77.799
	ρ	5.00	6.00	7.00	8.00	9.00	10.00	11.00	12.00	13.00	14.00	15.00	16.00	17.00	18.00	19.00	20.00
	c = 2	2.703	17.172	28.647	36.806	44.166	50.591	55.080	58.855	62.809	65.609	67.762	69.872	71.492	73.112	74.732	76.352
	ρ	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00
μ = 1	c = 3	4.687	16.615	27.162	35.529	42.149	48.020	52.708	56.146	59.504	62.068	64.598	66.320	67.840	69.360	70.880	72.400
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 4	1.354	13.443	25.868	34.636	41.198	46.841	51.562	55.210	58.460	61.401	63.641	65.632	67.212	68.792	70.372	71.952
	ρ	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	c = 5	1.667	13.019	21.759	30.499	38.126	44.271	48.299	52.881	55.815	58.623	61.271	63.313	64.833	66.353	67.873	69.393
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
	c = 1	0.0217	0.0367	0.0517	0.0683	0.0993	0.1191	0.1413	0.1887	0.3379	0.6087	1.8043	2.9276	4.3261	5.7826	7.3033	8.5000
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 2	0.0217	0.0367	0.0517	0.0679	0.0949	0.1222	0.1569	0.2039	0.3265	0.5127	1.0990	2.3261	4.1485	5.8850	7.2062	8.3913
	ρ	0.83	1.00	1.17	1.33	1.50	1.67	1.83	2.00	2.17	2.33	2.50	2.67	2.83	3.00	3.17	3.33
μ = 3	c = 3	0.0217	0.0337	0.0457	0.0577	0.0745	0.1001	0.1249	0.1923	0.2488	0.4596	0.7608	2.2826	4.1760	5.7173	6.9097	8.0000
	ρ	0.56	0.67	0.78	0.89	1.00	1.11	1.22	1.33	1.44	1.56	1.67	1.78	1.89	2.00	2.11	2.22
	c = 4	0.0217	0.0327	0.0437	0.0547	0.0661	0.0811	0.0989	0.1222	0.1600	0.2692	0.4565	1.4352	3.2336	4.8286	6.3239	7.3696
	ρ	0.42	0.50	0.58	0.67	0.75	0.83	0.92	1.00	1.08	1.17	1.25	1.33	1.42	1.50	1.58	1.67
	c = 5	0.0217	0.0367	0.0517	0.0681	0.0971	0.1249	0.1463	0.1691	0.2055	0.3046	0.5459	1.0216	2.4132	4.1304	5.3479	6.3696
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
	c = 1	0.000678	0.001468	0.002258	0.003048	0.003838	0.005083	0.008153	0.010978	0.015194	0.019720	0.025424	0.035144	0.046594	0.061040	0.081040	0.102500
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
	c = 2	0.000000	0.000660	0.001020	0.001556	0.002136	0.003373	0.004683	0.006623	0.008833	0.012513	0.016949	0.025596	0.035206	0.052916	0.073750	0.096669
	ρ	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
μ = 5	c = 3	0.000000	0.000660	0.001450	0.002308	0.003094	0.004274	0.006453	0.008743	0.012071	0.014941	0.020521	0.027464	0.038394	0.050542	0.067082	0.086252
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
	c = 4	0.000000	0.000660	0.001427	0.002285	0.003195	0.004765	0.007709	0.010604	0.014124	0.018937	0.023857	0.028777	0.034824	0.045133	0.051923	0.063123
	ρ	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	c = 5	0.000000	0.000260	0.000535	0.001277	0.002307	0.004201	0.005552	0.007325	0.010112	0.013662	0.019096	0.022857	0.030435	0.038655	0.047772	0.058982
	P	0.20	0.24	0.28	0.32	0.36	0.40	0.44	0.48	0.52	0.56	0.60	0.64	0.68	0.72	0.76	0.80
	c = 1	0.000000 00	0.00000430	0.000008 60	0.000054 16	0.000170 58	0.00028768	0.000394 88	0.0005510 1	0.000720 23	0.000857 93	0.00104689	0.001222 83	0.001403 73	0.0017142 1	0.002046 55	0.002395 75
	ρ	0.500	0.600	0.700	0.800	0.900	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
	c = 2	0.000000 00	0.00000320	0.000006 40	0.000009 60	0.000020 80	0.00004690	0.000098 36	0.0001914 2	0.000321 61	0.000514 16	0.00065622	0.000872 89	0.001096 69	0.0013517 2	0.001782 21	0.002278 61
	ρ	0.250	0.300	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800	0.850	0.900	0.950	1.000
μ = 10	c = 3	0.000000 00	0.00000520	0.000010 40	0.000030 70	0.000040 00	0.00004930	0.000078 62	0.0001210 8	0.000181 14	0.000304 92	0.00043509	0.000671 19	0.000940 65	0.0012864 7	0.001666 69	0.002239 59
	ρ	0.167	0.200	0.233	0.267	0.300	0.333	0.367	0.400	0.433	0.467	0.500	0.533	0.567	0.600	0.633	0.667
	c = 4	0.000000 00	0.00000210	0.000004 20	0.000006 30	0.000008 40	0.00001280	0.000037 90	0.0000630 0	0.000109 40	0.000224 00	0.00033860	0.000529 50	0.000758 55	0.0009780 5	0.001307 26	0.001625 00
	ρ	0.125	0.150	0.175	0.200	0.225	0.250	0.275	0.300	0.325	0.350	0.375	0.400	0.425	0.450	0.475	0.500
	c = 5	0.000000 00	0.00000420	0.000008 40	0.000029 15	0.000049 95	0.00007552	0.000101 62	0.0001803 7	0.000257 62	0.000317 72	0.00039062	0.000544 28	0.000664 58	0.0007916 7	0.000996 51	0.001213 51
	ρ	0.100	0.120	0.140	0.160	0.180	0.200	0.220	0.240	0.260	0.280	0.300	0.320	0.340	0.360	0.380	0.400

Follow Table (1) shows the average waiting time in row

The following are the data shows for the average waiting time in the row which confirm all of the above results: First, if we follow the results of the average waiting time for the grade in the case when setting the service rate ($\mu = 1$) as in Figure (5) we observe that the values tend to increase as the access rate increases and the average waiting time in the row decreases as the number of servers increases.

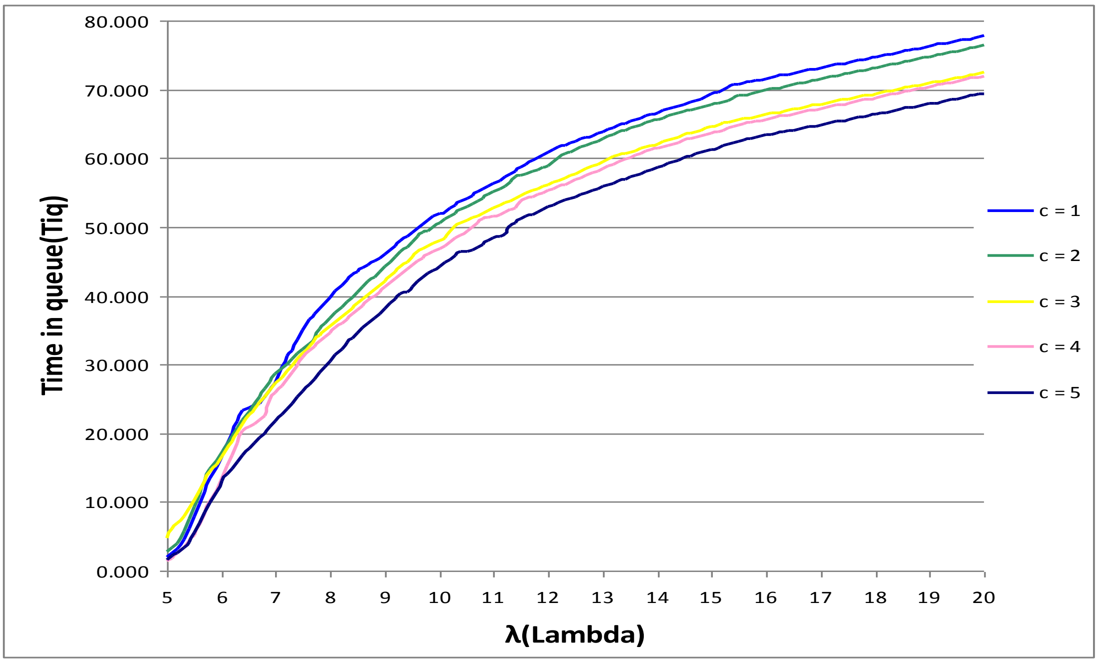

Fig-5: Average waiting time curve in row for each case of number of service providers c = 1,2,3,4,5 and constant service rate μ = 1

Second, if we also follow the results of the average values of the waiting time in the class in the case when the service rate is set: ($\mu = 3$) As in Figure 6 we observe that the values tend to be stable until the arrival rate is $\lambda = 12$ and thus the increase in the average waiting time The λ access rate has increased and the average waiting time in the class has decreased as the number of servers increases.

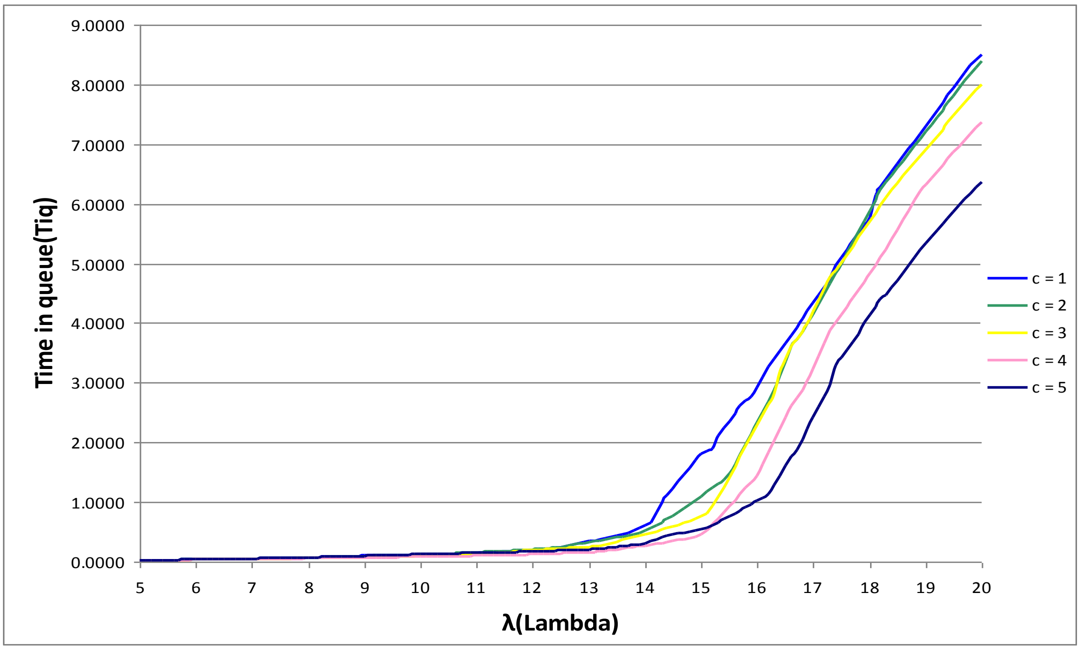

Fig-6: Average waiting time curve in row for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 3

Thirdly, if we also follow the results of the average values of the waiting time in the class in the case when the departure rate is set: ($\mu = 5$) As in Figure (7) we find the increase in the average waiting time as the access rate increases and the average waiting time in the row decreases as the number of servers increases.

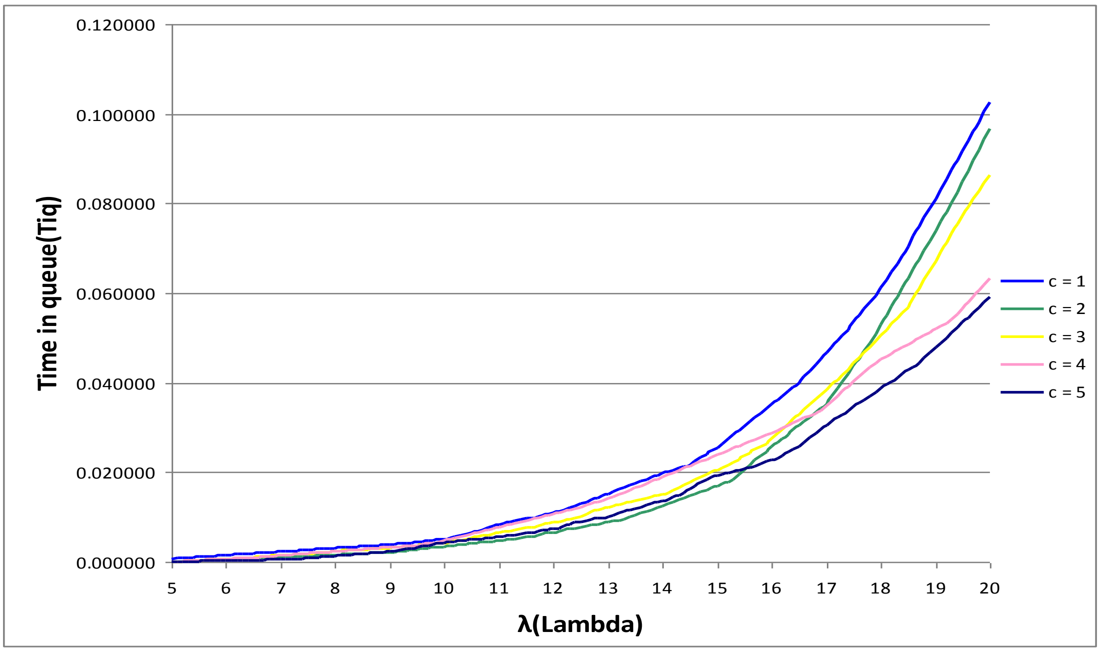

Fig-7: Average waiting time curve in row per case of number of service providers c = 1,2,3,4,5 and fixed service rate μ = 5

Fourthly, if we also follow the results of the average values of the waiting time in the class in the case when the departure rate is set: ($\mu = 10$) as in Figure 4, we also find the increase in the average waiting time as the access rate increases and the average wait in the row decreases as the number of servers increases.

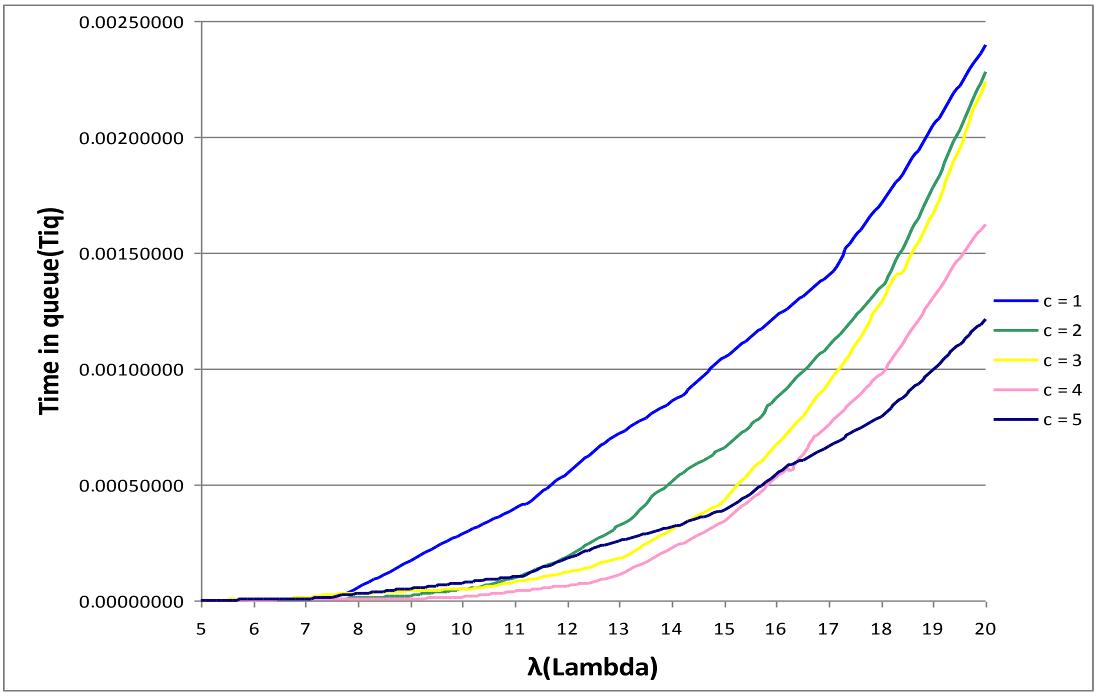

Fig-8: Average waiting time curve in row for each case of number of service providers c = 1,2,3,4,5 and constant service rate μ = 10

In each of the previous cases, we find that as the service rate increases, the average waiting time in the row decreases as shown in Figs. (1), (2), (3) and (4)

2. Standard 2: Average grade length

We mean that the average length of the row is clearly affected by both the number of servers, the access rate λ and the service rate μ. This results in a change in the value of ρ, since $\rho = \frac{\lambda}{\mu}$ we found the following

A. As the previous criterion shows, the greater the rate of λ access for the same number of servers, the average length of the row increases significantly, ie, there is a direct correlation between them. The increase of one of them leads to an increase in the other and we can observe this increase from Table (2).

B. Also, the higher the rate of service μ for the same value of the access rate and the number of servers itself, the lower the average length of the row, ie, there is an inverse relation between them and note this decrease from Table (2).

C. As for the number of servers, the greater the number of servers and the values of both the access and service rates, the lower the value of the average length of the row, and we notice this decrease from Table (2).

In general, the lower the value of ρ, the more stable the system is, and the values converge to a fixed value, where the average length of the row is determined as follows:

Table-2: follows the average grade length

	λ	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
μ = 1	c = 1	884.00	980.66	983.99	987.32	990.65	994.00	994.10	994.20	994.30	994.40	994.50	994.60	994.70	994.80	994.90	995.00
	ρ	5.00	6.00	7.00	8.00	9.00	10.00	11.00	12.00	13.00	14.00	15.00	16.00	17.00	18.00	19.00	20.00
	c = 2	853.00	986.88	992.00	993.01	993.11	993.21	993.31	993.41	993.51	993.61	993.71	993.81	993.91	994.01	994.11	994.21
	ρ	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00
	c = 3	914.00	980.00	990.00	990.00	990.00	991.00	991.00	991.00	992.00	992.00	992.00	992.00	992.00	992.00	992.00	992.00
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 4	813.00	975.00	985.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00	990.00
	ρ	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	c = 5	916.00	987.00	987.00	988.00	988.00	988.00	988.00	988.00	989.00	989.00	989.00	989.00	989.00	989.00	989.00	989.00
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
μ = 3	c = 1	41.00	71.00	95.00	143.00	228.00	335.00	441.00	563.00	678.00	825.00	915.00	969.00	979.00	984.05	984.53	985.01
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 2	39.00	69.00	110.00	161.00	234.00	326.00	430.00	537.00	657.00	789.00	914.00	979.67	984.20	986.05	986.53	987.01
	ρ	0.83	1.00	1.17	1.33	1.50	1.67	1.83	2.00	2.17	2.33	2.50	2.67	2.83	3.00	3.17	3.33
	c = 3	41.00	69.00	106.00	137.00	211.00	295.00	417.00	504.00	615.00	739.00	874.00	980.27	982.95	985.63	988.31	990.99
	ρ	0.56	0.67	0.78	0.89	1.00	1.11	1.22	1.33	1.44	1.56	1.67	1.78	1.89	2.00	2.11	2.22
	c = 4	32.00	54.00	101.00	135.00	193.00	263.00	363.00	461.00	588.00	722.00	847.00	965.00	978.00	988.05	988.53	989.00
	ρ	0.42	0.50	0.58	0.67	0.75	0.83	0.92	1.00	1.08	1.17	1.25	1.33	1.42	1.50	1.58	1.67
	c = 5	40.00	66.00	106.00	148.00	196.00	256.00	346.00	452.00	574.00	698.00	807.00	932.00	974.00	980.33	984.09	985.00
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
μ = 5	c = 1	5.00	5.77	11.00	20.17	29.00	45.00	70.66	98.00	135.00	165.00	223.00	296.00	352.00	413.00	463.33	549.00
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
	c =2	12.00	20.99	26.00	34.00	47.00	65.00	82.00	111.00	145.00	178.00	229.00	285.00	342.00	408.00	443.00	529.00
	ρ	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
	c = 3	5.00	9.00	19.00	32.00	43.00	66.00	93.00	108.00	138.00	161.00	208.00	257.00	316.00	373.00	444.00	502.00
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
	c = 4	3.00	5.73	19.00	27.00	41.00	58.00	82.00	101.00	128.00	159.00	193.00	232.00	275.00	330.00	400.00	457.00
	ρ	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	c = 5	6.00	8.00	17.00	29.00	42.00	62.00	88.00	109.00	142.00	166.00	196.00	231.00	270.00	318.00	369.00	441.00
	ρ	0.20	0.24	0.28	0.32	0.36	0.40	0.44	0.48	0.52	0.56	0.60	0.64	0.68	0.72	0.76	0.80
μ = 10	c = 1	0.000	0.400	0.800	3.330	6.143	9.000	14.000	19.644	20.004	24.660	29.572	32.400	38.500	45.000	55.000	64.000
	ρ	0.500	0.600	0.700	0.800	0.900	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
	c = 2	0.000	0.500	1.000	1.625	2.250	2.875	5.125	7.500	10.800	14.250	22.650	28.666	37.425	44.330	52.000	60.000
	ρ	0.250	0.300	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800	0.850	0.900	0.950	1.000
	c = 3	0.000	0.310	0.620	0.930	2.000	2.710	6.000	10.500	14.000	19.400	25.000	30.000	36.000	41.000	51.360	65.000
	ρ	0.167	0.200	0.233	0.267	0.300	0.333	0.367	0.400	0.433	0.467	0.500	0.533	0.567	0.600	0.633	0.667
	c = 4	0.000	0.250	0.500	0.750	1.000	2.249	3.330	4.400	12.340	18.000	23.000	27.000	34.800	39.000	51.500	57.000
	ρ	0.125	0.150	0.175	0.200	0.225	0.250	0.275	0.300	0.325	0.350	0.375	0.400	0.425	0.450	0.475	0.500
	c =5	0.000	0.400	0.800	1.355	2.053	2.583	3.125	4.600	5.800	9.500	15.000	19.400	21.200	27.000	34.000	43.500
	ρ	0.100	0.120	0.140	0.160	0.180	0.200	0.220	0.240	0.260	0.280	0.300	0.320	0.340	0.360	0.380	0.400

The following are the special graphs: Average row length:

First, if we follow the results of the average values of the length of the row in the case at the rate of service (μ = 1) we note that the values tend to increase as the access rate increases λ and then observe the stability of the average length of the row with increasing access rate and we note the decrease in the average length of the row, It is clear in form (9).

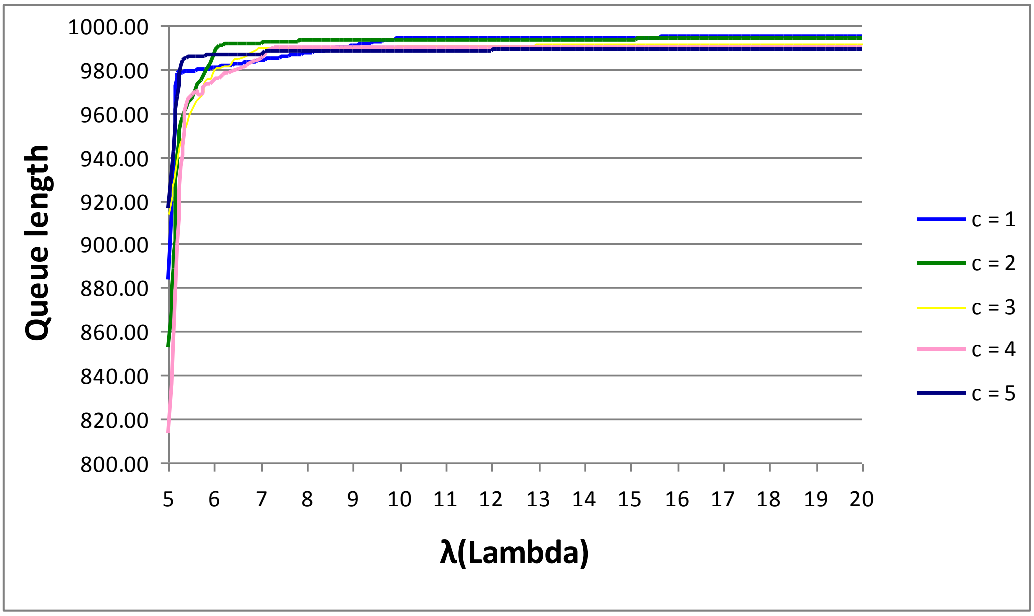

Fig-9: Average row length curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate $\mu = 1$

Second, if we also follow the results of the values of the average length of the row in the case when the rate of service is developed (μ = 3). Note that the values are increased until the arrival rate is λ = 17. Therefore, the average values of the length of the row are constant. The higher the access rate λ, the lower the average length of the row, the greater the number of servers.

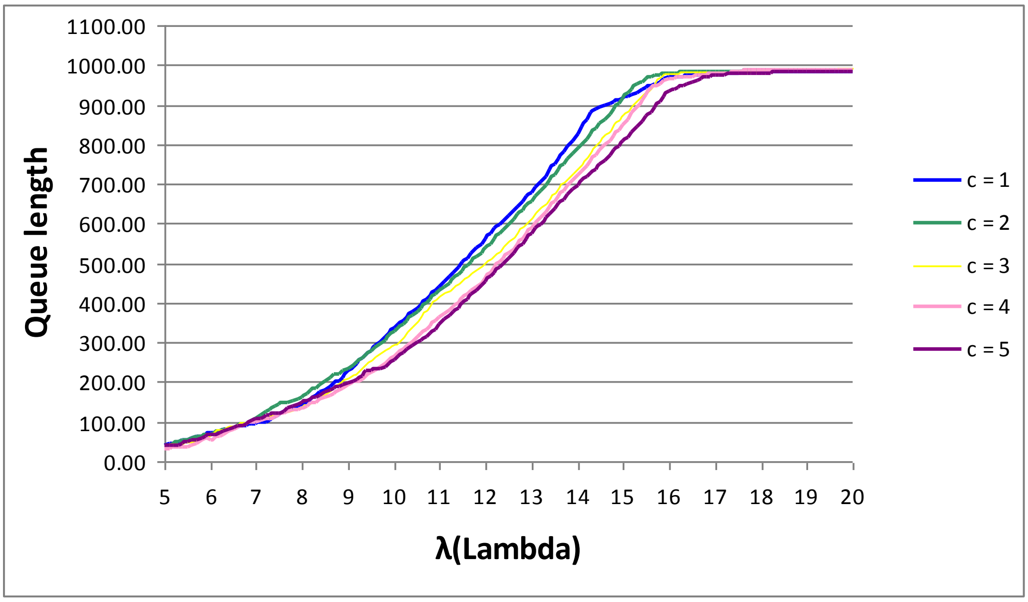

Fig-10: Average row length curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ= 3

Third: If we also follow the results of the average values of the length of the row in the case when setting the departure rate ($\mu = 5$) we find the increase in the average length of the row as the access rate increases and the average length of the row decreases as the number of servers increases as shown in figure (11).

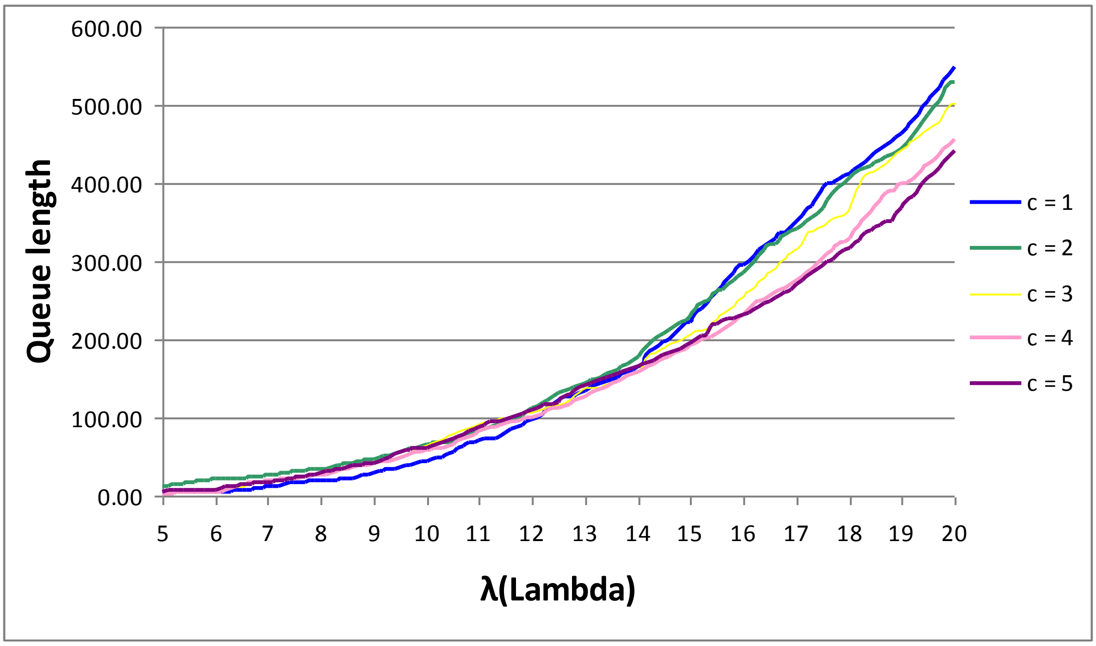

Fig-11: Average row length curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ= 5

Fourthly, if we also follow the results of the average values of the length of the row in the case when the service rate is set ($\mu = 10$), we also find the increase in the mean length of the row as the access rate increases, and the average length of the row decreases as the number of servers increases as shown in figure 12.

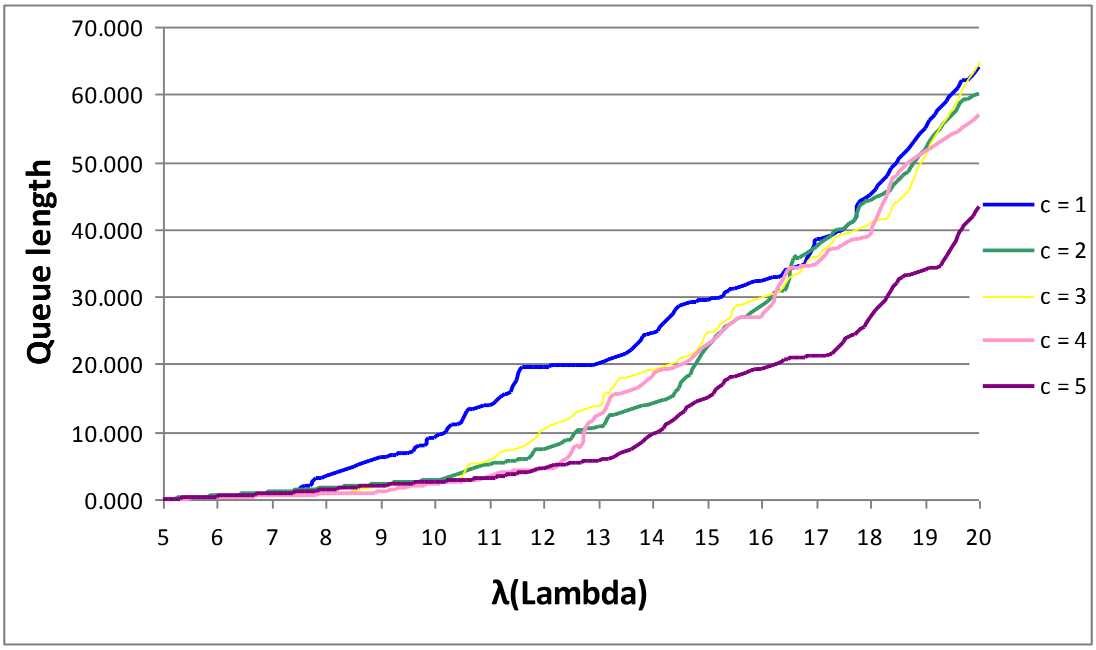

Fig-12: Average row length curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 10

In comparison with each of the previous cases, we find that with the increase in the rate of service μ, the average length of the row decreases as shown in the previous figures.

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