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Multi-Server Queues With Vacations According To Two System Performance Metrics (Average System Waiting Time, Average Vacation Time)

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Abstract: This study aims at introducing a multi-server queues with vacations according to two system performance metrics (average system waiting time, average vacation time). We examine how such a dynamic impacts common system measures such as stability, expected number of customers in the system, probability of an expected time. We have designed a method to calculate performance metrics for the previous queue system defined as average system uptime and average vacation time. We have changed the different features of access rates, service rates and number of servers to achieve the best results and try to improve and develop the system which makes it easier for servers to perform their service to obtain the appropriate vacations for them without being affected by the customer service and the system interest is disrupted where the results related to average system waiting time, average vacation time were found.
Keywords: Multi-Server, Queues, vacations, Performance Metrics, Waiting Time

1. Introduction

1.1 Queuing Systems

Queuing Systems theory is a mathematical study for the queuing rows or queuing lines. The queuing system consists of customers or units that require a type of service where the lines that customers enter when the service is not available immediately and there are cases where customers leave the system without considering the line and the service center that provides service to the customers.

The year 1910 was the real birth of the Queuing Systems theory by the Danish scientist Erlang, where he began that year to conduct experiments related to the problem of congestion in the center of telephone exchange through the workers in the telephone call center as it was found that callers are often subjected to some delays during the periods in which telephone calls are frequent because of the inability of the staff to meet orders in a synchronous manner with the speed with which they occur.

Erling was determined to calculate the delay duration for one worker in the division and then he circulated his study and the results to a number of workers. The development of the telephone traffic continued on the bases set by Erling and by the end of the Second World War, the use of this method was expanded and included a number of general cases, which characterized by Queuing lines [1].

Kendall [28] was one of the first to review the Queuing Systems theory from the perspective of random processes.

The last century has witnessed a tremendous development in queue theory, which has been highly regarded as a research area in both operational research and operational probabilities.

The concept of the Queuing Systems model is that when the customer reaches the server, he enters the queuing line, and then the service serves the customer who has to be served from this class. When the service is completed, the submission of service process is repeated to a new customer from the queue.

It is assumed that there is no time lag between the completion of the service to a particular customer and the commencement of service to the next customer in the queue if the service provider does not need time to be prepared to provide service to a new customer.

It is noticed that the key elements of the queuing model are the customer and the service provider. In queuing models, we are concerned about the interaction between the customer and the service provider that affects the time that the customer need to obtain the service.

Queuing systems cab be expressed as in Fig. (1-1), which shows the customers arriving at the queuing system expressed in a dashed rectangle and then standing in the queuing for their turn to get the service and then move to the service centers then the customers leave the system after obtaining the required service [2].

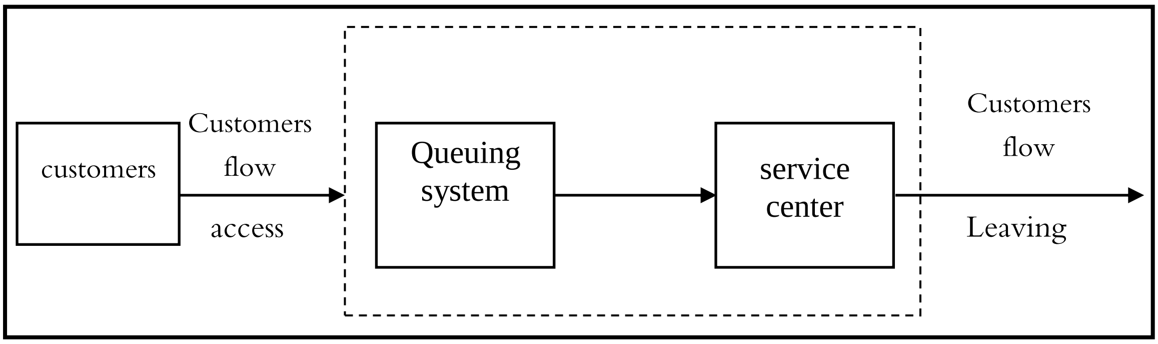

Figure (1): Queuing system

Customers are the ones who need the service and may be either individuals such as patients or buyers, and may be other types such as telephone calls, raw materials, etc. The queue consists of a group of customers (individuals or objects) who are for their turn to receive the service.
The service station is where customers go to get the service they need. If we combine service centers with queues, the result will be the system, and system management (the system here is the channel or service channels with rows) [3].

1.1.1 The Properties of queuing systems
Queuing systems have the following characteristics:

1. Arrival Process

The arrival process is a description of how customer service seekers enter the service stations [4].

2. Queue Capacity

It is the largest number that is allowed to be in the system [4].

3. Service Discipline

The service discipline is the way in which the service is provided to customers. The most popular customer service disciplines are:

a. "Service in order of access" means that it is the first who served or the first who arrives is the first to leave (First -In- First-Out) or (FIFO). This discipline is the common discipline in most of the queuing systems.
b. "Service by Reverse Access Order" or who finally arrives is served first (Last-In-First-Out) or (LIFO). This discipline is used in most storage systems where the products that are finally stored are consumed because they are within reach.
c. "Random service" where items are randomly selected to serve them regardless of their arrival time (Selection- In –Random-Order) or (SIRO) as in some of the data entry operations.
d. "service by priority" or (PRI), where the preference is given to some customers such as patients arriving at hospitals or clinics in serious condition and cases of important military messages. Or a preference according to a particular order or scale.

The percentage of time the server is busy providing service to the customer is given by the equation; $a = \frac{\lambda}{\mu}$,where a is the ratio between the access rate λ and the service rate μ that is a dimensionless quantity whose numerical value gives a measure of the service demand from the system and is known as the Erling units

The quantity $\rho = \frac{a}{c}$ is called the usage parameter where c is the number of servers where $c \geq 1$ and where there is one server, $\rho = a$ [2]

4. Service Time

The time spent in serving the customer [2].

1.2 Service Stages and Number of Service Channels

Queuing models in terms of the number of channels and stages of service are classified into the following models [5]:

A. Single Stage – Single Channel Model

Where in this case the customers form a single queue in front of one service center as in Fig. (1-2)

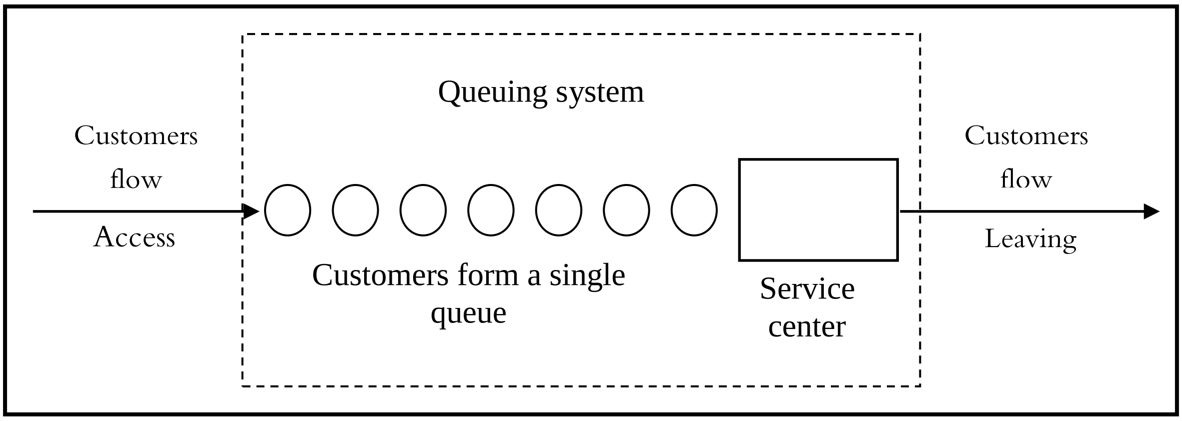

Fig-(1-2): one-channel and one-phase queue model

For example machines in front of a single maintenance unit.

B. Multiple Stages – Single Channel Model

In this case, the customers constitute more than one queuing line on more than one stage, as each stage offers a different type of service than the other stages as in Fig. (1-3) below:

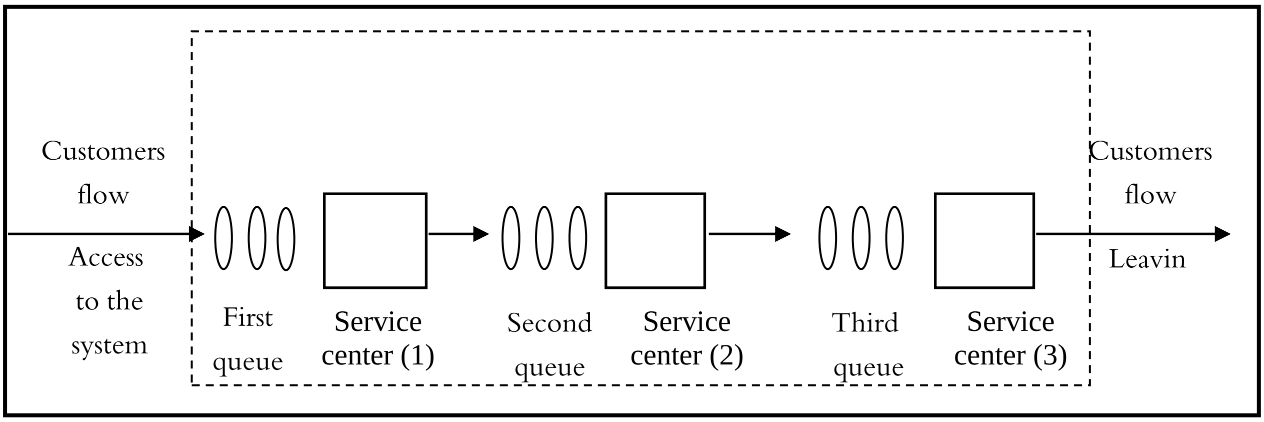

Fig-(1-3): Multiple Stages – Single Channel Model

For example, patients who move from one clinic to another for a comprehensive medical examination.

C. Single Stage - Multiple Channel Model

In this case, the service centers are multiple with a single queue

1.3 Queuing Applications

Queuing Systems theory ranks has taken the form of progress, development and prosperity both in the field of methodological theory and in practical applications.

In the field of methodological theories, we find that there is a large number of researchers who are interested in studying the models of queuing Systems theory from graduate and postgraduate students in the different related disciplines.

Queuing theory has become a target for practical applications. For example, the CPU is the server and the programs that are to be executed are the customers.

In the field of air traffic planning, aircraft in the air are for an empty passage that can be used for landing planes where more complex queue models have been developed to obtain the design of traffic lights at the crossroads. To overcome the problems that accompanies some of the work, these models are also applied to determine the optimal number of berths that receive vessels at ports in order to reduce total costs [6].

1.4 Birth -and -Death Process

Many systems that are interesting to be studied have evolved over time. Their study focuses on the dynamic behavior of these systems. These systems always include randomness in some way such as [7]:

- Study of Queuing Systems
- The number of successful students in a course over different years
- The temperature of the air in a city
- Number of data packets in an information network.

The random process x_t or $x(t)$ is defined as a sequence of random variables by index (t), which is often the time. Thus, the random process is plotted from the sample S space to a function in time t so that the $x_t(j)$ or each element$j \in S$ where it can be said that:

- For each value j, becomes $x_t(j)$ a function in time.
- For each value t, becomes $x_t(j)$a random variable.

The $x_t(\, j)$ function of a given j value is called realization of the random operation and defines the set of possible x_t values in the state space. The set of t values is defined as the parameter space.

Random processes are classified according to the fact that the state space the parameter space is connected or sporadic. For example, it is possible to talk about a random process that is time-related or time-spaced according to the parameter space.

The study of random processes is concerned with quantities, for example [8]:

A. Time Dependent Distribution

The probability where $x_t$takes a value in a subset of S at a known moment t.

B. Stationary Distribution

The probability where x_t takes a value in a subset of S when $t \to \infty$ (assuming there is an end).

C. Covariance between x_j , x_t

The correlation between x_j and x_t is determined at two different times t, j.

D. Hitting Probability

A probability of returning to a state from S.

E. First Passage Time

The moment when the random process enters for the first time in a case or set of cases when it starts from a certain initial state.

1.5 Queuing Systems Notations

As noted above, access and leaving have their own allocations and that there are various service principles. We have also indicated that customers may be come from an end-to-end source as we have indicated that system capacity may be limited (limited to a certain number of customers) and may be unlimited. Moreover, the system may contain one channel or several parallel channels. Kendall [28] suggested a symbolic way to denote all these things as follows:

A/B/C/X/Y/Z

Where the characters in this code mean the following:

A: stands for the distribution of the number of customers accessing to the system (or to the distribution of time between two sequential accessions). The symbol M is usually used to indicate that the number of customers follows the Poisson distribution or to indicate equivalently that the time between two sequential accesses follows the exponential distribution. This indicates to Markov's property (relative to Markov, whose name is associated with the random processes).

B: stands for the distribution of the number of customers who leave the system (or to divide the service time of a customer) and usually uses the symbol M to indicate that the distribution of the service time of a customer follows the exponential distribution.

C: stands for the number of parallel service channels *(C=1,2,3...,*$^{\infty}$*)*

X: stands for the principle of service, and some of the principles of service we have seen are, LIFO FIFO, SIRO, where GD symbol is used to denote the principle of general discipline, which can be any of the above principles or any other principle.

Y: stands for the capacity of the system and the number of customers that can be accommodated by the system (in row or queue and in the channel or service channels) so that *(Y=1,2,...,*$^{\infty}$*)*

Z: stands for the source energy, which is the number of source elements from which customers are generated. This energy may be limited or unlimited (*Z=1,2,,...,∞*)
Below are some of the known queue systems:
The symbol (*M/M/1/FIFO/∞/∞*) indicates a queue system characterized by:

The time interval between two consecutive access times and the time of a customer service following the exponential distribution and the system has one service channel and the service principle is "service by order of access" and the system power is unlimited and the power source is unlimited.
When the last three symbols are deleted, they indicate FIFO /∞/∞. For example, the M/M/1 symbol denotes the same concept as the previous code.
The symbol (*M/M/1/FIFO/K/∞*) has the same former concept except that the system energy is defined by the number K of elements.
The symbol (M^X/*M/S/N/FIFO*) means that customers arrive in the system in more than one time followed by the Poisson process and the service time follows the exponential distribution and the number of servers S server and the maximum number allowed to exist within system N, Service in order of access principle, and X batch size which can be fixed or follows a given distribution.
The symbol (M^X/E_K/*S/N/FIFO*) means that the time between the arrival of customer payments follows the exponential distribution. The service time is followed by the K-type distribution, the number of S servers and the maximum number allowed to exist within System N, Service in order of access principle, and X the batch size is fixed or follows a certain distribution.
Now, if we have a row system we mean that this system is a system of birth and death and in this case we will use the following symbols:
N(t): represents the random variable that represents the number of customers in the system up to the moment t where $t \geq 0$. We mean the number of customers in the queue system:

Number of customers in a queue system = number of customers in queue row or rows + number of customers served in a channel or service channels.

$P_n(t)$:The probability distribution of the number of customers in the system at time t

$$P_n(t) = P(N(t) = n) \qquad (1.5.1)$$

As we define the parameters N (t) and $P_n(t)$, the study of queuing systems is generally dependent on time t. However, many queue systems depend on the beginning of their work over time, but over time they reach stability.
In the stability state, the random variable N(t) and its probability distribution function $P_n(t)$ become independent from time and are denoted by N and P_n, where

$$P_n = P(n) = P(N = n) \qquad (1.5.2)$$

1.6 Performance Measures

The main objective of the study of queue systems is to achieve an appropriate level of service with a reasonable level of costs. To achieve this goal, we need to find measures called efficiency or performance measures by which we can study and analyze queue systems and then re-adjust or reconstruct these systems in such a way that we reach the desired goals. Such as determining the optimal number of service channels and the optimal speed of customer service [2].
There are some concepts that are used in performance measures such as:

1. Arrival Time

Is the time when the customer arrive the queue and is symbolized by AT.

2. Departure Time

This is the time when the customer completes the service and leaves the system and is symbolized by DT

3. Time of Beginning the Service

Is the time when the customer leaves the queue to start the service and is symbolized by TSB.

4. Time in Queue

Is the time in which the customer is in line to receive the service and is symbolized by TiQ.

5. Sojourn Time

Is the time in which the customer waitings in the system until he takes the service and leaves the system and is symbolized by SJ

1.6.1 Customer Performance Measures

Customer Performance Measures are the measures that measure the effectiveness of the customer such as:

a) **Time in a queue**: when the time in the queue is short, it is better for the customer.
b) **Service time**: Whenever service time is short, it is better for the customer.
c) **costs**P: If the costs are low, it is better for the customer.

1.6.2 Server Performance Measures

These measures measure the effectiveness of the server. Examples include:

a) **Service times** the faster the service time, the better for the server.
b) **Server downtime**: lower is better.
c) **The rate of leaving customers from the server**: large is better.

Sultan [29] calculated some of the following dimensions for the queuing system:

1. **Average customer time per queue.** The average time between the customer joining the queue and joining the service is the following:

$$W_Q = \frac{Total\ customer\ waiting\ times\ in\ queue}{The\ total\ number\ of\ customers\ in\ the\ system} \qquad (1.6.1)$$

2. **Average time for the customer in the system.** The average time between customer joining the system and completing the service is the following:

$$W_S = \frac{Total\ customer\ waiting\ times\ in\ the\ system}{The\ total\ number\ of\ customers\ in\ the\ system} \qquad (1.6.2)$$

3. **Average number of customers in the queue.** From the following relationship:

$$L_Q = \frac{Total\ customer\ waiting\ times\ in\ queue}{The\ total\ time\ of\ customers\ in\ the\ system} \qquad (1.6.3)$$

4. **Average number of customers in the system.** From the following relationship:

$$L_s = \frac{Total\ number\ of\ waiting\ customer\ \ in\ the\ system}{The\ total\ time\ of\ customers\ in\ the\ system} \qquad (1.6.4)$$

5. **The probability of busy service centers during the arrival of any customer to the system.** From the following relationship**:**

$$P_B = \frac{Average\ number\ of\ busy\ servers}{The\ total\ number\ of\ servers\ in\ the\ system} \qquad (1.6.5)$$

1.7 Little's Law

Assume that λ denotes to the access rate in the unit of time, W the average time in the system, and L is the average number of customers in the system. The Little formula is considered as a useful outcome in queue systems and this formula gives the relationship between L and W as follows [9]:

$$L = \lambda W \qquad (1.7.1)$$

The relationship between the average length of the queue L_Q and the time W_Q can be written as follows:

$$L_Q = \lambda\, W_Q \qquad (1.7.2)$$

The relationship between the average time of the customer in W_S and the average number of customers in the L_S system can also be written as follows:

$$L_S = \lambda W_S \qquad (1.7.3)$$

The significance of Little's formula is that the average time can be calculated from the average queue length and vice versa. Thus, for example, it is enough to measure the length of the queue and thus we can obtain the average time so that:

$$W_Q = \frac{L_Q}{\lambda} \qquad (1.7.4)$$

Since we are going to study the M/M/c queue model, we will address the potential distributions used for this type.

2. Probability distributions used in the study

2.1 Poisson Distribution

The Poisson process of potential probability distributions is common in many applications, and the Poisson process of discrete probability distributions is named after Poisson.

The Poisson experiment is a process that produces numerical values for a random variable X representing the number of times of an accident (or phenomenon) in a time period symbolized by the symbol t. The exponential distribution is characterized by loss of memory meaning that if we had a system, the time required for the available vacations to the servers does not depend on the previous vacation or the following in the sense that they are independent.

2.2 The Exponential Distribution

The exponential distribution is one of the important distributions because it has many applications, most notably the queuing systems. The time period between the arrival of a customer and another of the service center follows the exponential distribution. The reason for this designation is that this distribution is based on an exponential mathematical equation and this equation is $f(t) = \lambda\, e^{-\lambda t}$, $\lambda > 0$, $t \geq 0$ Where t is the time period between the arrival of a customer and another and λ is the customer access rate in the unit of time.

The exponential distribution has a relation with the distribution of Poisson. If events occur following Poisson distribution, the period between two events follows the exponential distribution. For example, if customer access to a service center follows the Poisson distribution, the time between arrival follows the exponential distribution.

2.3 Phase Type Distribution (PH)

PH is a probability distribution that results in the probability of one or more Poisson operations, it is a distribution that can generate random variables for time that achieve Markov as it is ble to achieve a Markov property for each phase. All random variables generated by this distribution are positive values.

If the distribution of PH is one phase, it can be considered an exponential distribution, but if the distribution is more than one stage, it can be considered a generalization of the Erling process.

Since we will study the multi-server wait queues with vacations, the vacation time follows PH distribution and this distribution was chosen because it is a mixture of continuous and intermittent distribution [10].

4. Simulation of Queuing Systems

Sometimes, in practice, there are many complex problems that are difficult to put into mathematical models or mathematical models are difficult to solve analytically. In such cases, we have to search for optimal solutions and methods that succeed in finding such optimal approximation solutions simulation method.
There are cases where simulation is recommended [11]:

1. Difficulty in obtaining data about the real problem or the high cost of obtaining such data.
2. The difficulty of constructing a mathematical formulation that can be used to find an analytical solution to the problem.
3. There is no direct analytical way to solve the problems in question.
4. The difficulty of verifying results in some experiments

4.1 Randomized Trials Method

The randomized trials method expresses the simulation method using the sample. These samples are taken from a theoretical population, where the probability distribution of the variable we are studying is determined. We then generate random variables representing these samples. Is a simulation of the random processes that arise in the system, which leads to the development of an algorithm to formulate the functions of the system studied in doing various work, in addition to monitoring and analysis of random processes that arise in the system studied and the preparation and dissemination of monitoring results [12].

4.2 Generating Random Variables Methods

Several methods have been designed to generate random variables, including:

4.2.1 Linear Congruential Method

The linear congruential method was used by lehmer in 1951, where it presents a series of integers where we find a set of random numbers that follow the regular distribution $U[0,1]$. This method is based on generating a set of numbers between 0 and m-1. The same sequence of random numbers can be produced by the same method when required by the following repetitive relationship:

$$X_{i+1} = (a\, x_i + c) mod\, m \quad , \quad i = 0,1,2, \ldots. \qquad (2.11.1)$$

Where the initial value X_0 is called the original value being expanded and the constant a is called *a* fixed multiplier and the constant *c* is called an added value and *m* is called the modulus.

4.2.2 Inverse Transformation Method

The Inverse Transformation Method can be applied if the probability density function f(x) and the cumulative distribution function F(x) have an inverse distribution function F^{-1}(x), it is known that the cumulative function field F(x) Period $0 \leq F(x) \leq 1$ If *y* is a random number in the period, it will be:

$$F(x) = Y \qquad (2.11.2)$$

Then:

$$x = F^{-1}(Y) \qquad (2.11.3)$$

Where $F^{-1}(Y)$ is inverse of function F(x) and $0 \leq F(x) \leq 1$

If we want to generate random variables X that, follow the exponential distribution of the parameter $\lambda > 0$ and whose probability density p.d.f is given as follows:

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x} & ; \ x > 0 \\ 0 & ; \ otherwise \end{cases} \quad (2.11.4)$$

The cumulative distribution function is:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = 1 - e^{-\lambda x} \quad , x \ge 0 \quad (2.11.5)$$

To generate random variables from exponential distribution using Inverse Transformation Method, we find the cumulative distribution function of the random variable X, as in equation (2.11.5). Then we put $F(x) = R$ on the field X and thus the relationship become:

$$R = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - R$$

$$-\lambda x = \ln(1 - R) \quad (2.11.6)$$

$$x = \frac{-1}{\lambda} \ln(1 - R)$$

By taking the logarithm of the parties

where $R \sim U(0,1)$

$$\therefore \ x = \frac{-1}{\lambda} \ln R$$ then $1 - R \sim U(0,1)$

We generate the required number of random numbers R_i and then calculate the observations from the random variable [13].

4.2.3 Acceptance Rejection Method

The method of acceptance and rejection is used to generate random variables from different distributions. The basis of this method is initially based on the generation of random numbers that follow the regular distribution in a repetitive manner and only accept them to meet certain conditions. These conditions are designed to accept the random variables that follow the given distribution and the required is to generate random numbers from it.

1. Convolution Method

The Convolution Methodis a method for variables that consist of two or more independent random variables. If we assume that we have X, a random variable is the sum $Y_1 + Y_2 + \cdots + Y_m$ where Y_j , $j = 1, ..$ m are constant independent identical i.i.d variables and m is a fixed constant number, then $Y_1 + Y_2 + \cdots + Y_m$ is called a convolution of variable m [13].

2. Composition Method

This method is a direct method for generating random variables. This method is used when the function consists of a sum of m of probability distributions as follows:

$$f(x) = \sum_{i=1}^{m} p_i f_i(x) \quad , \quad 0 \le p_i \le 1 \quad (2.11.9)$$

The method is to generate a separate random variable k that follows the distribution:

$$P[K = i] = p_i \quad , \quad i = 1, 2, ..., n \quad (2.11.10)$$

Then we generate $Y_i \sim F(x)$ where $F(x)$ is the cumulative distribution function of the function $f(x)$ and then put $X(u) = Y_i$ to calculate the value of u [15].

3. Mid-square Method

This method was introduced by Von-Neumann in the mid-1940s, where this method starts with an initial number and then squared. The random number is then selected after excluding the fractions and then quadrature this number to in turn generate the second random number and so on [14].

4. Constant Multiplier Technique

In this method, two points (X_0, X_1) are chosen so that the number X_0 is fixed and then multiplied by the number X_1 and then exclude the fractions to get the random number and continue as follows:

Assuming that $X_1 = 7223$ and $X_0 = 3987$ for random numbers we follow the following:

$$V_0 = X_0 X_1 = (3987)(7223) = 28798101 \Rightarrow X_2 = 7981$$

$$R_1 = 0.7981$$

$$V_1 = X_0 X_2 = (3987)(7981) = 31820247 \Rightarrow X_3 = 8202$$

$$R_2 = 0.8202$$

The success of this method depends on the choice of the constant [14].

5. Terminating Trile Methods

There are several criteria by which the simulation of a system under study can be terminated [16].

1. Determine a certain number of events e.g. 100 customer service.
2. Determine the operating time e.g. ended after 3 hours on the computer i.e. 3 hours simulation.
3. According to real time such as the start of the Hajj season
4. Internal event within the system such as malfunction of all devices
5. An external event such as that the user decides to change the experiment.

4.3 M/M/c Queue with Asynchronous Vacations

There are two vacation mechanisms [9]:

1. service station vacation
2. server vacation

Chao and Zhao [17] studied the previous classification of the vacation. The study aimed at finding the stable probability of the different states. The algorithms were created to arrive at numerical solutions. Both types of analysis were analyzed and compared by numerical observations of the both two systems, below we present both types as follows:

4.3.1 Service Station Vacation

All servers in the station get a vacation at the same time, and this happens whenever the system is empty, and then all the system servers leave the system for a vacation and return to the system at one time when the vacation is over.

4.3.2 Server Vacation

It is the second mechanism of vacation and is often found in practice, which is the most common. In this case, each server unit is independent and can take its own vacation when the customer service is completed and can be used in any other secondary work.

This phenomenon occurs, for example, in post offices where and when there is a window without a queue of customers. Here the writer goes to other types of work (sorting, distribution, etc.).

Note that if c=1, where c is the number of servers, i.e., one server system, then the two vacation types match. Tian el al. [18] studied this case.

They analyzed the wait queue of the type GI/M/1. The objective was to study the model taking into account the vacation time of the service providers followed by the exponential distribution and the arrival time of customers follows the general distribution where there was one service provider.

If we compare multi-server system vacation models with single-server vacation models, we find that multi-server vacation models are more complex and have different dynamics than single-server vacation models [9].

4.4 Vacation Policies in Multi- Server Models

There are several vacation queuing policies for many queue applications. We will review four different policies here [19]:

a) Asynchronous All Server Vacation Policy

The definition of the asynchronous vacation is that all servers in the multi-servers M/M/c queue must start independently and randomly, where the server can take the vacation when it ends up serving one of the customers with another customer to be served. Now, there are servers that provide the service for other customers, on vacation or idle do not work (in the case of the only vacation).

This vacation may take the so-called multiple vacation as the server or servers take another vacation if there is no customer or customers in the queue for the completion of the previous vacation, and repeat the vacation until a customer appears in the queue within the system. This type is symbolized as (Asynchronus Multiple Vacation) (AS, MV).
In the case of single-server servers, the server or servers take only one vacation when the queue is empty. After completing this vacation, either the server or the servers will remain idle or serve up to customers. This type is symbolized as (asynchronus single vacation) (AS, SV).

The difference between a multiple vacation and one vacation is that the server or servers on multiple vacations repeat the vacation in the absence of a customer or customers on the queue. On the only vacation, after the vacation ends, the server or servers will be idle if there is no customer or customers on the queue.

However, if the server or servers took time to prepare before the customer or customer service, this type is called the (asynchronous setup) (AS, SU).
It is noted that the non-inclusive vacation is not comprehensive where the servers take a single vacation on a regular basis, as long as the queue is zero. The concept of asynchronous vacation is like the concept of a server vacation.

b) Synchronous All-Server Vacation Policy

First, I will refer to the definition of synchronous vacation, which means that all the multiple-servers M/M/C server queue start simultaneously at the same time, for example, in the multi-user terminal system, where the super computer is shut down or weak Power supply can be considered a synchronized vacation.

This vacation may take the so-called multi-vacation, where all servers take another vacation together if the system is not busy for the completion of the previous vacation. The vacation process is repeated until the customer appears in the queue within the system where you resume customer service and servers c continue in the customer service. This type is known as (Synchronous Multiple Vacation) (SY, MV).

In the case of unicast servers, all c-servers take only one vacation and together when the system becomes empty once, the last customer service is completed in the system. After completing this vacation either the servers will remain idle or serve up from the customers. This type is known as (Synchronous Single Vacation) (SY, SV).
Another type of synchronous vacation is that the synchronous form is timed for Setup which is known as (synchronous setup) (SY, SU). In such a system, the system becomes empty once the last customer service is completed in the system. In this case, All servers are out of service or out of business. When one or more customers arrive, c servers are going to open or run and there is time to set up before starting to work in the customer or customers service.
It is noted that the vacation is inclusive, meaning that all servers are included in the vacation.
The concept of synchronous vacation is like the concept of service station vacation.

c) Some-Server Vacation Policy

It is the number of servers allowed to take the vacation. The maximum number of servers on the vacation should not exceed the number d where $d \leq c < 0$ where c is the number of servers. For example, doctors in the hospital must specify the number of doctors allowed to take the vacation. Doctors take the vacation so there is no harm to the patients so we need to determine the number of doctors who take the vacation.

d) Threshold Vacation Policy

All servers start a vacation at the moment of service completion when the system becomes busy and the servers continue to take the synchronous vacation until the number of customers in the system is at least N, i.e. the system starts to serve customers when the number of customers in the system N number where N = 1,2,....
To be able to identify queuing systems with vacations, we need to explain almost the Birth-and-Death Process.

4.5 Quasi-Birth-and-Death Process

Most studies of multi-server vacation models have focused on M/M/C systems, which use Markov series for queues, the most commonly used mathematical tool that changes in time. The use of Markov's method is the advantage of this method in terms of ease of use and simplicity. Since the multi-server system serving a number of customers is a system in which the number of customers changes in time, Markov's method is ideal for designing a system that demonstrates how the system handles customers.

The process of birth and death (QBD) is similar to the process of birth and death described previously. The difference between them is that the process of birth and death is one-dimensional. While Quasi-Birth-and-Death Process is two-Dimensional in the process of birth and death, this dimension represents the number of customers in almost the process of birth and death. In addition to the number of customers, there is a second dimension is the servers state, and we need to add this dimension to describe the system.

The infinitesimal generator (the smallest matrix with all probability) is a QBD represented by a three-tiered matrix whose elements are matrices within the original matrix.

$$\Omega = \{(k, j) : \ k = 0,1,2,3,\dots, \ j = 1,2,3,\dots m\}$$

The process {X(t), J(t), t≥0} can be defined on the given state space Ω through

Where k stands for the number of customers, and j represents the state of the server in terms of the presence of a vacation or its absence and m represents the maximum limit of states.
The transition process to k is limited only to the adjacent sate, but is not restricted to the state after j, and in more detail where the system is in the $(k, j) \in \Omega$ state, where Ω represents the state space, it can move to one of the following situations:

$$k, j)\text{،}(k\text{-}1, j)\text{،}(k+1, j\ \{(\ $$

While you cannot move to the position with the shape
$\{(k \pm n, j) \in \Omega : n \geq 2\}$.

When the QBD transition method does not depend on k, it is called a homogenous QBD process or a semi-spontaneous process of birth and death.

If the transition process is dependent on k, we call it a heterogeneous QBD process, or a keratogenesis and k-dependent process.

The operation {X(t), J(t), t≥0} is called QBD if the finite generator of this process is given as

$$Q = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & B_3 & A_3 & C_3 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \qquad (3.3.1)$$

As all the partial arrays of matrix Q are a square matrix of class c where c is the total number of servers [20, 21].
Note that the matrix A_k where $k \geq 0$ represents a matrix in which the arrival and leave rates and the diametric elements in this matrix are negative numbers and the non-Qatari elements are positive numbers.
The matrix B_k where $k \geq 1$ represents a matrix with only the leave rates and all its elements positive numbers.
The C_k matrix where $k \geq 0$ represents a matrix with only access rates and all its elements is positive.
The matrix Q is required to be

$$(A_0 + C_0)e = (B_k + A_k + C_k)e = 0, k \geq 1$$

Where e is the vector that all its elements equals number one [10].

The group {0,1}, ..., (0,c)} expresses the absence of customers, and called the boundary levels
The group {k, 1}, ..., (k, c)} expresses the existence of a number of customers, k where k = 1,2,3 ..., called non-boundary levels, and the matrices A_0, C_0, B_1, are called the border matrices.

In some applications, if the micro-generator (infinite) matrices are not defined and independent and do not depend on k, the matrix Q is written as[19] .

$$Q = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (3.3.2)$$

Where

A_0 is a matrix of $m_1 \times m_1$grade

C_0 A matrix of $m_1 \times c$ grade

B_1 A matrix of $c \times m_1$ grade

m_1 represents the number of cases at the level k = 1

Matrices of $A_0, C_0, B_{1,}A, B, C$ can be explained by the following QBD model:

Figure (3-1) shows the status transition diagram of the M/M/1 queue model

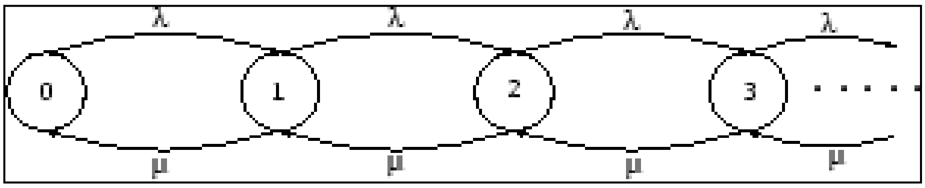

Figure (3-1): Example of Operation (QBD)

In this case, the matrix Q is can be written as:

$$Q = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Where;

$$A_0 = (-\lambda) \quad , \quad A = \begin{pmatrix} -(\mu+\lambda) & \mu \\ 0 & -(\mu+\lambda) \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \mu \\ 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$$

$$C_0 = (\lambda \quad 0) \quad , \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

The QBD process can be analyzed in the canonical form:

(QBD) is a positive recurrence process, that is, the same state can be accessed more than once through a process that is controlled by Markov chains. The number of system accesses follows Poisson distribution at λ, and the number of servers in the system also follows the distribution. The time between each successive access follows the exponential distribution at λ. The wait time in the queue follows the exponential distribution with an average of μt, and $\pi_{ij}^{(n)}$ is the probability of moving from state i to state j in *n* is a step given by the relationship

$$\pi_{ij}^{(n)} = (X_n = j \,/\, X_0 = i) \quad , \quad n \geq 0\,, i, j = 0,1,2, \ldots$$

Where X_0 represents state i, which is a stand-alone random variable, X_n represents state j, which is a stand-alone random variable.

If X (t) represents, the number of customers at the moment t and J(t) represents the number of busy servers.

Assuming that $\lim_{t\to\infty} X(t) = X$ and that $\lim_{t\to\infty} J(t) = J$. Therefore, the stability probabilities can be defined as follows:

$$\pi_{ij} = P(X = i, J = j) = \lim_{t\to\infty}(X(t) = i, J(t) = j)$$

Where π_{ij}: is the probability of being in state j of the previous state i, where $\pi_{\text{i}} = (\pi_{i1}\pi_{i2} \ldots \pi_{ic})$ is the queue i in the transition matrix where $(i, j) \in \Omega$, Ω is the states space (Neuts, 1981).
Below are some of the theories of QBD without proof [19];

4.5.1 First theory

The process of birth and death is irreducible (QBD) where any case can be accessed from any other state that is a recurring positive means that we can reach the same state more than once during Markov's process, if and only if:

1. Equation of matrices: $R^2B + RA + C = 0$ has the least one non-negative (R), the matrix rate and the A, B and C matrices mentioned earlier (3.3.2).
2. The set of homogeneous linear equations on the form $\pi_0(A_0 + RB_1) = 0$ has a positive solution and meets the normal condition: $\pi_0(I - R)^{-1}e = 1$, where e is the vector which all its elements equal to the number one.
3. Stable probability distribution can be expressed in a sequence of geometric matrices As follows: $\pi_l = \pi_{l-1}R \quad , \quad l = 1,2,3 \ldots$

   The final formula is thus: $\pi_l = \pi_0 R^l \quad , \quad l \geq 0$

In many practical applications, the properties and variables defined in the QBD process are used in the standard form in the QBD process in the non-standard form (non-canonical) process called QBD with complex boundaries (complex boundary behavior), which does not depend on the fact that the level *l* = 0 is the limit only, but the value *l* changes and takes values of *l* = 0,1,2, ... c, that is, the limits become variable levels where we cane note the following:

The infinite probability generation matrix Q has the same pattern as the previous, where;

$$Q = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Where;

A_0 A matrix of $m_1 \times m_1$ grade

B_1 Non-square matrix $c \times m_1$

C_0 Non-square matrix $m_1 \times c$

Where m_1 represents the number of states at the level k = 1

The following is the theory that deals with the non-standard form:

4.5.2 Second theory

Near-birth and death (QBD) (irreducible) can be positive recurrent if and only if:

1. Equation of the following matrices: $R^2B + RA + C = 0$ has the least non-negative solution R, since all matrices in the previous equation were defined by theory (3-3-1).
2. The homogeneous linear equations on the form $(\pi_0, \pi_1)B[R] = 0$ have a positive solution, where: $B[R] = \begin{bmatrix} A_0 & C_0 \\ B_1 & A + RB \end{bmatrix}_{(c+m_1)\times(c+m_1)}$ is a Square matrix and (π_0, π_1) meets the normal condition: $\pi_0 e + \pi_1(I - R)^{-1}e = 1$
3. Stable probability distribution can be expressed in a sequence of geometric matrices where

$$\pi_l = \pi_{l-1}R \quad , \quad l = 2,3, \ldots$$

   Then, the final formula is thus: $\pi_l = \pi_1 R^{l-1} \quad , \quad l = 2,3, \ldots.$

4. The probabilistic probability distribution of the previous state π_l where $l \geq 0$ is called the modified geometric matrix distribution.

In this case, the probability generation matrix *Q* differs from that in the case of the standard process only in the probability of transition from the border situations.

When analyzing multiple server queues with a M/M/c vacation we will use QBD in a more complex and variable way. There is no single boundary level l = 0, but the limits are multiple to l = 1,2,3, ..., c-1, and those levels contain a different number of states, and the number of states at level l will be denoted by $m_l$where: $0 \le l \le c-1$ attain the infinite micro generator on the form:

$$Q = \begin{bmatrix} A_0 & C_0 & & & & & \\ B_1 & A_1 & C_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_{C-1} & A_{C-1} & C_{C-1} & & \\ & & & B_C & A & C & \\ & & & & B & A & C \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (3.3.3)$$

Where

A_k is matrix $m_k \times m_k$, $0 \le k \le c-1$ grade

B_k A matrix of $m_k \times m_{k-1}$, $0 \le k \le c-1$ grade

C_k matrix of $m_k \times m_{k+1}$, $0 \le k \le c-2$ grade

C_{c-1} matrix of $m_{c-1} \times c$ grade

B_C matrix of $c \times m_{C-1}$ grade

A, B, C square matrices of c × c grade

The matrix Q in (3.3.3) can be divided as follows:

$$(3.3.4) \quad Q = \left[\begin{array}{c|cccc} \mathcal{A}_0 & \mathcal{C}_0 & & & \\ \hline \mathcal{B}_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & \ddots & \ddots & \ddots \end{array}\right]$$

Where: $\mathcal{A}_0$ is a square matrix of $m^* = m_0 + \dots + m_{c-1}$ grade

$$\mathcal{B}_1 = (0, B_k, k = C) \text{ ، } \mathcal{C}_0 = \begin{bmatrix} 0 \\ C_{c-1} \end{bmatrix}$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{C-2} & A_{C-2} & C_{C-2} \\ & & & & B_{C-1} & A_{C-1} \end{bmatrix}$$

Where:

$\mathcal{B}_1$ Non-square matrix of $c \times m^*$grade

$\mathcal{C}_0$ A non-square matrix of $m^* \times c$ grade

According to the above, the previous theory can be reformulated as follows:

4.5.3 Third theory

Semi-reduced birth and death (QBD) are frequent if and only if:

1. Equation of the following matrices: $B + RA + C = 0 \;\; R^2$ has the least non-negative solution R.
2. Homogeneous linear equations on the form

$$(\pi_0, \pi_1, \ldots, \pi_C)B[R] = 0$$

It has a positive solution$(\pi_0, \pi_1, \ldots, \pi_C)$, where:

$$B[R] = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{C-1} & A_{C-1} & C_{C-1} \\ & & & & B_C & A+RB \end{bmatrix}$$

The positive solution $(\pi_0, \pi_1, \ldots, \pi_c)$ meets the standard requirement where;

$$\sum_{l=0}^{c-1} (\pi_l e + \pi_c [I-R]^{-1} e) = 1$$

3. In addition, the stable probability distribution of the M/M/c queues model with vacations can be expressed mathematically as a geometric matrix expression as follows:

$$\pi_l = \pi_c R^{l-c}, l \geq c$$

4.6 Conditional Stochastic Decomposition

Assuming that the two-dimensional non-negative vector (X, J) has a common probability distribution:

$$\pi_{kj} = P(X = k, J = j), k \geq 0, 0 \leq j \leq c$$

Assuming that: $\pi_k = (\pi_{k0}, \pi_{k1}, \ldots, \pi_{kc}), k \geq 0$

Also assuming that: (X, J) follows the geometric matrix distribution (multiplying a matrix by matrices follows the geometric distribution), and there is a non-negative matrix R of grade c + 1 and then:

$$\pi_k = \beta R^k, \quad k \geq 0 \quad ; \quad \beta(I-R)^{-1} e = 1$$

when $k = 0$, then $\beta = \pi_0 = (\beta_0, \beta_1, \ldots, \beta_c)$

If R is a top trigonometric matrix (all elements below the main diameter are equal to zero) it can be divided as follows:

$R = \begin{bmatrix} H & \eta \\ 0 & r \end{bmatrix}$ where H is a square matrix c x c and η is vertical c x 1 vector and r is a real number.

Then, we will know the conditional random variable $X^{(c)}$ next $X^{(c)} = \{X | J = c\}$ that is, $X^{(c)}$ is a random variable equal to X provided that J is equal to c.

The variable $X^{(c)}$ is the sum of two independent random variables: $X^{(c)}=X_0 + X_d$

Where X_0 follows the geometrical distribution with parameter r, and X_d follows the distribution of PH of grade c and has the function generating the following probabilities:

$$X_d(z) = \frac{1}{\sigma}\{\beta_c + z(\beta_0, \beta_1, \ldots, \beta_{c-1})(I - zH)^{-1}\eta\}$$

Where;

$$\sigma = \beta_c + (\beta_0, \beta_1, \ldots \beta_{c-1})(I - H)^{-1}\eta \qquad (3.4.1)$$

If R is a trigonometric low matrix, $R = \begin{bmatrix} r & 0 \\ \xi & H \end{bmatrix}$ (all elements above the main diameter are zero), where H is a square matrix c × c and ζ is a c x1 vertical vector and r is a real number located in the category (0,1).

Thus we know the conditional random variable $X^{(c)}$ as follows:

$X^{(c)} = \{X | J = 0\}$, that is, $X^{(c)}$ is a random variable equal to X if J equals 0.

The variable $X^{(c)}$ is the sum of two independent random variables: $X^{(c)}=X_0 + X_d$

Where X_0 follows the engineering distribution with parameter r, and X_d follows the distribution of PH of grade c and has the function generating the following probabilities:

$$X_d(z) = \frac{1}{\sigma}\{\beta_c + z(\beta_0, \beta_1, \ldots, \beta_{c-1})(I - zH)^{-1}\zeta\}$$

Where;

$$(3.4.2) \quad \sigma = \beta_c + (\beta_0, \beta_1, \ldots \beta_{c-1})(I - H)^{-1} \zeta$$

(Sengupta, [22])

We will also discuss the different models of the non-stop queue systems, which include:

3. Multiple vacation model
4. Single vacation models and setup time

5. Multiple Vacation Model

In this case, we find that in a multi-service system M/M/c at the λ access rate and μ service rate, any server starts the vacation once a customer has finished serving and no customer is in the system.

At the end of the vacation for a server, if there is no customer in the queue, the server starts on another vacation, but if there are customers in the queue, the server starts to serve them immediately.

Since servers start their own vacations independently from each other, this system is called the inertial vacation model and is symbolized by the symbol M/M/c (AS, MV).

We will assume the following:

1. Vacation time follows

- Non-square matrix $(c+1) \times c^*$, $\mathcal{B}_1 = (0 \quad B_C)$
- $\mathcal{C}_0$ Non-square matrix $c^* \times (c+1)$, $\mathcal{C}_0 = \begin{pmatrix} 0 \\ C_{C-1} \end{pmatrix}$
- A_k square matrix of $(k+1)$ grade, $0 \le k \le c-1$

2. Know the next:

$$A_k = \begin{pmatrix} -h_0 & c\theta & & & \\ & -h_1 & (c-1)\theta & & \\ & & \ddots & \ddots & \\ & & & -h_{k-1} & (c-k-1) \\ & & & & -(\lambda + k\mu) \end{pmatrix}$$

Where; h_k is a function in λ ، μ ، θ ، defined as

$$h_k = h_k(\lambda, \mu, \theta) = \lambda + k\mu + (c-k)\theta, \quad 0 \le k \le c$$

- $C_0 = (\lambda \quad 0)$ ، $B_1 = \begin{pmatrix} 0 \\ \mu \end{pmatrix}$ ، $A_0 = -\lambda$

- B_k Non-square matrix of $(k+1) \times k$, $0 \le k \le c-1$ grade

$$B_k = \begin{pmatrix} 0 & & & & \\ 0 & \mu & & & \\ \vdots & & 2\mu & & \\ & & & \ddots & \\ & & & & (k-1)\mu \\ 0 & & & & k\mu \end{pmatrix}_{(k+1)\times k}$$

- C_k Non-square matrix of $(k+1) \times (k+2)$, $0 \le k \le c-1$ grade

$$C_k = \begin{pmatrix} \lambda & & & & 0 \\ & \lambda & & & \vdots \\ & & \ddots & & \\ & & & \lambda & 0 \end{pmatrix}_{(k+1)\times(k+2)}$$

- *A* square matrix of $(c+1)$ grade

$$A = \begin{pmatrix} -h_0 & c\theta & & & \\ & -h_0 & (c-1)\theta & & \\ & & \ddots & \ddots & \\ & & & -h_{c-1} & \theta \\ & & & & -h_c \end{pmatrix}_{(c+1)\times(c+1)}$$

- B Square diagonal matrix of c + 1 grade

$$B = diag(0, \mu, 2\mu, \ldots c\mu) = \begin{pmatrix} 0 & & & & \\ & \mu & & & \\ & & 2\mu & & \\ & & & \ddots & \\ & & & & c\mu \end{pmatrix}_{(c+1)\times(c+1)}$$

- C Square diagonal matrix of c + 1 grade

$$C = \lambda I = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix}_{(c+1)\times(c+1)}$$

(Levy and Yechiali, [23]; Vinod, [24]; Tian and Li, [25])

The analysis of the system of any queue to access the system stability state and to achieve the state of the system stability in the used model is that the access rate is lower than the leaving rate. This is achieved if $\lambda < c\mu$, by dividing the two vales on cμ results in: $\rho = \lambda / c\mu < 1$, which is the stability condition, also the functions matrix R is the solution of the equation of the famous matrices and has to be determined:
$B + RA + C = 0 \ R^2$, and in order to reach them, the following theories must be used [19]:

5.1 First theory

If $\rho = \lambda / c\mu < 1$, then he following quadratic equation has two different solutions

$k\mu z^2 - [\lambda + k\mu + (c-k)\theta]z + \lambda = 0 \,; 1 \le k \le c$ ، the two solutions $r_k < r_k^*$ achieve the condition : $0 < r_k < 1 , r_k^* \ge 1$.

5.2 Second theory

If ρ <1 is the least negative solution R (where R is defined as in theory 1 and given by the following square matrix of (*c+1*s) matrix*:*

$$R = \begin{pmatrix} r_0 & r_{01} & & r_{0c} \\ & r_1 & \cdots & r_{1c} \\ & & \ddots & \vdots \\ & & & r_c \end{pmatrix} \quad (3.5.2)$$

Where;

- $0 < r_0 = \lambda \big/ (\lambda + c\theta) < 1$ ، $0 < r_k < 1, 1 \le k \le c-1$ Are roots that were previously defined in Theory 1.
- $r_c = \rho < 1$.
- Non-diametric elements (i.e., not located on the main diameter) achieve the following frequency relationship:

$$j\mu = \sum_{i=k}^{j} r_{ki} r_{ij} + (c-j+1)\theta r_{k,(j-1)} - [\lambda + j\mu + (c-j)\theta] r_{kj} = 0 \qquad (3.5.3)$$

It is the relationship that gives the value of the element r_{kj} in terms of the next element in the same column directly $r_{k\,(j+1)}$ which implies that j> k, so that:

$$0 \le k \le c-1 \,, k+1 \le j \le c$$

It is clear that the calculation of $r_{kk} = r_k$ values can be computed directly from the square root value in theory (3.5.1) as follows:

$$r_k = [\lambda + k\mu + (c-k)\theta - \sqrt{(\lambda + k\mu + (c-k)\theta)^2 - 4\lambda k\mu}] \;/[2k\mu]$$

However, the calculation of non-diametric values r_{kj} is very difficult because it is calculated sequentially by calculating its previous values and the following figure (3-2) shows an example of the state of c = 4:

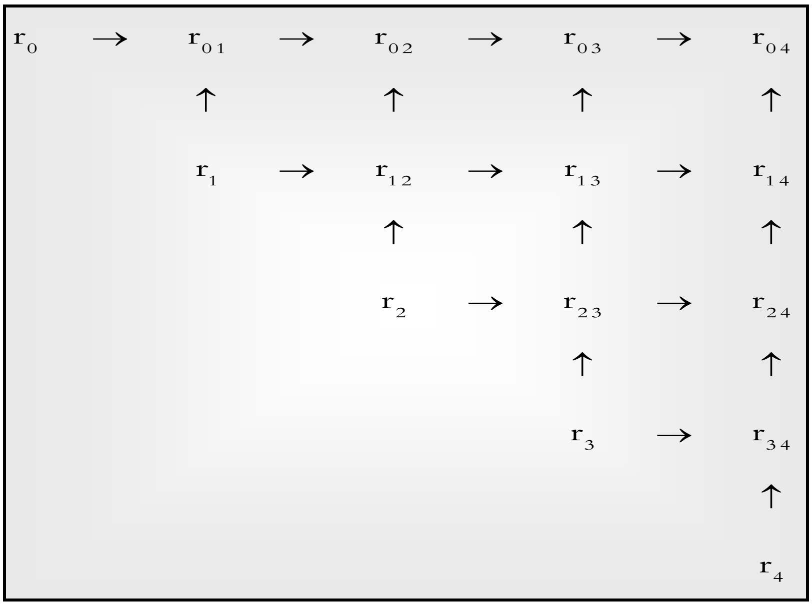

Figure (3-2): Sequential calculation of the non-diametric values of matrix R in the state of c = 4

After all the above we assume that:

The stable end of the random operation: $\{[L_v(t), J(t)], t \ge 0\}$ of M/M/c (AS, MV) system is (L_v, J) when $t \to \infty$

Then the stable probability of transition from state k to state j is defined as follows:

$$\pi_{kj} = P\{L_V = k, J = j\} = \lim_{t\to\infty} P\{L_V(t) = k, J(t) = j\}, (k,j) \in \Omega$$

Where Ω is the state's space

The following is the vector form of the transition probability:

$$\pi_0 = \pi_{00}, \pi_1 = (\pi_{10}, \pi_{11}), \dots \pi_k = (\pi_{k0}, \pi_{k1}, \dots, \pi_{kk}), 0 \le k \le c$$

In the case of k> c, all probability vectors are equal:

$$\pi_k = (\pi_{k0}, \pi_{k1}, \dots, \pi_{kc})$$

The following theory illustrates the sequential relationship of probability vectors:

5.3 Third theory

If $\rho < 1$: then the stable probabilistic vectors achieve the relationship

$$\lambda\, \pi_k e = \pi_{k+1} T_{k+1}^0$$

Where;

$$T_k^0 = (0, \mu, 2\mu, \dots, k\mu)^T, \qquad 0 \le k \le c-1$$
$$T_k^0 = (0, \mu, 2\mu, \dots, c\mu)^T, \qquad k \ge c$$

It is the relation that gives the vector π_{k+1} by the vector π_k.

To compare M/M/c (AS, MV) system and the traditional M/ M/c system we have to:

First, know the probability of a number of customers in the M/ M/c (AS, MV) (L_v - c) system if all the servers are busy and can be written as:

$$L_v^{(c)} = P\{L_v - c / J = c\}$$

Second, know β_k: which is the positive solution to equation matrices:

$$(\pi_0, \pi_1, \dots, \pi_c) B[R] = 0 \quad (3.5.4)$$

$$B[R] = \begin{pmatrix} \mathcal{A}_0 & \mathcal{C}_0 \\ \mathcal{B}_1 & A + RB \end{pmatrix} = \begin{pmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{C-1} & A_{C-1} & C_{C-1} \\ & & & & B_C & A + RB \end{pmatrix} \quad (3.5.5)$$

Third, represent βc, B[R] as follows:

$$\beta_c = (\beta_{c0}, \beta_{c1}, \dots, \beta_{cc}) = (\delta, \beta_{cc}), \delta = (\beta_{c0}, \beta_{c1}, \dots, \beta_{c(c-1)})$$

$$R = \begin{pmatrix} H & \eta \\ 0 & \rho \end{pmatrix}, H = \begin{pmatrix} r_0 & r_{01} & \cdots & r_{0(c-1)} \\ & r_1 & \cdots & r_{1(c-1)} \\ & & \ddots & \vdots \\ & & & r_{c-1} \end{pmatrix}_{c\times c}, \eta = \begin{pmatrix} r_{0c} \\ r_{1c} \\ \vdots \\ r_{(c-1)c} \end{pmatrix} \quad (3.5.6)$$

The following theories illustrate the relationship between M/M/c (AS, MV) and the traditional M/M/c system:

5.4 Fourth theory

If $\rho < 1$, then, $L_v^{(c)}$ can be divided into two parts : $L_v^{(c)} = L_0^{(c)} + L_d$ where;

$L_0^{(c)}$ is the random variable that represents the corresponding variable in M/M/c, the number of customers in queue, if all the servers are busy, it follows the geometrical distribution
L_d is the length of the extra queue resulting from the vacation, and follows the distribution of PH of c grade
Where;

$$P\{L_d = k\} = \begin{cases} \frac{1}{\sigma}\beta_{cc} & , \quad k = 0 \\ \frac{1}{\sigma}\delta H^{k-1}\eta , & k \geq 1 \end{cases} \quad (3.5.7)$$

Where;

$$\sigma = \beta_{cc} + \delta(I - H)^{-1}\eta$$

Sengupta [22] has demonstrated that the PH distribution in L_d must be a sporadic PH distribution of class c and where he has derived its distribution function.

It was also concluded that predicting the conditional random variable $L_v^{(c)}$ is:

$$E[L_v^{(c)}] = \frac{1}{1-\rho} + \frac{1}{\sigma}\delta(I - H)^{-2}\eta$$

We also define the conditional time: $W_V^{(c)} = (W_V | J = c)$. The following is the theory of conditional time:

5.5 Fifth theory

If ρ < 1, then, $W_v^{(c)}$ can be divided into two parts : $W_v^{(c)} = W_0^{(c)} + W_d$ where;

$W_0^{(c)}$ is the random variable that represents to the corresponding variable in M/M/c. time in the queue without vacation if all the servers are busy, which follows the exponential distribution with the parameter cμ(1-ρ).
W_d is the delay time of the return of the vacation and follows the distribution of exponential matrices where:

$$P\{W_d \leq x\} = 1 - \frac{1}{\sigma}\delta.\exp\{-c\mu(I - H)x\}(I - H)^{-1}\eta \quad (3.5.8)$$

It was also possible to conclude the expectation of conditional time:

$$E[W_v^{(c)}] = \frac{1}{c\mu(1-\rho)} + \frac{1}{c\mu\sigma}\delta(I - H)^{-2}\eta = \frac{1}{c\mu}E[L_v^{(c)}]$$

(Sengupta, [22]).

6. Single Vacation and Setup Time Models

Assuming that we are studying the M/M/c (AS,SV) system model as a single asynchronous vacation. In this system, every server gets a single vacation with a fixed time at the moment it is finished the service of a customer, and then returns to the service of a customer if someone is in the queue, or remains idle.
Thus, there are only three cases per server:

1. To serve a customer
2. Have a vacation
3. To be without work

6.1 M/M/c Queue with Synchronous Vacations

We have mentioned earlier that there are multiple vacation policies for queue servers that are suitable for many applications and we have introduced one of these policies, which is the vacation policy of all the servers that are not synchronized, and here we will be interested in studying another policy, which is the vacation policy of all synchronized servers whether this vacation multiple or single or having prepare time and then build a system to evaluate the performance of this type of synchronized vacation.

6.2 M/M/c Queue with Synchronous Vacations

In this section, we will learn about the different models of queuing systems with synchronized vacations that include:

- Multiple vacation model
- Single vacation models and setup time

6.3 Multiple Vacation model

The M/M/c multi-server queue system has the λ access rate, the μ service rate, the FIFO service provisioning system, and the c number of server system. The stability condition for this system is [26]:

$$\rho = \lambda(c\mu)^{-1} < 1$$

in the M/M/c (SY,MV) system:

The number of customers at the moment t in this system is indicated by $L_v(t)$

Then, we define J(t) as: the function that indicates the existence of a vacation for servers or not as follows:

$$J(t) = \begin{cases} 0 & if\ servers\ are\ not\ on\ vacation, \\ j & if \text{ servers are on vacation at phase j, } j = 1,2..c \end{cases}$$

Because vacations are synchronized, at least one server is busy during the non-vacation period, and some servers may not be working.

In the single synchronous vacation system, M/M/c (SY, SV), all unoccupied servers take a single vacation together at the same time as the completion of the service for all customers, at which point the system becomes empty and at the end of the vacation either servers remain idle, or serve customers if a customer appears to request service from the system [27].

6.4 Computational Analysis of Synchronous Vacations Systems

Multi-server queue model analysis the with synchronous vacations is based on:

1. Structure of the system under study
2. System operation policies

6.5 System Structure

The structure of the system is the basic rules on which the system is based:

1. The system consists of a set of servers and their number s where s = 1,2....
2. The system allows an infinite number of customers with i where i = 1,2....
3. Customers arrive at the system in a random way.
4. The queue capacity is not limited.
5. Customers queuing in the line to request service from multiple stations or service centers.
6. Service time is random.
7. Generate access times for customers from exponential distribution at λ.
8. Generation of service times according to exponential distribution at μ_s.
9. Randomly generate vacation times from exponential distribution at a rate of μ_V.

6.6 Operating Policies

In order to run this system we are working on a model to simulate this system and running this system by the following steps:

1. We assume that the number of customers requesting service is i where i = 1,2, ... then the arrival time for customers from the exponential distribution at λ according to the equation [16]:

   AT (i + 1) = AT (i) + IAT (i), i = 1,2, ... (4.3.1)

   Where AT(i) is the time of customer access IAT (i), time i between customer i and customer i + 1 AT (i + 1) i + 1 customer arrival time.
2. Service time varies from one customer to another depending on the status of the service provider where the customer can receive the service directly if the service provider is idle or in the row until the end of the service provider to provide service to the previous customer.
3. The principle of service which will be used from first comes first served (FIFO)
4. Randomly generate vacation times from exponential distribution at a rate of μ_V.
5. Customer service from customer 1 to customer S where S is the number of servers in the system without to assume that the system receives the first S of customers without giving a vacation to the system at the start of the system
6. The service of the remaining customers from customer S + 1 to the last customer can indicate the timing of the completion of the provision of server i for the service of the customer to whom this server provides the required service

   With Dts (i) where i = 1,2, ... S. Therefore, we have a vector of S elements, DTs, which represents each of its elements. When each S server can provide the service to a new customer after its end to provide its current customer service.

7. Results and Discussion

This section shows the results and evaluation of the performance of the queuing with vacations system. We have sought to obtain the best results for the calculation of performance measures. We have worked to change the λ access rate, which varies between 5-20 customer> also, we assume different values for leaving rates that take μ = 1,3,5,10 and the number of servers c takes values $c = 1,2,3,4,5$

We have designed a method to calculate performance metrics for the previous queue system defined as average system uptime and average vacation time. We have changed the different features of access rates, service rates and number of servers to achieve the best results and try to improve and develop the system which makes it easier for servers to perform their service to obtain the appropriate vacations for them without being affected by the customer service and the system interest is disrupted where we have obtained the following results:

7.1 Standard 1: Average waiting time in the system

It means that the average waiting time of the system is clearly affected by both the number of servers, the λ access rate and the service rate μ. This results in a change in the value of ρ where $\rho = \frac{\lambda}{\mu}$, We found that:

A. The higher the access rate λ for the same number of servers we find that the average waiting time of the system is increasing clearly that there is a correlation between them, where the increase of one is the increase in the other and we can note this increase from Table (1).

B. Also, the higher the rate of service μ for the same value of the access rate and the number of servers itself, the lower the average waiting time in the system, i.e., there is an inverse relation between them where this decrease can be seen in Table (1).

C. As for the number of servers, the greater the number of servers and the values of both the arrival and leaving rates, the lower the value of the average time to waiting in the system this decrease can be seen in Table (1).

In general, the lower the value of ρ, the more stable the system, and the values converge to a fixed value, where the average waiting time is determined by the system as follows:

	λ	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
μ = 1	c = 1	3.19	19.84	30.71	40	47.34	52.9	57.66	61.56	65	67.59	70.2	72.38	74.18	75.98	77.78	79.46
	ρ	5.00	6.00	7.00	8.00	9.00	10.00	11.00	12.00	13.00	14.00	15.00	16.00	17.00	18.00	19.00	20.00
	c = 2	3.94	18.7	29.99	38.38	45.96	52	56.38	60.72	63.55	66.6	69.15	71.17	72.77	74.37	75.97	77.57
	ρ	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00
	c = 3	5.85	14.91	26.42	35.68	42.66	49.36	53.56	4.57	4.90	5.24	5.57	5.90	6.24	6.57	6.90	7.24
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 4	2.34	14.52	26.9	35.33	42.33	48.03	53.67	56.95	59.75	62.23	64.95	68.35	69.55	70.75	71.95	73.15
	ρ	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	c = 5	2.66	13.72	23.51	32.44	40	45	49.36	52.66	56.26	59.84	62.07	64.57	66.67	68.07	69.47	70.87
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
μ = 3	c = 1	0.35	0.36	0.37	0.38	0.4	0.425	0.455	0.528	0.604	0.77	1.9	3.2	4.5	5.82	7.63	8.88
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 2	0.35	0.36	0.37	0.38	0.4	0.43	0.518	0.56	0.65	0.82	1.15	2.38	4.3	5.96	7.47	8.75
	ρ	0.83	1.00	1.17	1.33	1.50	1.67	1.83	2.00	2.17	2.33	2.50	2.67	2.83	3.00	3.17	3.33
	c = 3	0.33	0.34	0.35	0.36	0.37	0.385	0.41	0.501	0.63	0.84	1.04	2.34	4.3	5.82	7.12	8.33
	ρ	0.56	0.67	0.78	0.89	1.00	1.11	1.22	1.33	1.44	1.56	1.67	1.78	1.89	2.00	2.11	2.22
	c = 4	0.33	0.34	0.35	0.36	0.37	0.384	0.404	0.454	0.51	0.58	0.78	1.67	3.37	5.06	6.52	7.66
	ρ	0.42	0.50	0.58	0.67	0.75	0.83	0.92	1.00	1.08	1.17	1.25	1.33	1.42	1.50	1.58	1.67
	c = 5	0.33	0.34	0.35	0.36	0.37	0.381	0.401	0.423	0.49	0.63	0.87	1.77	3.07	4.37	5.5	6.64
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33

μ = 5	c = 1	0.20447	0.20557	0.20667	0.20777	0.20887	0.21017	0.21223	0.21515	0.21929	0.22411	0.22977	0.23939	0.25074	0.26503	0.28192	0.29682
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
	c = 2	0.20596	0.20646	0.20696	0.20746	0.20809	0.21023	0.21253	0.21487	0.2176	0.22111	0.22583	0.2331	0.2428	0.25565	0.27128	0.29066
	ρ	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
	c = 3	0.19702	0.19732	0.19762	0.19792	0.19819	0.19939	0.20044	0.20232	0.20458	0.20834	0.21342	0.22021	0.23298	0.2493	0.26851	0.29319
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
	c = 4	0.19809	0.19839	0.19869	0.19905	0.19995	0.2013	0.20285	0.20468	0.20758	0.21106	0.21532	0.22102	0.22766	0.23549	0.24508	0.25634
	ρ	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	c = 5	0.192	0.1925	0.19292	0.1937	0.1947	0.1957	0.1986	0.2014	0.2053	0.2101	0.21515	0.22065	0.22563	0.23373	0.24244	0.25404
	ρ	0.20	0.24	0.28	0.32	0.36	0.40	0.44	0.48	0.52	0.56	0.60	0.64	0.68	0.72	0.76	0.80
μ = 10	c = 1	0.103021	0.103031	0.103041	0.103051	0.103061	0.103095	0.103149	0.103191	0.103341	0.103491	0.103646	0.103938	0.104158	0.104378	0.104668	0.104958
	ρ	0.500	0.600	0.700	0.800	0.900	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
	c = 2	0.102500	0.102520	0.102540	0.102555	0.102685	0.102815	0.102945	0.103083	0.103208	0.103278	0.103557	0.103917	0.104197	0.104397	0.104685	0.104958
	ρ	0.250	0.300	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800	0.850	0.900	0.950	1.000
	c = 3	0.098896	0.098906	0.098916	0.098926	0.098936	0.098946	0.098965	0.098985	0.099051	0.099171	0.099229	0.099399	0.099638	0.099896	0.100192	0.100542
	ρ	0.167	0.200	0.233	0.267	0.300	0.333	0.367	0.400	0.433	0.467	0.500	0.533	0.567	0.600	0.633	0.667
	c = 4	0.098187	0.098197	0.098207	0.098217	0.098251	0.098291	0.098339	0.098391	0.098451	0.098511	0.098577	0.098727	0.098877	0.099027	0.099229	0.099417
	ρ	0.125	0.150	0.175	0.200	0.225	0.250	0.275	0.300	0.325	0.350	0.375	0.400	0.425	0.450	0.475	0.500
	c = 5	0.095750	0.095760	0.095770	0.095780	0.095790	0.095800	0.095830	0.095898	0.095970	0.096180	0.096254	0.096428	0.096653	0.097066	0.097437	0.098100
	ρ	0.100	0.120	0.140	0.160	0.180	0.200	0.220	0.240	0.260	0.280	0.300	0.320	0.340	0.360	0.380	0.400

Table (1) Average waiting time in the system

The following are the graphs for the average waiting time in the system:

First, if we follow the results of the average waiting time of the system in the case when the rate of service is (μ = 1), we note that the values tend to increase as the access rate increases and the average waiting time of the system decreases as the number of servers increases as shown in Figure (1).

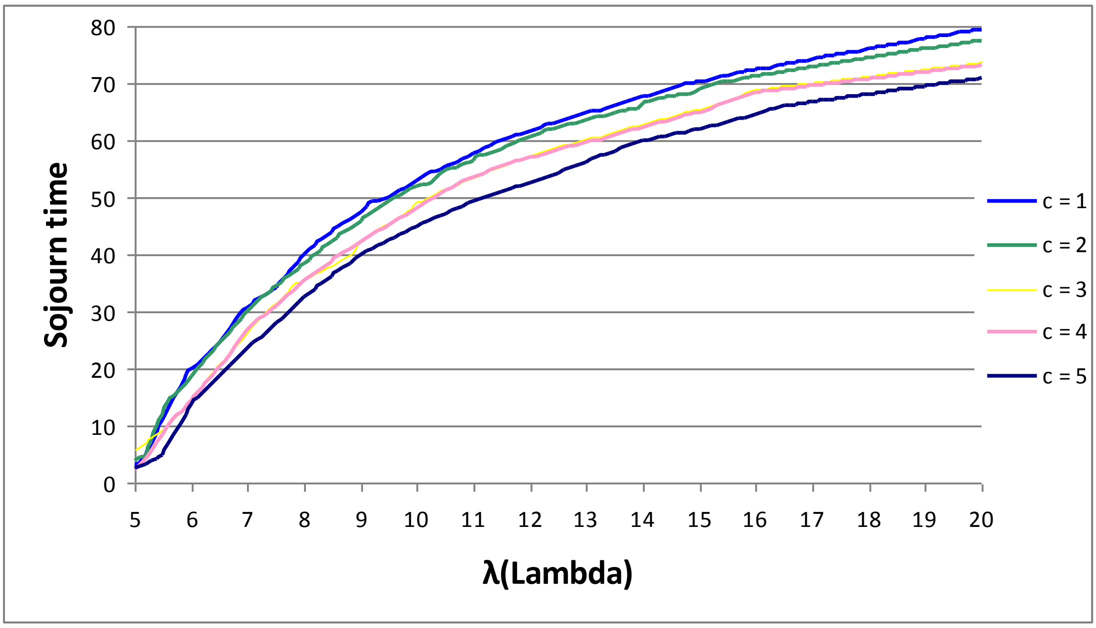

Fig-1: the curve representing average waiting time in then system for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 1

Second, if we also follow the results of the average values of the waiting time of the system in the case when the service rate is set (μ = 3), we note that the values tend to be somewhat stable until the arrival rate is λ = 14 and therefore the increase in the average waiting time λ as well as the average waiting time of the system decreases with the number of servers as shown in Figure (2).

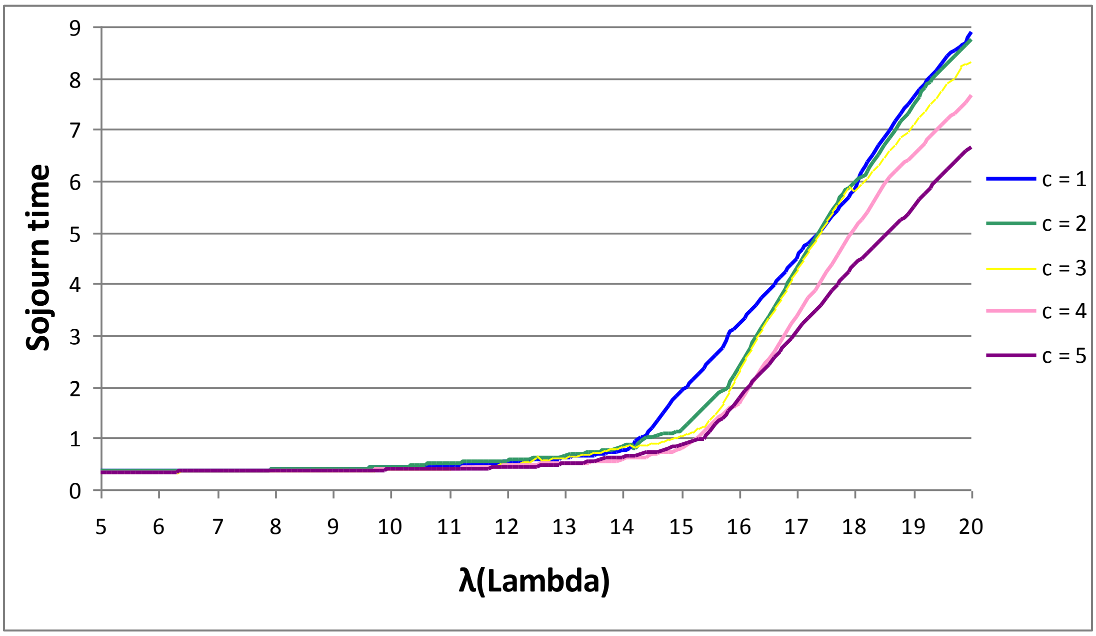

Fig-2: the curve representing average waiting time in then system for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 3 μ=3

Thirdly, if we also follow the results of the average values of the system waiting time in the case when the service rate is set (μ = 5), the increase in the average waiting time of the system increases as the access rate increases and the average waiting time decreases as the number of servers increases,

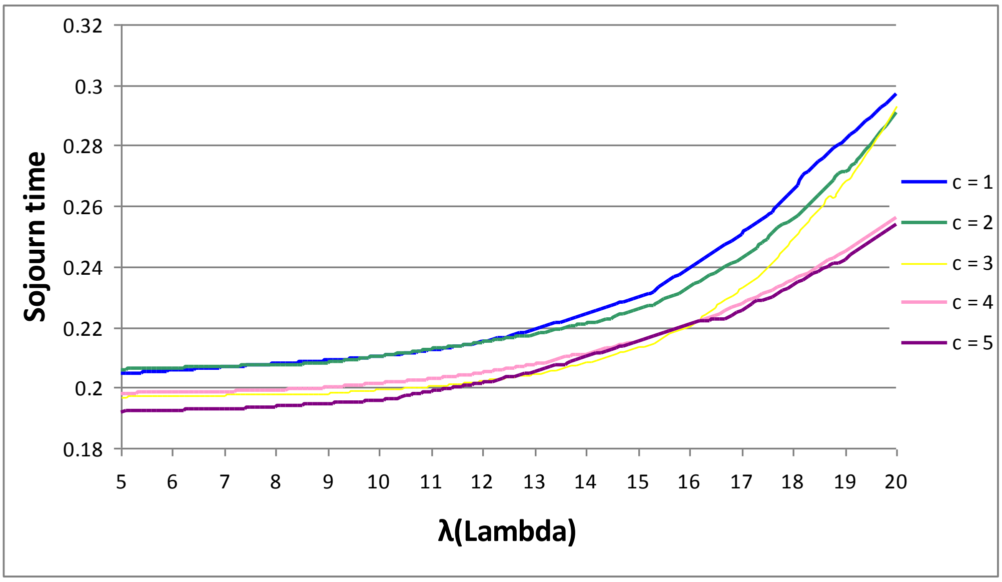

Fig-3: the curve representing average waiting time in then system for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 5

Fourth, if we also follow the results of the average waiting time of the system in the situation at the rate of service (μ = 10) we also find the observed increase in the average time of waiting of the system the greater the rate of access λ and significantly decreases the average waiting time of the system as the number of servers as evident from Figure (4).

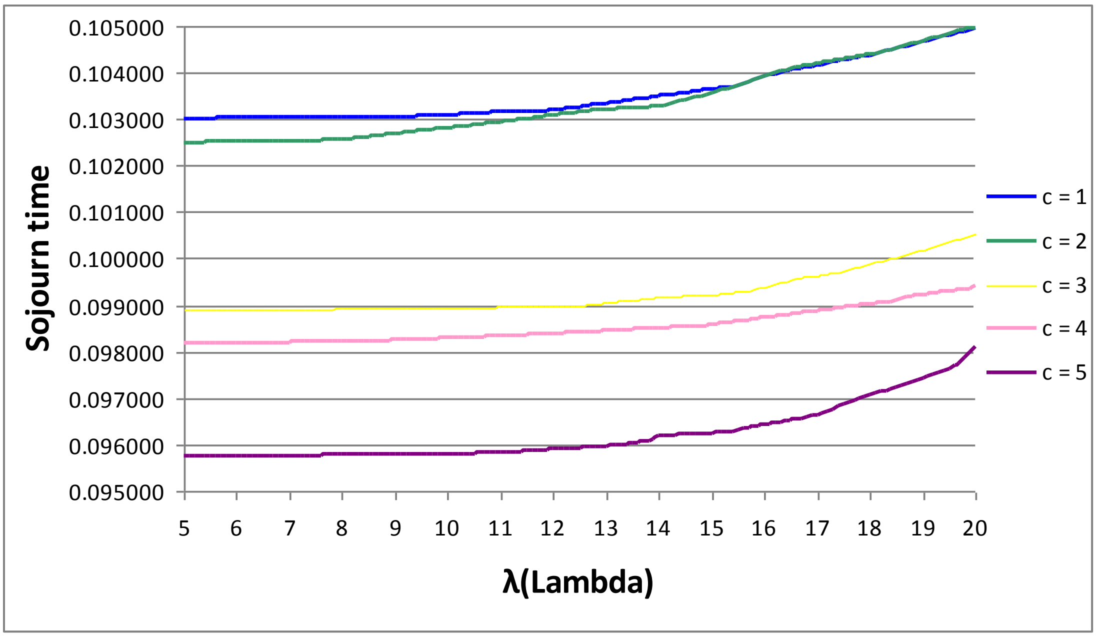

Fig-4: the curve representing average waiting time in then system for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 10

In each of the previous cases, we find that as the service rate increases, the average waiting time of the system decreases as shown in Figures 1, 2, 3, and 4.

7.2 Standard 2: Average vacation time

This measure is concerned with the convenience of servers and their interest and helps to satisfy all servers to give them a vacation so as not to affect the workflow and know the extent of the damage on the system in the case of giving a vacation to servers at certain periods and under certain circumstances, we found that the average vacation time is clearly affected by each of the number of servers, The λ and the service rate μ and this would result in a change in the value of ρ as $\rho = \frac{\lambda}{\mu}$ we found the following:

a) With the increase in the value of λ, the number of servers and the stability of the service rate μ, we find that the vacation time is very low.
b) With the increase in the value of the service rate μ, the stability of the access rate and the stability of the number of servers, there is a chance of a vacation to a certain extent.
c) The increased number of servers and the stability of both access and service rates also increase the chance of a very large vacation.
d) Thus, the lower the value of ρ, the more stable the system, then increasing the chance of the vacation and then increasing the average vacation time.

Table (2) shows the average vacation time in different cases as follows:

	λ	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
μ = 1	c = 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	ρ	5.00	6.00	7.00	8.00	9.00	10.00	11.00	12.00	13.00	14.00	15.00	16.00	17.00	18.00	19.00	20.00
	c = 2	0.013	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	ρ	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00
	c = 3	0.077	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 4	0.2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	ρ	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	c = 5	0.457	0.342	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
μ = 3	c = 1	0.1872	0.1476	0.1178	0.1009	0.0899	0.0718	0.0627	0.0410	0.0168	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	ρ	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00	4.33	4.67	5.00	5.33	5.67	6.00	6.33	6.67
	c = 2	0.1931	0.1501	0.1224	0.1168	0.1060	0.0935	0.0937	0.0792	0.0686	0.0251	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	ρ	0.83	1.00	1.17	1.33	1.50	1.67	1.83	2.00	2.17	2.33	2.50	2.67	2.83	3.00	3.17	3.33
	c = 3	0.1910	0.1607	0.1302	0.1146	0.1092	0.0989	0.1007	0.0843	0.0734	0.0462	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	ρ	0.56	0.67	0.78	0.89	1.00	1.11	1.22	1.33	1.44	1.56	1.67	1.78	1.89	2.00	2.11	2.22
	c = 4	0.2021	0.1773	0.1536	0.1448	0.1408	0.1225	0.1217	0.1101	0.1027	0.1048	0.1463	0.0000	0.0000	0.0000	0.0000	0.0000
	ρ	0.42	0.50	0.58	0.67	0.75	0.83	0.92	1.00	1.08	1.17	1.25	1.33	1.42	1.50	1.58	1.67
	c = 5	0.2378	0.2000	0.1862	0.1667	0.1445	0.1396	0.1217	0.1101	0.1027	0.1048	0.0867	0.0739	0.0637	0.0567	0.0517	0.0000
	P	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
μ = 5	c = 1	0.2000	0.1584	0.1362	0.1176	0.0985	0.0879	0.0761	0.0675	0.0632	0.0558	0.0535	0.0475	0.0436	0.0397	0.0309	0.0239
	ρ	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80	4.00
	c = 2	0.1984	0.1596	0.1285	0.1199	0.1073	0.0887	0.0765	0.0696	0.0682	0.0673	0.0600	0.0611	0.0543	0.0456	0.0400	0.0280
	ρ	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
	c = 3	0.1989	0.1655	0.1383	0.1159	0.1075	0.0975	0.0781	0.0742	0.0697	0.0654	0.0655	0.0648	0.0576	0.0575	0.0479	0.0399
	ρ	0.33	0.40	0.47	0.53	0.60	0.67	0.73	0.80	0.87	0.93	1.00	1.07	1.13	1.20	1.27	1.33
	c = 4	0.2080	0.1691	0.1477	0.1262	0.1176	0.1069	0.1004	0.0848	0.0858	0.0841	0.0835	0.0751	0.0694	0.0664	0.0570	0.0500
	ρ	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	c = 5	0.2293	0.1975	0.1617	0.1447	0.1323	0.1229	0.1204	0.1088	0.1029	0.1037	0.0847	0.0876	0.0837	0.0680	0.0637	0.0641
	ρ	0.20	0.24	0.28	0.32	0.36	0.40	0.44	0.48	0.52	0.56	0.60	0.64	0.68	0.72	0.76	0.80
μ = 10	c = 1	0.19583	0.17167	0.14488	0.12509	0.10832	0.10111	0.09001	0.07931	0.07183	0.06873	0.06435	0.06100	0.05490	0.05040	0.04590	0.04420
	ρ	0.500	0.600	0.700	0.800	0.900	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
	c = 2	0.19583	0.16973	0.14484	0.12958	0.11473	0.10195	0.09039	0.08195	0.07245	0.06625	0.06375	0.06105	0.05737	0.05392	0.04915	0.04543
	ρ	0.250	0.300	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800	0.850	0.900	0.950	1.000
	c = 3	0.20750	0.16875	0.14375	0.12491	0.10997	0.09954	0.08794	0.08136	0.07542	0.06967	0.06341	0.05751	0.05541	0.05331	0.05041	0.04791
	ρ	0.167	0.200	0.233	0.267	0.300	0.333	0.367	0.400	0.433	0.467	0.500	0.533	0.567	0.600	0.633	0.667
	c = 4	0.20167	0.16417	0.14507	0.12767	0.11599	0.10458	0.09342	0.08522	0.08066	0.07333	0.06739	0.06215	0.05962	0.05751	0.05542	0.05292
	ρ	0.125	0.150	0.175	0.200	0.225	0.250	0.275	0.300	0.325	0.350	0.375	0.400	0.425	0.450	0.475	0.500
	c = 5	0.22250	0.18892	0.16507	0.14208	0.12833	0.11775	0.10526	0.09976	0.09167	0.08337	0.07826	0.07267	0.06986	0.06770	0.06375	0.06165
	ρ	0.100	0.120	0.140	0.160	0.180	0.200	0.220	0.240	0.260	0.280	0.300	0.320	0.340	0.360	0.380	0.400

Table (2): the average vacation time

The following are the graphs for the average vacation time:

First, if we follow the results of the average values of the vacation time in the case when setting the service rate (μ = 1), we note that the values tend to decrease as the access rate increases λ in the sense that there is a reverse correlation between the vacation time and the arrival rate is almost fading and decay and the average vacation time increases The number of servers is as shown by Figure (5).

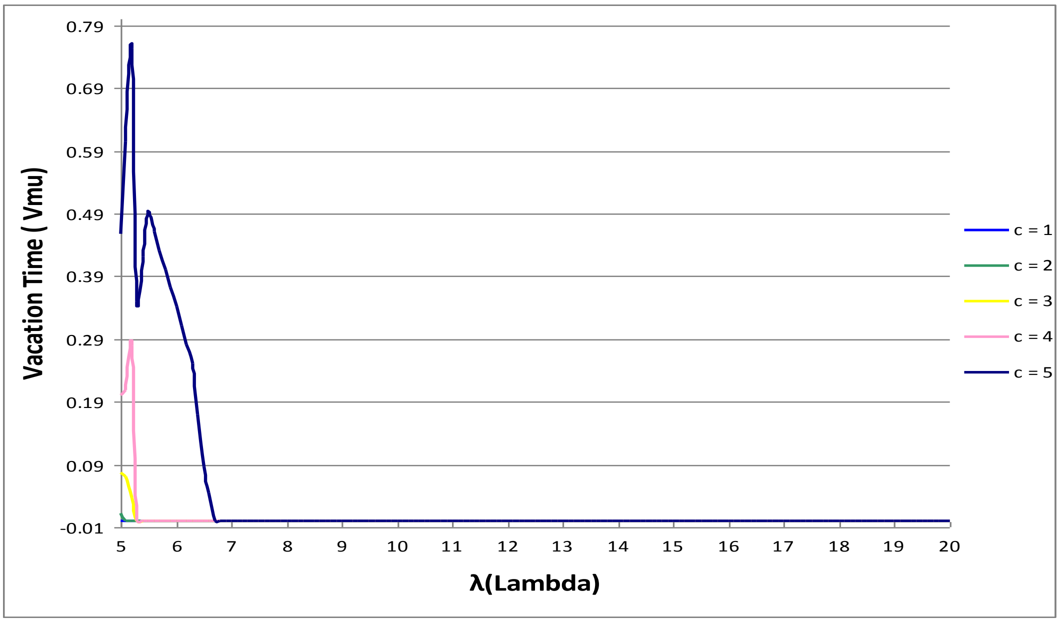

Fig-5: the curve representing average duration for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 1

Second, if we also follow the results of the average time of vacation in the case when the rate of service is developed (μ = 3), we note that values tend to decrease and fluctuate sometimes between increase and decrease. Thus, the increase in the average vacation time is found when the arrival rate is less λ and the average vacation time increases with the number of servers as shown in Figure (6).

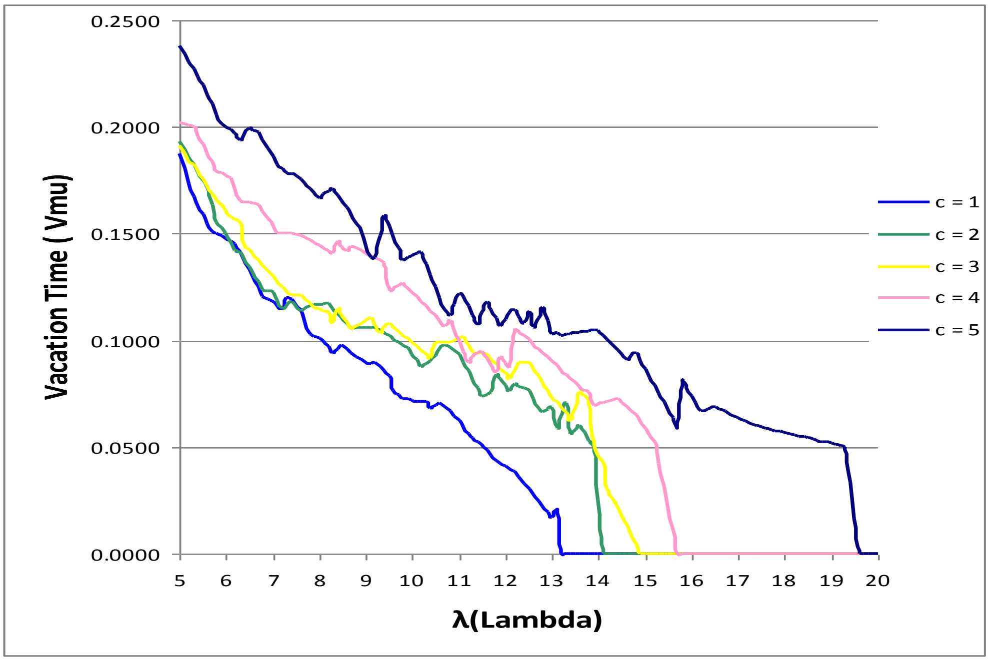

Fig-6: the curve representing average duration curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 3

Third: If we also follow the results of the average values of the vacation time in the case when the rate of service is developed (μ = 5), we find the decrease in average vacation time as the arrival rate increases, and the average vacation time increases with the number of servers as shown in Figure (7).

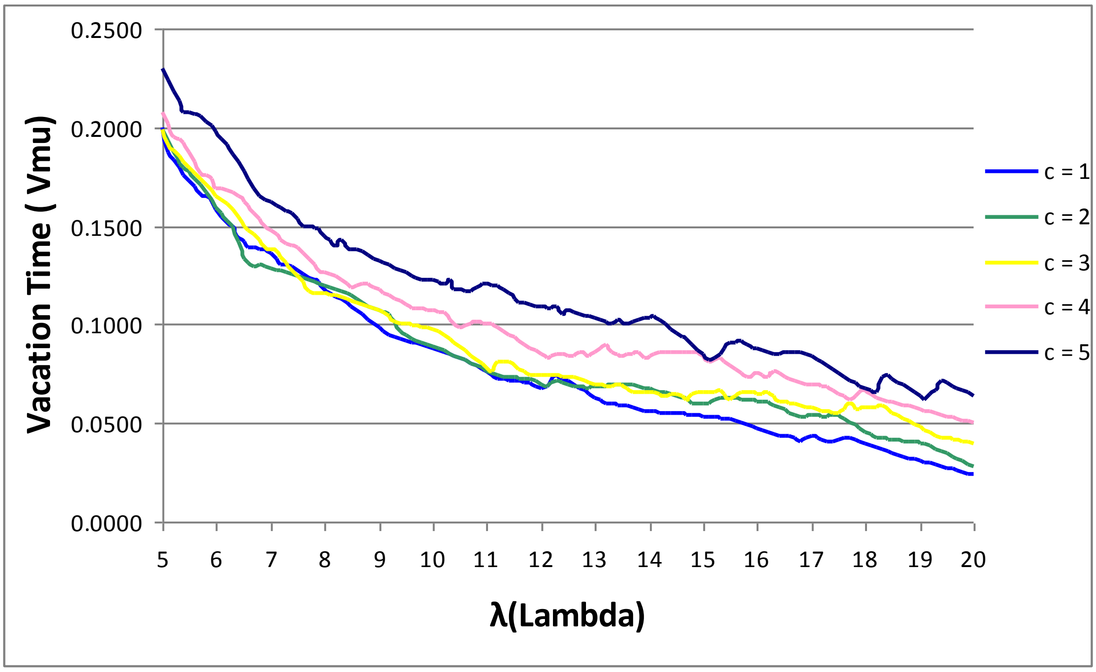

Fig-7: the curve representing average duration curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 5

Four: If we also follow the results of the average values of the vacation time in the case when the rate of service is developed (μ = 10). We also find the increase in the average vacation time when the arrival rate is less than λ, and the average vacation time increases with the number of servers as shown in Figure 8.

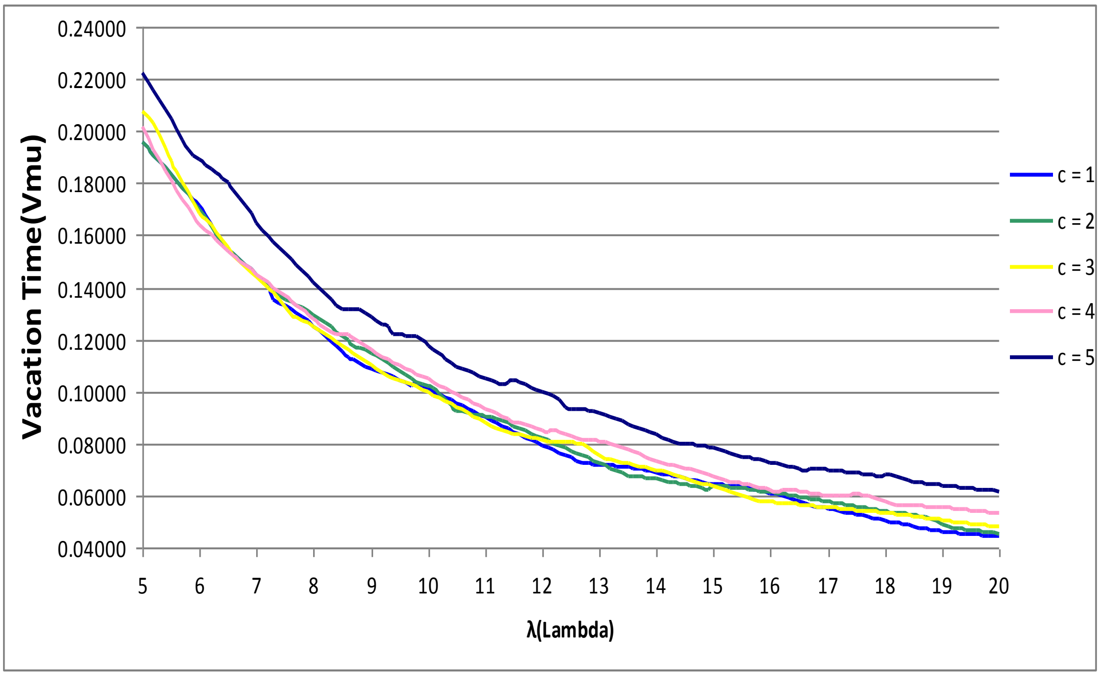

Fig-8: the curve representing average duration curve for each case of the number of service providers c = 1,2,3,4,5 and constant service rate μ = 10

In each of the previous cases, in comparison, it is found that with the increase in the average service interval, the average vacation time is increased as shown in Figures (5), (6), (7) and (8).

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