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Existence and Uniqueness of Solution for Knowledge Accumulation Model

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Abstract: This paper is based on the development system. After considering new learning knowledge, knowledge forgetting rate and repeated contact rate, we build knowledge accumulation model. With the help of characteristics methods, we prove the existence and uniqueness of solution.
Keywords: knowledge accumulation model characteristics methods.

INTRODUTION

Recently, knowledge competition shows are more and more popular all over the world, such as “Wheel of Fortune” in USA, “Chinese Poetry Competition” in China, “Newsdog” in India and so on. When answering questions, we always can’t remember the whole knowledge clearly, even though we learned it before. So we are interested in how knowledge accumulate. After looking for many papers, I found there was few math model about this, paper[1] study knowledge accumulation progress by data statistics,so we built knowledge accumulation model based on development system [2], in order to studying how knowledge accumulate.

MODEL BUILDING

We introduce knowledge accumulation function $F(t,x)$, here, t denotes time, x denotes knowledge memory period, so $F(t,x)$ denotes knowledge accumulation for memory period less than x at time t.

By the realistic significance of knowledge accumulation, we assume that $F(t,x) \geq 0$, and $F(t,x)$ is smooth about t and x. for $x_1 > x_2$, $F(t,x_1) \geq F(t,x_2)$, i.e $F(t,x)$ is monotonically increasing about x.
x_m denotes maximum memory period, so $f(t,0) = 0$, $f(t,x_m) = f(t,\infty)$.

Assuming $f(t,x) = \dfrac{\partial F(t,x)}{\partial x}$, $f(t,x)$ denotes distribution density function of knowledge accumulation about x at t. Since $F(t,x)$ is monotonically increasing about x, we have $f(t,x) \geq 0$.

In unit time, in interval $[x, x+\Delta x]$, we define the forgotten knowledge to be $K(t,x,\Delta x)$, and we define knowledge forgetting rate: $\lambda(t,x) = \lim\limits_{\Delta x \to 0} \dfrac{K(t,x,\Delta x)}{f(t,x)\Delta x}$, we know $0 \leq \lambda(t,x) \leq 1$. so for sufficiently small Δt and Δx, from t to $t+\Delta t$, in interval $[x, x+\Delta x]$, the forgotten knowledge can be denoted as: $K(t,\Delta t,x,\Delta x) = \lambda(t,x) f(t,x)\Delta t \Delta x$.

From t to $t+\Delta t$, in interval $[x, x+\Delta x]$, apart from forgotten knowledge, the unforgotten knowledge will be accumulated $f(t+\Delta t, x+\Delta x)\Delta x$ at memory period $x+\Delta x$ at moment $t+\Delta t$. By practical significance of memory period and time, whenever time change Δt, memory period will change the same value Δx as Δt, thus, we have $\Delta x = \Delta t$

We introduce distribution density function of new learning knowledge: $g(t,x)$
From t to $t+\Delta t$, in interval $[x, x+\Delta x]$, knowledge accumulation progress satisfies:

$$f(t+\Delta t, x+\Delta x)\Delta x - f(t,x)\Delta x = -\lambda(t,x) f(t,x)\Delta t\Delta x + g(t,x)\Delta t\Delta x$$

It can be rewritten as:

$$[f(t+\Delta t, x+\Delta x)\Delta x - f(t+\Delta t, x)\Delta x] + [f(t+\Delta t, x)\Delta x - f(t,x)\Delta x] = -\lambda(t,x) f(t,x)\Delta t\Delta x + g(t,x)\Delta t\Delta x$$

Dividing both sides by $\Delta t \Delta x$, we have

$$\frac{f(t+\Delta t, x+\Delta x)-f(t+\Delta t, x)}{\Delta x}+\frac{f(t+\Delta t, x)-f(t, x)}{\Delta t}=-\lambda(t, x) f(t, x)+g(t, x)$$

Let $\Delta t \to 0$, $\Delta x \to 0$, we have

$$\frac{\partial f(t, x)}{\partial x}+\frac{\partial f(t, x)}{\partial t}=-\lambda(t, x) f(t, x)+g(t, x)$$

Its initial condition is given as: $f(0, x)=f_0(x)$, here $f_0(x) \geq 0$

Its boundary condition is given as: $f(t,0)=\gamma(t)\int_0^{x_m} f(t, x) dx$. Here $\gamma(t)$ denotes repeated contact rate, and it satisfies: $0 \leq \gamma(t) \leq 1$.

Considering all the above things, we get the knowledge accumulation model:

$$\begin{cases} \dfrac{\partial f(t, x)}{\partial x}+\dfrac{\partial f(t, x)}{\partial t}=-\lambda(t, x) f(t, x)+g(t, x) & 0 \leq t, 0 \leq x \leq x_m \quad (1.1) \\ f(0, x)=f_0(x) & 0 \leq x \leq x_m \quad (1.2) \\ f(t,0)=\gamma(t)\int_{x_0}^{x_m} f(t, x) dx & 0 \leq t \quad (1.3) \end{cases}$$

EXISTING AND UNIQUESS OF SOLUTION FOR KNOWLEDGE ACCUMULATION MODEL

In this section, we have the following basic assumption [3]:

- In interval $[0, x_m]$, $f_0(x)$ is boundary value function and has proper smoothness.
- In region $\Omega=\{(t, x) \mid 0 \leq t, 0 \leq x \leq x_m\}$, $\lambda(t, x)$ is boundary value function and has proper smoothness.
- In region $\Omega=\{(t, x) \mid 0 \leq t, 0 \leq x \leq x_m\}$, $g(t, x)$ is boundary value function and has proper smoothness.

We have the following compatibility condition:

Zero order compatibility condition: $f_0(x)=\gamma(0)\int_{x_0}^{x_m} f_0(\xi) d\xi$

First order compatibility condition: $-d(0) f_0(0)+f_0'(0)+g(0,0)=\int_{x_0}^{x_m}[\gamma'(0) f_0(\xi)+f_0'(\xi)\gamma(0)] d\xi$

In region $\Omega_1=\{(t, x) \mid 0 \leq t \leq x_0, t \leq x \leq x_m\}$, we apply following transformation:

$$h(t, x)=f(t, x) e^{\int_{x-t}^{x} d(\tau) d\tau}-\int_{x-t}^{x} g(t-x+\tau, \tau) e^{\int_{x-t}^{\tau} d(v) dv} d\tau \quad (1.4)$$

We have

$$\begin{aligned} \frac{\partial h(t, x)}{\partial t} = & \frac{\partial f(t, x)}{\partial t} e^{\int_{x-t}^{x} d(\tau) d\tau}+f(t, x) e^{\int_{x-t}^{x} d(\tau) d\tau} d(x-t)-g(t-x+x-t, x-t) e^{\int_{x-t}^{x-t} d(v) dv} \\ & -\int_{x-t}^{x}\left[g_t'(t-x+\tau, \tau) e^{\int_{x-t}^{\tau} d(v) dv}-g(t-x+\tau, \tau) e^{\int_{x-t}^{\tau} d(v) dv} d(x-t)\right] d\tau \\ = & \frac{\partial f(t, x)}{\partial t} e^{\int_{x-t}^{x} d(\tau) d\tau}+f(t, x) e^{\int_{x-t}^{x} d(\tau) d\tau} d(x-t)-g(0, x-t) \\ & -\int_{x-t}^{x}\left[g_t'(t-x+\tau, \tau) e^{\int_{x-t}^{\tau} d(v) dv}-g(t-x+\tau, \tau) e^{\int_{x-t}^{\tau} d(v) dv} d(x-t)\right] d\tau \end{aligned} \quad (1.5)$$

$$\begin{aligned}\frac{\partial h(t,x)}{\partial x} &= \frac{\partial f(t,x)}{\partial x} e^{\int_{x-t}^{x} d(\tau)d\tau} + f(t,x)e^{\int_{x-t}^{x} d(\tau)d\tau}\left(d(x)-d(x-t)\right) + g(t-x+x-t,x-t)e^{\int_{x-t}^{x-t} d(v)dv} \\ &\quad - g(t-x+x,x)e^{\int_{x-t}^{x} d(v)dv} - \int_{x-t}^{x}\left[-g'_x(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} + g(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} d(x-t)\right]d\tau \\ &= \frac{\partial f(t,x)}{\partial x} e^{\int_{x-t}^{x} d(\tau)d\tau} + f(t,x)e^{\int_{x-t}^{x} d(\tau)d\tau}\left(d(x)-d(x-t)\right) - g(t,x)e^{\int_{x-t}^{x} d(v)dv} + g(0,x-t) \\ &\quad - \int_{x-t}^{x}\left[-g'_x(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} - g(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} d(x-t)\right]d\tau\end{aligned} \tag{1.6}$$

Adding (1.5) and (1.6), since (1.1) and $g'_x(t-x+\tau,\tau) = -g'_t(t-x+\tau,\tau)$, we have

$$\begin{aligned}\frac{\partial h(t,x)}{\partial t} + \frac{\partial h(t,x)}{\partial x} &= \frac{\partial f(t,x)}{\partial t} e^{\int_{x-t}^{x} d(\tau)d\tau} + \frac{\partial f(t,x)}{\partial x} e^{\int_{x-t}^{x} d(\tau)d\tau} + f(t,x)e^{\int_{x-t}^{x} d(\tau)d\tau} d(x) + -g(t,x)e^{\int_{x-t}^{x} d(v)dv} \\ &= \left(\frac{\partial f(t,x)}{\partial t} + \frac{\partial f(t,x)}{\partial x} + d(x)f(t,x) + -g(t,x)\right)e^{\int_{x-t}^{x} d(v)dv} \\ &= 0\end{aligned}$$

When $t = 0$, we have

$$\begin{aligned}h(0,x) &= f(0,x)e^{\int_{x}^{x} d(\tau)d\tau} - \int_{x}^{x} g(-x+\tau,\tau)e^{\int_{x}^{x} d(v)dv} d\tau \\ &= f(0,x) \\ &= f_0(x)\end{aligned}$$

Thus, $(1.1)-(1.3)$ can be transformed as:

$$\begin{cases}\dfrac{\partial h(t,x)}{\partial t} + \dfrac{\partial h(t,x)}{\partial x} = 0 & 0 \le t \le x_0,\ t \le x \le x_m \quad (1.7) \\ h(0,x) = f_0(x) & t \le x \le x_m \quad (1.8)\end{cases}$$

We will solve $(1.7)-(1.8)$ with the characteristics methods [4] in the following.
Parameterizing boundary conditions, we have

$$\begin{cases} t = 0 \\ x = v \end{cases}$$

Calculating its Jacobian determinants, we have

$$J\,|_{\Gamma} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \ne 0$$

Thus (1.7) is uncharacteristic, we characterize it

$$\begin{cases}\dot{t}(u) = 1 & \dot{x}(u) = 1 & \dot{z}(u) = 0 \\ t(0) = 0 & x(0) = v & z(0) = f_0(x)\end{cases} \tag{1.9}$$

We get the solution of (1.9)

$$\begin{cases} t(u,v) = u \\ x(u,v) = u + v \\ z(u,v) = f_0(v) \end{cases}$$

We get $v = x(u,v) - u = x(u,v) - t(u,v)$, thus, we get the solution $h(t,x)$ in Ω_1

$$h(t,x) = f_0(x-t) \tag{1.10}$$

Combining (1.4) and (1.10), we get the solution of $(1.1)-(1.3)$ in Ω_1

$$f(t,x) = e^{-\int_{x-t}^{x} d(\tau)d\tau}\left[f_0(x-t) + \int_{x-t}^{x} g(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} d\tau\right] \tag{1.11}$$

3.2 In region $\Omega_2 = \{(t,x) \mid 0 \le t, t - x_0 \le x \le t, 0 \le x \le x_m\}$, we apply the following transformation:

$$h(t,x) = f(t,x)e^{\int_{x-t}^{x} d(\tau)d\tau} - \int_{x-t}^{x} g(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} d\tau \tag{1.12}$$

We have

$$\frac{\partial h(t,x)}{\partial t} = \frac{\partial f(t,x)}{\partial t} e^{\int_0^x d(\tau)d\tau} - \int_0^x g_t'(t-x+\tau,\tau)e^{\int_0^{\tau} d(v)dv} d\tau \tag{1.13}$$

$$\frac{\partial h(t,x)}{\partial x} = \frac{\partial f(t,x)}{\partial x} e^{\int_0^x d(\tau)d\tau} + f(t,x)e^{\int_0^x d(\tau)d\tau} d(x) - g(t,x)e^{\int_0^x d(v)dv} - \int_0^x g_x'(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} d\tau \tag{1.14}$$

Adding (1.12) and (1.13), we have

$$\begin{aligned}\frac{\partial h(t,x)}{\partial t} + \frac{\partial h(t,x)}{\partial x} &= \frac{\partial f(t,x)}{\partial t} e^{\int_0^x d(\tau)d\tau} + \frac{\partial f(t,x)}{\partial x} e^{\int_0^x d(\tau)d\tau} + f(t,x)e^{\int_0^x d(\tau)d\tau} d(x) - g(t,x)e^{\int_0^x d(v)dv} \\ &= \left(\frac{\partial f(t,x)}{\partial t} + \frac{\partial f(t,x)}{\partial x} + d(x)f(t,x) - g(t,x)\right)e^{\int_0^{\tau} d(v)dv} \\ &= 0\end{aligned} \tag{1.15}$$

When $t = 0$, we have

$$\begin{aligned} h(t,0) &= f(t,0) \\ &= \gamma(t)\int_{x_0}^{x_m} f(t,x)dx \\ &= \int_{x_0}^{x_m} \gamma(t)e^{-\int_0^{\xi} d(\xi)d\xi}\left(h(t,\xi) + \int_0^{\xi} g(t-\xi+\tau,\tau)e^{\int_0^{\tau} d(v)dv} d\tau\right)d\xi \end{aligned} \tag{1.16}$$

Thus, $(1.1) - (1.3)$ can be transformed as:

$$\begin{cases} \dfrac{\partial h(t,x)}{\partial t} + \dfrac{\partial h(t,x)}{\partial x} = 0 & 0 \le t \le x_0, t \le x \le x_m \\ h(t,0) = \displaystyle\int_{x_0}^{x_m} \gamma(t)e^{-\int_0^{\xi} d(\xi)d\xi}\left(h(t,\xi) + \int_0^{\xi} g(t-\xi+\tau,\tau)e^{\int_0^{\tau} d(v)dv} d\tau\right)d\xi & t \le x \le x_m \end{cases} \tag{1.17}$$

Applying characteristics methods to (1.17), we get the solution of (1.17) in Ω_2

$$\begin{aligned} h(t,x) &= h(t-x,0) \\ &= \int_{x_0}^{x_m} \gamma(t-x)e^{-\int_0^{\xi} d(\xi)d\xi}\left(h(t-x,\xi) + \int_0^{\xi} g(t-x-\xi+\tau,\tau)e^{\int_0^{\tau} d(v)dv} d\tau\right)d\xi \end{aligned} \tag{1.18}$$

Since $t - x \in \Omega_1$, $h(t-x,\xi)$ has been solved in Ω_1 , combining (1.12)and(1.18), we get the solution of $(1.1) - (1.3)$ in Ω_2

$$f(t,x) = e^{-\int_0^x d(\tau)d\tau}\left[\int_{x_0}^{x_m} \gamma(t-x)h(t-x,\xi)d\xi + \int_0^x g(t-x+\tau,\tau)e^{\int_0^{\tau} d(v)dv} d\tau\right] \tag{1.19}$$

3.3 In region $\Omega_3 = \{(t,x) \mid 0 \le t,\ \ t-2x_0 \le x \le t-x_0, 0 \le x \le x_m\}$,
$\Omega_4 = \{(t,x) \mid 0 \le t,\ \ t-3x_0 \le x \le t-2x_0, 0 \le x \le x_m\}, \cdots$
we repeat the same method, we get the same solution as(1.19), so we get the solution of $(1.1) - (1.3)$ in region Ω .

$$f(t,x) = \begin{cases} e^{-\int_{x-t}^{x} d(\tau)d\tau}\left[f_0(x-t) + \displaystyle\int_{x-t}^{x} g(t-x+\tau,\tau)e^{\int_{x-t}^{\tau} d(v)dv} d\tau\right] & 0 \le t, t \le x \le x_m \\ e^{-\int_0^x d(\tau)d\tau}\left[\displaystyle\int_{x_0}^{x_m} \gamma(t-x)f(t-x,\xi)d\xi + \int_0^x g(t-x+\tau,\tau)e^{\int_0^{\tau} d(v)dv} d\tau\right] & 0 \le t, x \le t \le x_m \end{cases} \tag{1.20}$$

The formula (1.20) is a recursion formula of solution, by discussing the smoothness of the formula, we have the following theorem.

Theorem1: If the knowledge accumulation model $(1.1) - (1.3)$ satisfies the basic assumptions and the compatibility conditions, then the recursion formula given by (1.20) is the unique solution of the model.

***Proof:* The existence of the solution for knowledge accumulation model.**

Firstly, let's prove the global continuity of $f(t,x)$ given by (1.20). On the line $x=t$, in $\Omega_1=\{(t,x)|0\le t, t\le x\le x_m\}$, we have

$$f(t,x)|_{\Omega_1}=e^{-\int_0^x d(\tau)d\tau}\left[f_0(0)+\int_0^x g(\tau,\tau)e^{\int_0^\tau d(v)dv}d\tau\right]$$

In $\Omega'=\{(t,x)|0\le t, x\le t\le x_m\}$, we have

$$f(t,x)|_{\Omega'}=e^{-\int_0^x d(\tau)d\tau}\left[\int_{x_0}^{x_m}\gamma(0)f(0,\xi)d\xi+\int_0^x g(\tau,\tau)e^{\int_0^\tau d(v)dv}d\tau\right]$$

By zero order compatibility condition, we have $f(t,x)|_{\Omega_1}=f(t,x)|_{\Omega'}$ on line $x=t$, thus, $f(t,x)$ is continuous on line $x=t$. By the assumptions and recursion formula (1.20), we have the global continuity of $f(t,x)$.

Secondly, let's prove the continuous differentiability of $f(t,x)$ given by (1.20). On the line $x=t$, and in $\Omega_1=\{(t,x)|0\le t, t\le x\le x_m\}$, we have

$$\begin{aligned}\frac{\partial f(t,x)}{\partial t}|_{\Omega_1}=e^{-\int_0^x d(\tau)d\tau}\Bigg[&-d(0)f_0(0)-d(0)\int_0^x g(\tau,\tau)e^{\int_0^\tau d(v)dv}d\tau+\frac{\partial f_0(0)}{\partial t}+g(0,0)\\&+\int_0^x\frac{\partial g(t-x+\tau,\tau)}{\partial t}e^{\int_0^\tau d(v)dv}d\tau+\int_0^x g(t-x+\tau,\tau)e^{\int_0^\tau d(v)dv}d\tau\Bigg]\end{aligned}$$

In $\Omega'=\{(t,x)|0\le t, x\le t\le x_m\}$, we have

$$\frac{\partial f(t,x)}{\partial t}|_{\Omega'}=e^{-\int_0^x d(\tau)d\tau}\left[\int_{x_0}^{x_m}\left(\frac{\partial\gamma(0)}{\partial t}f(0,\xi)+\gamma(0)\frac{\partial f(0,\xi)}{\partial t}\right)d\xi+\int_0^x\frac{\partial g(\tau,\tau)}{\partial t}e^{\int_0^\tau d(v)dv}d\tau\right]$$

By first order compatibility condition, we have $\frac{\partial f(t,x)}{\partial t}|_{\Omega_1}=\frac{\partial f(t,x)}{\partial t}|_{\Omega'}$ on line $x=t$, thus, $f(t,x)$ is continuously differentiable on line $x=t$. By the assumptions and recursion formula (1.20), we have the global continuous differentiability of $f(t,x)$. Thus, the existence of solution for knowledge accumulation is proved.

The uniqueness of the solution for knowledge accumulation model

Assuming $f_1(t,x)$ and $f_2(t,x)$ are two solutions satisfied $(1.1)-(1.3)$, In $\Omega_1=\{(t,x)|0\le t, t\le x\le x_m\}$, we have

$$\begin{aligned}f_1(t,x)-f_2(t,x)&=e^{-\int_{x-t}^x d(\tau)d\tau}\left[f_0(x-t)+\int_{x-t}^x g(t-x+\tau,\tau)e^{\int_{x-t}^\tau d(v)dv}d\tau\right]\\&\quad-e^{-\int_{x-t}^x d(\tau)d\tau}\left[f_0(x-t)+\int_{x-t}^x g(t-x+\tau,\tau)e^{\int_{x-t}^\tau d(v)dv}d\tau\right]\\&=0\end{aligned}$$

Thus, knowledge accumulation model $(1.1)-(1.3)$ has the unique solution in Ω_1.

In region $\Omega_2=\{(t,x)|0\le t, t-x_0\le x\le t, 0\le x\le x_m\}$, we notice

$$\begin{aligned}f_1(t,x)-f_2(t,x)&=e^{-\int_0^x d(\tau)d\tau}\left[\int_{x_0}^{x_m}\gamma(t-x)f_1(t-x,\xi)d\xi+\int_0^x g(t-x+\tau,\tau)e^{\int_0^\tau d(v)dv}d\tau\right]\\&\quad-e^{-\int_0^x d(\tau)d\tau}\left[\int_{x_0}^{x_m}\gamma(t-x)f_2(t-x,\xi)d\xi+\int_0^x g(t-x+\tau,\tau)e^{\int_0^\tau d(v)dv}d\tau\right]\\&=e^{-\int_0^x d(\tau)d\tau}\int_{x_0}^{x_m}\gamma(t-x)(f_1(t-x,\xi)-f_2(t-x,\xi))d\xi\end{aligned}$$

Since $t-x\in\Omega_1$, we have proved $f_1(t-x,\xi)=f_2(t-x,\xi)$ in Ω_1, thus we have $f_1(t,x)-f_2(t,x)=0$ in Ω_2, repeating in $\Omega_3,\Omega_4\cdots$, we have $f_1(t,x)=f_2(t,x)$ in Ω, i.e. the knowledge accumulation model $(1.1)-(1.3)$ has the unique solution in Ω.

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