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On Question and Answer: HIV Modelling in Mathematical Epidemiology

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Abstract: The basic reproduction number $[R_0]$ importantly summaries the dynamics of an infectious disease in human populations. This parameter is called a threshold parameter. For an infection to invade a population $[R_0]$ should be greater than 1. In epidemiological studies, intervention seeks to lower the value of $[R_0]$ to below 1 so that the disease dies out. However, simple modelling of malaria revealed that, theoretically it is possible for an infection to persist indefinitely even if an intervention has reduced $[R_0]$ below 1. If this happens, we call it bistable equilibrium. In this work, we focused mainly on the analysis of the Disease-Free Equilibrium Point [DFEP] to see the circumstances surrounding the propagation of an infectious disease. We found that the disease dies out if the reproduction number is less than 1.
Keywords: HIV, Reproduction Number, Model, Variables and Parameters, Epidemiology.

INTRODUCTION

The Human Immunodeficiency Virus [HIV] weakens the immune system so that it cannot fight off foreign bodies such as germs, viruses and fungi. This virus causes AIDS, [Acquired Immune Deficiency Syndrome]. People with HIV are very susceptible to certain uncommon diseases and illnesses. In U.S. about 1.2 million people are living with HIV infection, this is with reference to the CDC, and there are about 50,000 new HIV cases observed each year [12].

HIV attacks and destroys the T-cells which fight diseases in the body. HIV feeds on the proteins in the cell to make a copy of it and then kills the cell. The secondary stage of HIV infection is AIDS, which occur when CD4 count in a blood test is less than 200. In most cases, people contract HIV from:

- Having unprotected sex with an infected person
- Sharing a needle to take drugs
- Dirty needles used for a tattoo or in body piercing

METHODOLOGY

Question 1: Identify a disease of your choice.
Answer: The disease of our choice is HIV epidemic.
Question 2: Describe the disease using the four phases of epidemiology.

Answer:

Phase I: Descriptive, what is the problem? Who is involved? Where and when?
The problem here is the infection of people by HIV, a virus that destroys the immune system of the body, rendering it susceptible to any infection or disease. The virus enters the body through sexual contacts; it enters the leucocytes and introduces its information in the cell. The process will lead to cell budding, a process where a cell develops pimples on its surface. Thereafter, the viruses from the destroyed cell leave and each attacks a new cell until all the cells are destroyed if there is no treatment. The problem involves all people in a community if they interact in a random manner. The infection occurs all the time since if treatment is not included, more and more people get infected.

Phase II: Analytical, analyze the cause or determinants; how the disease is caused and why is it continuing. The infection is caused, in most cases by sexual contacts. It is a sexually transmitted infection (STI). If infected by the virus, the virus enters the leucocytes, stays in the cell and introduces its information in the nucleus. After some time, the destroyed cell releases new viruses which each attacks a new cell if there is no intervention or treatment. In the absence of treatment, the process continues till all the white blood cells are finished. The body becomes susceptible to any type of infection or disease which eventually leads to AIDS and death.

Phase III: Intervention, these are clinical and community trials that are used to ask questions about the effectiveness of the new methods for controlling the disease or for improving underlying conditions. Intervention for this epidemic encompasses administering of antiretroviral drugs which boost the immune system; in this case the differential equations describing the disease dynamics include treatment. Members of the community will be oriented about the cause and effects of the disease. These include teaching people in schools, at health centers and carrying out workshops and programs that orient people. Also, people should be taught to stick to one partner and using condoms.

Phase IV: Evaluation, this will attempt to measure the effectiveness of health service programs.

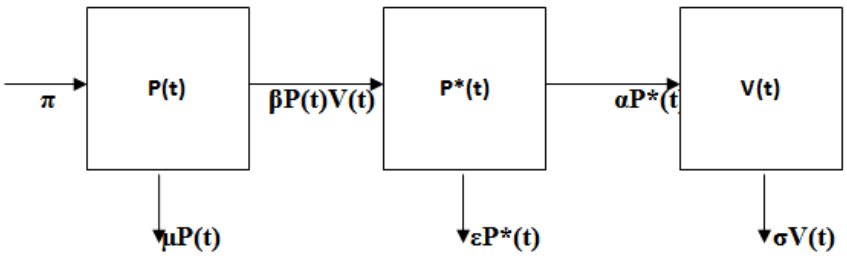
Question 3: Write down questions which need to be asked about the disease in the form of a questionnaire.
Answer:
Q1: How many cases have you recorded so far with HIV?
Q2: Have these cases received treatment?
Q3: What is the prevalence rate so far?
Q4: Of those cases you recorded, are there any who got full blown AIDS?
Q5: Are the drugs readily available?
Q6: How is the Government responding to this epidemic? Does it include it on the national budget?
Q7: Did you orient people about the ways of reducing the risk of contraction?
Q8: Since you started your intervention, is the rate of new cases increasing or decreasing?

Question 4: Identify the main variables, parameters, and assumptions of the disease.
Answer:
Main variables: P(t)= population of the uninfected people.
P*(t)= population of the infected people.
V(t)= population of the free virus particles.

Parameters: π=rate at which the population grows.
β=rate of infection.
μ=rate at which uninfected people die.
α=rate at which infected people succumb to the disease.
ε=rate of virus particle multiplication.
σ=natural death rate of virus particles.

Assumptions:
(a) The parameters are constants.
(b) The disease does not kill people instantly.
(c) Demographic parameters do not change with time.
(d) Males and females mix in a random manner.
(e) The rate of infection is proportional to the product of the number of uninfected and infected.
(f) Transmission occurs as a result of sexual contact.

Question 5: Draw a flow chart of your model.
Answer:



Question 6: Write down the equations of the system.

Answer:

$$\frac{dP(t)}{dt} = \pi - \beta P(t)V(t) - \mu P^*(t) \quad (1)$$
$$\frac{dP^*(t)}{dt} = \beta P(t)V(t) - \alpha \quad (2)$$
$$\frac{dV(t)}{dt} = \varepsilon P^*(t) - \sigma V(t) \quad (3)$$

Question 7: Carry-out a mathematical analysis of the model.

Answer: Mathematical Analysis of the Model

The equilibrium states occur when

$$\frac{dP}{dt} = \frac{dP^*}{dt} = \frac{dV}{dt} = 0.$$

Thus, the disease-free equilibrium point occurs when

$$V^* = 0 \rightarrow P^* = 0 \text{ and } P = \frac{\pi}{\mu}.$$

Therefore, the Disease-Free Equilibrium Point (**DFEP**) becomes:

$$\textbf{DFEP} = \left(\frac{\pi}{\mu}, 0, 0 \right).$$

Thus, the General Jacobian of the system is: $\boldsymbol{J} = \begin{pmatrix} -\beta V^* - \mu & 0 & -\beta P \\ \beta P & -\alpha & \beta P \\ 0 & \varepsilon & -\sigma \end{pmatrix}$.

Analysis of the Disease-Free Equilibrium (DFEP) Point

$$\boldsymbol{J}_{(\textbf{DFEP})} = \begin{pmatrix} -\mu & 0 & -\frac{\beta\pi}{\mu} \\ \frac{\beta\pi}{\mu} & -\alpha & \frac{\beta\pi}{\mu} \\ 0 & \varepsilon & -\sigma \end{pmatrix}.$$

The eigenvalues are obtained from the characteristic polynomial of:

$$\left| J - \lambda I \right| = 0,$$

that is,

$$\det \begin{pmatrix} -\mu - \lambda & 0 & -\frac{\beta\pi}{\mu} \\ \frac{\beta\pi}{\mu} & -\alpha - \lambda & \frac{\beta\pi}{\mu} \\ 0 & \varepsilon & -\sigma - \lambda \end{pmatrix} = 0.$$

This gives us:

$$\left(-\mu - \lambda\right)\left[\left(-\alpha - \lambda\right)\left(-\sigma - \lambda\right) - \frac{\varepsilon\beta\pi}{\mu}\right] - \frac{\beta\pi}{\mu}\left(\frac{\varepsilon\beta\pi}{\mu}\right) = 0.$$

Simplifying this equation we obtain:

$$-\lambda^3 - (\mu + \alpha + \sigma)\lambda^2 - \left(\mu\alpha + \mu\sigma + \alpha\sigma - \frac{\varepsilon\beta\pi}{\mu}\right)\lambda - \left(\alpha\mu\sigma - \varepsilon\beta\pi + \frac{\varepsilon\beta^2\pi^2}{\mu^2}\right) = 0,$$

or

$$\lambda^3 + (\mu + \alpha + \sigma)\lambda^2 + \left(\mu\alpha + \mu\sigma + \alpha\sigma - \frac{\varepsilon\beta\pi}{\mu}\right)\lambda + \left(\alpha\mu\sigma - \varepsilon\beta\pi + \frac{\varepsilon\beta^2\pi^2}{\mu^2}\right) = 0.$$

This implies that $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$

where

$$a_1 = \mu + \alpha + \sigma,$$
$$a_2 = \mu\alpha + \mu\sigma + \alpha\sigma - \frac{\varepsilon\beta\pi}{\mu},$$

$$a_3 = \alpha\mu\sigma - \boldsymbol{\varepsilon}\beta\pi + \frac{\varepsilon\beta^2\pi^2}{\mu^2}.$$

We now apply the Routh-Hurwitz criterion to determine the stability of the disease-free equilibrium point. The disease-free equilibrium point is stable if $a_1\, a_2 - a_3 > 0$, that is, it is stable if

$$(\mu + \alpha + \sigma)\left(\mu\alpha + \mu\sigma + \alpha\sigma - \frac{\varepsilon\beta\pi}{\mu}\right) - \left(\alpha\mu\sigma - \boldsymbol{\varepsilon}\beta\pi + \frac{\varepsilon\beta^2\pi^2}{\mu^2}\right) > 0.$$

Now we look at the Epidemic Equilibrium Point (**EEP**). This point is obtained from the set of equations:

$$\pi - \beta PV^* - \mu P = 0 \qquad (4)$$
$$\beta PV^* - \alpha P^* = 0 \qquad (5)$$
$$\boldsymbol{\varepsilon}P^* - \sigma V^* = 0 \qquad (6)$$

From equation (6)

$$P^* = \frac{\sigma}{\varepsilon}V^* \qquad (7)$$

Substituting equation (7) into equation (5) we obtain:

$$\beta P^* V^* - \frac{\alpha\sigma}{\varepsilon}V^* = 0 \;\rightarrow\; V^* = 0 \text{ or } P^* = \frac{\alpha\sigma}{\beta\varepsilon}.$$

From equation (4), when $P = \frac{\alpha\sigma}{\beta\varepsilon}$, then

$$\pi - \beta\left(\frac{\alpha\sigma}{\beta\varepsilon}\right)V^* - \mu\left(\frac{\alpha\sigma}{\beta\varepsilon}\right) = 0.$$

Simplifying this gives:

$$V^* = \frac{\varepsilon\pi}{\alpha\sigma} - \frac{\mu}{\beta} \text{ and hence, } P^* = \frac{\pi}{\alpha} - \frac{\mu\sigma}{\varepsilon\beta}.$$

Thus, the Endemic Equilibrium Point (**EEP**) becomes:

$$\textbf{EEP} = (P, P^*, V^*) = \left(\frac{\alpha\sigma}{\beta\varepsilon}, \frac{\pi}{\alpha} - \frac{\mu\sigma}{\varepsilon\beta}, \frac{\varepsilon\pi}{\alpha\sigma} - \frac{\mu}{\beta}\right).$$

Analysis of the Endemic Equilibrium Point (**EEP**)

Mathematical models were developed to describe the dynamics of infectious diseases in human populations, and were applied to specific diseases [13]. There are threshold theorems that involve the basic reproduction number $[R_0]$, the contact numberγ, and the replacement number *R*. These theorems were reviewed for the classic SIR epidemic and endemic models. In much the same way, some results with new expressions for R_0 were obtained for endemic models such as MSEIR and SEIR, with either continuous age or age groups.

In the system of equations (1-3), the value of the reproduction number, R_0 is given by$R_0 = \frac{\beta\varepsilon\pi}{\alpha\mu\sigma}$, which determines the stability of the **EEP** if $R_0 > 1$. we now need to show that this is so. The endemic equilibrium point is stable if:

$$P^* > 0$$
$$\rightarrow \frac{\pi}{\alpha} - \frac{\mu\sigma}{\varepsilon\beta} > 0$$
$$\rightarrow \frac{\beta\varepsilon\pi}{\alpha\mu\sigma} - 1 > 0$$
$$\rightarrow R_0 - 1 > 0 \;\rightarrow R_0 > 1,$$

where $R_0 = \frac{\beta\varepsilon\pi}{\alpha\mu\sigma}$.

CONCLUSION

Since the reproduction number$R_0 > 1$, we therefore conclude that the endemic equilibrium point is stable.

Question 8: Include the likely intervention in the model and carry-out a mathematical analysis.

Answer: The likely intervention is to introduce treatment of the infected individuals. System of differential equations describing the dynamics of the infection with treatment becomes:

$$\frac{dP(t)}{dt} = \pi - \beta P(t)V(t) - \mu P(t) \quad (8)$$

$$\frac{dP^*(t)}{dt} = \beta P(t)V(t) - \alpha P^*(t) \quad (9)$$

$$\frac{dP_T^*(t)}{dt} = \delta P^*(t) - \tau P_T^*(t) \quad (10)$$

$$\frac{dV(t)}{dt} = \gamma P_T^*(t) - \varphi V(t) \quad (11)$$

Finding Equilibrium Points

Similar to the system without treatment, we realise that the equilibrium points occur when

$$\frac{dP(t)}{dt} = \frac{dP^*(t)}{dt} = \frac{dP_T^*(t)}{dt} = \frac{dV(t)}{dt} = 0.$$

The only equilibrium point that exists is the Disease-Free Equilibrium Point (**DFEP**), that is

$$\textbf{DFEP} = \left(\frac{\pi}{\mu}, 0,0,0\right).$$

Analysis of the DFEP

The general Jacobian Matrix is given by

$$\boldsymbol{J} = \begin{pmatrix} -\beta V^* - \mu & 0 & 0 & -\beta P^* \\ \beta V^* & -\alpha & 0 & \beta P^* \\ 0 & \delta & \tau & 0 \\ 0 & 0 & \gamma & -\varphi \end{pmatrix}$$

Therefore

$$\boldsymbol{J}_{(\boldsymbol{DFEP})} = \begin{pmatrix} -\mu & 0 & 0 & -\frac{\beta\pi}{\mu} \\ 0 & -\alpha & 0 & \frac{\beta\pi}{\mu} \\ 0 & \delta & \tau & 0 \\ 0 & 0 & \gamma & -\varphi \end{pmatrix}$$

The eigenvalues are obtained from the characteristic polynomial of:

$$|J - \lambda I| = 0,$$

that is,

$$\det\begin{pmatrix} -\mu - \lambda & 0 & 0 & -\frac{\beta\pi}{\mu} \\ 0 & -\alpha - \lambda & 0 & \frac{\beta\pi}{\mu} \\ 0 & \delta & \tau - \lambda & 0 \\ 0 & 0 & \gamma & -\varphi - \lambda \end{pmatrix} = 0.$$

Simplifying the determinant gives characteristic polynomial

$$\lambda^4 + a_4\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1 = 0,$$

where

$$a_1 = \frac{\beta\delta\gamma\pi}{\mu},$$

$$a_2 = \alpha\mu\left(\tau\varphi + \tau + \varphi\right),$$

$$a_3 = \alpha\mu + \mu\tau\varphi + \mu\tau + \mu\varphi + \alpha\varphi\,\tau + \alpha\tau + \alpha\varphi\,,$$

$$a_4 = \alpha + \mu + \tau\varphi + \tau + \varphi\,.$$

CONCLUSION

We now apply the Routh-Hurwitz criterion to determine the stability of the disease-free equilibrium point. The disease-free equilibrium point is stable if

$$a_1 a_2 - a_3 a_4 > 0.$$

Observation

We observed from this research that, calculations to reduce the basic reproduction number can be achieved. This necessitates the disease to die out instead of continuing indefinitely. The reduction in R_0 is achieved through interventions.

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