Scholars Journal of Physics, Mathematics and Statistics
Abbreviated Key Title: Sch. J. Phys. Math. Stat.
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

Modelling HIV Viral Propagation using the First Generation and Eigen-value Approaches

Confess Matete*, Memory Mandiudza
Department of Mathematics and Computer Science, Great Zimbabwe University, Masvingo, Zimbabwe

***Corresponding author**
Confess Matete

Article History
Received: 26.02.2018
Accepted: 10.03.2018
Published: 30.03.2018

DOI:
10.21276/sjpms.2018.5.2.5

Abstract: The parameter,[R_0], called the basic reproduction number, can be obtained by the methods called First Generation and Eigen-value Approaches. The significance of this parameter is to summarize the dynamics of an infection in human populations. This parameter is called a threshold parameter. A population gets infected by an infectious disease if $R_0 > 1$. Epidemiologists thrive to have $R_0 < 1$ so that the disease dies out. However, some simple models of bacterial-caused diseases (such as malaria) revealed that, in theory, it is possible for an infection to persist indefinitely even if we have $R_0 < 1$. If we encounter such a scenario, we call it a bitable equilibrium. This paper focuses mainly on the analysis of the Disease-Free Equilibrium Point [DFEP], using the first generation and eigen-value approaches to see the circumstances surrounding the propagation of an infectious disease. The results displayed the fact that, the disease dies out if the reproduction number is $R_0 < 1$.

Keywords: Virus Propagation, First Generation Approach, Eigen-value Approach, Dimensionalising Original Variables and Parameters.

INTRODUCTION

Epidemiologists fight hard to arrest the Human Immunodeficiency Virus [HIV], which weakens the immune system. Should this happens, the immune system cannot fight off foreign bodies such as germs, viruses and fungi [1].

If there is no intervention, the viruses invade the white blood cells and introduce their information. Without treatment, this will lead to cell budding which causes AIDS, (Acquired Immune Deficiency Syndrome). HIV infected people are very susceptible to certain uncommon diseases and illnesses because of the weaker immune system. In U.S. about 1.2 million people are living with HIV infection, this is with reference to the CDC, and there are about 50,000 new HIV cases observed each year [2].

The Model

The simple model that describe the dynamics of the propagation of HIV in the human body is given by

$$\begin{pmatrix}\lambda_w S\\ \lambda_r S\end{pmatrix} = \begin{pmatrix}\dfrac{\beta S(I_w+\eta_w A_w)}{N}\\ \dfrac{\beta S(I_r+\eta_r A_r)}{N}\end{pmatrix}$$

We now find R_0 using the First Generation method

But $F = \begin{pmatrix}\lambda_w S\\ \lambda_r S\end{pmatrix} \Rightarrow F = \begin{pmatrix}\dfrac{\beta S(I_w+\eta_w A_w)}{N}\\ \dfrac{\beta S(I_r+\eta_r A_r)}{N}\end{pmatrix}$

$$V = \begin{pmatrix}(\alpha_w+\mu)I_w\\ (\alpha_r+\mu)I_r\end{pmatrix}$$

$$\frac{\partial F}{\partial x_j} = \begin{pmatrix} \frac{\beta S^*}{N} & 0 \\ 0 & \frac{\beta S^*}{N} \end{pmatrix}$$

$$V = \begin{pmatrix} \alpha_w + \mu & 0 \\ 0 & \alpha_r + \mu \end{pmatrix}$$

$$V^{-1} = \frac{1}{|V|}\begin{pmatrix} \alpha_r + \mu & 0 \\ 0 & \alpha_w + \mu \end{pmatrix} = \frac{1}{(\alpha_w+\mu)(\alpha_r+\mu)}\begin{pmatrix} \alpha_r + \mu & 0 \\ 0 & \alpha_w + \mu \end{pmatrix}$$

$$So\ FV^{-1} = \begin{pmatrix} \frac{\beta S^*}{N} & 0 \\ 0 & \frac{\beta S^*}{N} \end{pmatrix}\frac{1}{(\alpha_w+\mu)(\alpha_r+\mu)}\begin{pmatrix} \alpha_r + \mu & 0 \\ 0 & \alpha_w + \mu \end{pmatrix}$$

$$= \frac{1}{(\alpha_w+\mu)(\alpha_r+\mu)}\begin{pmatrix} \frac{\beta S^*}{N}(\alpha_r + \mu) & 0 \\ 0 & \frac{\beta S^*}{N}(\alpha_w + \mu) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\beta S^*}{N(\alpha_w+\mu)} & 0 \\ 0 & \frac{\beta S^*}{N(\alpha_r+\mu)} \end{pmatrix}$$

$$\therefore R_{0=}\rho FV^{-1}$$

The eigenvalues are obtained from | FV^{-1} - λI |=0. The eigenvalues are

$$\lambda_1 = \frac{\beta S^*}{N\alpha_w + \mu}, \lambda_2 = \frac{\beta S^*}{N\alpha_r + \mu} \Rightarrow R_0 = \max\left\{\frac{\beta S^*}{N\alpha_w + \mu}, \frac{\beta S^*}{N\alpha_r + \mu}\right\}.$$

Finding the Wild Strain Equilibrium Point

This exist if $\frac{dS}{dt} = \frac{dI_w}{dt} = \frac{dA_w}{dt} = 0.$

$$\pi - \lambda_w S^* - \lambda_r S^* - \mu S^* = 0 \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{...}(i)$$

That is $\lambda_w S^* - (\alpha_w + \mu) I_w^* = 0 \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{.........}(ii)$

$$\alpha_w S^* - (\delta_w + \mu) A_w^* = 0 \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{.........}(iii)$$

From (iii) $A_w^* = \frac{\alpha_w}{\delta_w + \mu} I_w^* \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{.........}(iv)$

From the equation $\frac{\beta\left(I^*{}_w + \eta_w A^*{}_w\right) S^*}{N} - (\alpha_w + \mu) I_w^* = 0 \text{..........} \text{..........} \text{..........} \text{.........}(v)$

Substituting (iv) into (v) and simplifying we get:

$$I_w^* = 0, S^* = \frac{N(\delta_w + \mu)(\alpha_w + \mu)}{\beta(\delta_w + \mu + \eta_w \alpha_w)} \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..........} \quad \text{..}(vi)$$

From (i) $\pi - \lambda_w S^* - \mu S^* = 0,$

that is the absence of the resistant strain

$$\Rightarrow \pi - \frac{\beta}{N}\left(I_w^* + \frac{\eta_w}{\delta_w} \right)$$

Simplifying in terms of S* we get

$$I_w^* = \frac{N(\delta_w + \mu)(\pi - \mu S^*)}{\beta S^*(\delta_w + \mu + \eta_w \alpha_w)} \quad \text{and} \quad A_w^* = \frac{N\alpha_w(\pi - \mu S^*)}{\beta S^*(\delta_w + \mu + \eta_w \alpha_w)} \quad .$$

Therefore

$$\left(S^*, I_w^*, A_w^*\right) = \left(\frac{N(\delta_w + \mu)(\alpha_w + \mu)}{\beta(\delta_w + \mu + \eta_w \alpha_w)}, \frac{N(\delta_w + \mu)(\pi - \mu S^*)}{\beta S^*(\delta_w + \mu + \eta_w \alpha_w)}, \frac{N\alpha_w(\pi - \mu S^*)}{\beta S^*(\delta_w + \mu + \eta_w \alpha_w)} \right).$$

Finding the Resistant Strain Equilibrium Point

This exist if $\frac{dS}{dt} = \frac{dI_w}{dt} = \frac{dA_w}{dt} = 0$. That is

$$\pi - \lambda_w S^* - \lambda_r S^* - \mu S^* = 0 \text{.......... ....}(i)$$

$$\lambda_r S^* - (\alpha_r + \mu) I_r^* = 0 \text{.......... ....}(ii)$$

$$\alpha_r I_r^* - (\delta_r + \mu) A_r^* = 0 \text{.......... ....}(iii)$$

From (iii) $A_r^* = \frac{\alpha_w}{\delta_r + \mu} I_r^*$ ....(iv)

From the equation

$$\frac{\beta\left(I^*_w + \eta_w A^*_w\right) S^*}{N} - (\alpha_w + \mu) I_w^* = 0 \text{.......... ....}(v)$$

Substituting equation (iv) into equation (v) and simplifying we get:

$$I_r^* = 0, S^* = \frac{N(\delta_r + \mu)(\alpha_r + \mu)}{\beta(\delta_r + \mu + \eta_r \alpha_r)} \text{.......... ....(}vi$$

From (i) $\pi - \lambda_r S^* - \mu S^* = 0,$

That is, in the absence of the resistant strain

$$\Rightarrow \pi - \frac{\beta}{N}\left(I_r^* + \frac{\eta_r \alpha_r}{\delta_r + \mu} I_r^* \right) = 0 .$$

Simplifying in terms of S* we get

$$I_r^* = \frac{N(\delta_r + \mu)(\pi - \mu S^*)}{\beta S^*(\delta_r + \mu + \eta_r \alpha_r)} \quad \text{and} \quad A_r^* = \frac{N\alpha_r(\pi - \mu S^*)}{\beta S^*(\delta_r + \mu + \eta_r \alpha_r)} \quad .$$

Therefore $\left(S^*, I_r^*, A_r^*\right) = \left(\frac{N(\delta_r + \mu)(\alpha_r + \mu)}{\beta(\delta_r + \mu + \eta_r \alpha_r)}, \frac{N(\delta_r + \mu)(\pi - \mu S^*)}{\beta S^*(\delta_r + \mu + \eta_r \alpha_r)}, \frac{N\alpha_r(\pi - \mu S^*)}{\beta S^*(\delta_r + \mu + \eta_r \alpha_r)} \right)$.

Finding the Coexistence Equilibrium Point

This exist if

$$\frac{dS}{dt} = \pi - \frac{\beta S^*}{N}\left(I_w^* + \eta_w A_w^* - \frac{\beta S^*}{N}\left(I_w^* + \eta_w A_w^{\ *}\right) - \mu S^* \right) = 0 \text{.......... ....}(i)$$

$$\frac{dI_w^*}{dt} = \frac{\beta S^*}{N}\left(I_w^* + \eta_w A_w^{\ *} - (\alpha_w + \mu)I_w^*\right) = 0 \text{.......... ....}(ii)$$

$$\frac{dI_r^*}{dt} = \frac{\beta S^*}{N}\left(I_r^* + \eta_r A_r^{\ *} - (\alpha_r + \mu)I_r^*\right) = 0 \text{.......... ....}(iii)$$

$$\frac{dA_w^*}{dt} = \alpha_w I_w^* - (\delta_w + \mu)A_w^{\ *} = 0 \text{.......... ....}(iv)$$

$$\frac{dA_r^*}{dt} = \alpha_r I_r^* - (\delta_r + \mu)A_r^{\ *} = 0 \text{.......... ....}(v)$$

In addition, the Coexistence Equilibrium Point exist if $I_w^*,\ I_r^* > 0$. From the equations (iv) and (v) we get

$$A_w^* = \frac{\alpha_w}{\delta_w + \mu} I_w^*,\ A_r^* = \frac{\alpha_r}{\delta_r + \mu} I_r^* \ .$$

From equation (i) we simplify to get

$$S^* = \frac{N_\pi}{\beta\left(I_w^* + \eta_w A_w^*\right) + \beta\left(I_r^* + \eta_r A_r^*\right) - \mu N} \ .$$

Therefore, expressing in terms of I_w^* and I_r^* we have: $(S^*,\ I_w^*,\ I_r^*,\ A_w^*,\ A_r^*)$

$$= \left(\frac{N_\pi}{\beta\left(I_w^* + \eta_w A_w^*\right) + \beta\left(I_r^* + \eta_r A_r^*\right) - \mu N},\ I_w^*, I_r^*, \frac{\alpha_w}{\delta_w + \mu} I_w^*, \frac{\alpha_r}{\delta_r + \mu} I_r^* \right) \ .$$

Finding the Diseases-Free Equilibrium Point (DFEP)

The **DFEP** exists if $I_w^* = I_r^* = 0 \Rightarrow S^* = \frac{\pi}{\mu}$. Therefore we have **DFEP** being equivalent to

$$(S^*,\ I_w^*,\ I_r^*,\ A_w^*,\ A_r^*) = \left(\frac{\pi}{\mu}, 0, 0, 0, 0 \right).$$

There are several ways of controlling the disease, such as

- Introducing treatment.
- Abstaining from sex or stick to one partner.
- Practising protective sex, such as using condoms.
- Orienting the society about the cause of the disease and how to reduce transmission.

Expressing the model in dimensionless variables

Let

$$s=\frac{S}{N_h},t_1=\frac{T}{N_h},i=\frac{I}{N_h},r=\frac{R}{N_h},s_m=\frac{S_m}{N_m},i_m=\frac{I_m}{N_m}$$

$$\frac{ds}{dt}=\frac{1}{N_h}\frac{dS}{dt},\frac{dt_1}{dt}=\frac{1}{N_h}\frac{dT}{dt},\frac{di}{dt}=\frac{1}{N_h}\frac{dI}{dt},\frac{dr}{dt}=\frac{1}{N_h}\frac{dR}{dt},\frac{ds_m}{dt}=\frac{1}{N_m}\frac{dS_m}{dt},\frac{di_m}{dt}=\frac{1}{N_m}\frac{dI_m}{dt}$$

$$\frac{ds}{dt}=\frac{1}{N_h}\left[\mu(1-p)N_h-\left(\beta_1\frac{i_mN_m}{N_m}+\mu\right)sN_h+\theta t_1N_h+vrN_h\right]$$

$$=\mu(1-p)N_h-(\beta_1 i_m+\mu)s+\theta t_1+vr.$$

$$\frac{dt_1}{dt}=\frac{1}{N_h}\left[\mu pN_h-(1-\phi)\left(\beta_1 t_1\frac{i_mN_mN_h}{N_m}+\mu\right)\_(\mu-\theta)t_1N_h\right]$$

$$=\mu p-(1-\phi)(\beta_2t_1i_m)-(\mu+\theta)t_1.$$

$$\frac{di}{dt}=\frac{1}{N_h}\left[\beta_1 sN_h\frac{i_mN_m}{N_m}+(1-\phi)\beta_2t_1N_h\frac{i_mN_m}{N_m}-(\gamma+\mu)iN_h\right]$$

$$=\beta_1 si_m+(1-\phi)\beta_2t_1i_m-(\gamma-\mu)i\mu.$$

$$\frac{dr}{dt}=\frac{1}{N_h}[\gamma iN_h-(\mu-v)rv_h]$$

$$=\gamma i-(\mu+v)r.$$

$$\frac{ds_m}{dt}=\frac{1}{N_m}\left[\delta N_m-\delta s_mN_m-\beta_3s_m\frac{iN_hN_m}{N_m}\right]$$

$$=\delta-\delta s_m-\beta_3s_mik,k=\frac{N_h}{N_m}.$$

Therefore

$$\frac{di_m}{dt}=\frac{1}{N_m}\left[\beta_3s_mN_m\frac{iN_h}{N_m}-\delta i_mN_m\right]$$

$$=\beta_3s_mik-\delta i_m.$$

Finding Equilibria

The Disease-free equilibrium point exists if $i^*=0$. From the last equation

$\beta_3s_m^*i^*k-\delta i_m^*=0\Rightarrow i_m^*=0$ since $i^*=0$

From the fifth equation

$$\delta-\delta s_m^*-\beta_3s_m^*\beta i^*k=0\Rightarrow s_m^*=1$$

From the fourth equation

$$\gamma\ \gamma i^*-(\mu+v)r^*=0\Rightarrow r^*=0$$

From the second equation

$$\mu p-(1-\phi)\beta_2t_1^*i_m^*-(\mu+\theta)t_1^*=0\Rightarrow t_1^*=\frac{\mu p}{\mu+\theta}.$$

From the first equation

$\mu(1-p)-\mu s^{*}+\frac{\theta\mu p}{\mu+\theta}=0$ since $r^{*}=i^{*}_{m}=0 \Rightarrow s^{*}=(1-p)\frac{\theta p}{\mu+\theta}$.

The disease-free equilibrium point becomes

$$(s^{*},t_1^{*},i^{*},r^{*},s_m^{*},i_m^{*})=\left((1-p)+\frac{\theta p}{\mu+\theta},\frac{\theta\mu p}{\mu+\theta},0,0,1,0\right).$$

Finding the Endemic Equilibria Point (EEP)

This equilibrium point exists if $i^{*}0$. From the fourth equation we have

$$\gamma i^{*}-(\mu+v)r^{*}=0 \Rightarrow r^{*}=\frac{\gamma}{\mu+v}i^{*}$$

From the fifth equation we have

$$\delta-\delta s_m^{*}-\beta_3 s_m^{*}i^{*}k=0 \Rightarrow s_m^{*}=\frac{\delta}{\delta+\beta_3 i^{*}k}$$

From the sixth equation

$$\beta_3 s_m^{*}i^{*}k-\delta i_m^{*}=0 \Rightarrow i_m^{*}=\frac{\delta}{\delta+\beta_3 i^{*}k}.$$

From the second equation we get:

$$\mu p-(1-\phi)\beta_2 t_1^{*}\left(\frac{k\beta_3}{\delta+\beta_2 ki^{*}}i^{*}\right)-(\mu+\theta)t_1^{*}=0$$

$$\mu p=t_1^{*}\left[(1-\phi)\left(\frac{k\beta_3 t^{*}}{\delta+\beta_2 ki^{*}}\right)+(\mu+\theta)\right]$$

$$\Rightarrow t_1^{*}=\frac{\mu p(\delta+\beta_2 ki^{*})}{(1-\phi)k\beta_3 t^{*}+(\mu+\theta)(\delta+\beta_2 ki^{*})}$$

From the first equation we obtain

$$\mu(1-p)-(\beta_1 i_m^{*}+\mu)s^{*}+i^{*}+vr^{*}=0$$

Solving for s^{*} we obtain $s^{*}=$

$$\frac{\delta+\beta_3 ki^{*}}{(\beta_1\beta_3 ki^{*}+\mu(\delta+\beta_3 ki^{*}))}\left[\mu(1-p)\frac{\theta\mu p(\delta+\beta_2 ki^{*})}{(1-\phi)k\beta_3 i^{*}+(\mu+\theta)(\delta+\beta_2 ki^{*})}+\frac{\gamma v}{\mu+v}i^{*}\right]$$

The **EEP** becomes

$$\left(\frac{\delta+\beta_3 ki^{*}}{(\beta_1\beta_3 ki^{*}+\mu(\delta+\beta_3 ki^{*}))}\left[\mu(1-p)\frac{\theta\mu p(\delta+\beta_2 ki^{*})}{(1-\phi)k\beta_3 i^{*}+(\mu+\theta)(\delta+\beta_2 ki^{*})}+\frac{\gamma v}{\mu+v}i^{*}\right],\right.$$
$$\left.\frac{\mu p(\delta+\beta_2 ki^{*})}{(1-\phi)k\beta_3 t^{*}+(\mu+\theta)(\delta+\beta_2 ki^{*})},\frac{\gamma}{\mu+v}i,\frac{\delta}{\delta+\beta_3 i^{*}k},\frac{\delta}{\delta+\beta_3 i^{*}k}\right)$$

CONCLUSION

The disease-free equilibrium point is stable if all the eigen-values are less than zero.

OBSERVATION

We observed that, the EEP is stable if the eigen-values are less tha zero. This necessitates the disease to die out instead of continuing indefinitely. The reduction in R_0 is achieved through interventions.

REFERENCES

1. Matete C, Chirima J. On Question and Answer: HIV Modelling in Mathematical Epidemiology. Sch. J. Phys. Math. Stat. 2018; 5(1):78-84
2. Smith DL, McKenzie FE, Snow RW, Hay SI. Revisiting the basic reproductive number for malaria and its implications for malaria control. PLoS biology. 2007 Feb 20;5(3):e42.