Scholars Journal of Physics, Mathematics and Statistics
Abbreviated Key Title: Sch. J. Phys. Math. Stat.
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

High Order Commutator of Marcinkiewicz Integrals with Rough Kernel Associated with Schrödinger Operators

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Article History
Received: 20.12.2017
Accepted: 20.01.2018
Published: 30.03.2018

DOI:
10.21276/sjpms.2018.5.2.6

Abstract: In this paper, by using the classical inequalities and properties of the Marcinkiewicz integrals operator, we establish the boundedness of the high order commutator of Marcinkiewicz integrals with rough kernel associated to Schrödinger operators on Lebesgue spaces.
Keywords: Marcinkiewicz integral, rough kernel, Schrödinger operator, commutator, Lebesgue space.

INTRODUCTION

We consider the Schrödinger operator $L=-\Delta+V$, where V is a nonnegative potential belonging to the reverse Hölder class RH_q for $q\ge\frac{n}{2}, n\ge 3$.

Let S^{n-1} be the unit sphere on P^n $(n\ge 2)$ equipped with the normalized Lebesgue measure $d\sigma=d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on P^n, that is

$$\Omega(\lambda x)=\Omega(x), \tag{1.1}$$

for all $\lambda>0$ and $x\in \mathrm{P}^n$.

(ii) Ω has mean zero on S^{n-1}, that is

$$\int_{S^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.2}$$

where $x'=\frac{x}{|x|}$ for any $x\neq 0$.

As in [1], for a given potential $V\in RH_q$ with $q\ge\frac{n}{2}$, we define the auxiliary function as

$$\rho(x)=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)dy\le 1\right\},\quad x\in \mathrm{P}^n.$$

Obviously, $0<\rho(x)<\infty$ for any $x\in \mathrm{P}^n$.

Let $\theta>0$ and $0<\beta<1$. The new Campanato class $\Lambda_\beta^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\frac{\beta}{n}}}\int_{B(x,r)}|b(y)-b_B|dy\le C\left(1+\frac{r}{\rho(x)}\right)^\theta, \tag{1.3}$$

for all $x\in \mathrm{P}^n$ and $r>0$. A seminorm of $b\in\Lambda_\beta^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in (1.3). Note that if $\theta=0$, $\Lambda_\beta^\theta(\rho)$ is the classical Campanato space; if $\beta=0$, $\Lambda_\beta^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [2].

The Marcinkiewicz integrals with rough kernel associated with the Schrödinger operator L is defined by

$$\mu_{j,\Omega}^L(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\Omega(x-y)|K_j^L(x,y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where $K_j^L(x,y)=\tilde{K}_j^L(x,y)|x-y|$ and $\tilde{K}_j^L(x,y)$ is the kernel of $R_j=\left(\frac{\partial}{\partial x_j}\right)L^{-\frac{1}{2}}$, $j=1,\cdots,n$. Let b be a locally integrable function and m be a positive integer. The m -order commutator generated by $\mu_{j,\Omega}^L$ and b is defined by

$$\left[b^m,\mu_{j,\Omega}^L\right]f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\left|\Omega(x-y)\right|K_j^L(x,y)(b(x)-b(y))^m f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

The Marcinkiewicz integral associated with the Schrödinger operator has been extensively studied by many authors. For example, GAO and Tang [3] showed that Marcinkiewicz integral μ_j^L are bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ and from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. Meanwhile they also proved that μ_j^L are bounded on $BMO(\mathbb{R}^n)$, and from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. When b belongs to $BMO_\theta(\rho)$, Chen and Zou [4] proved that the commutator $\left[b,\mu_j^L\right]$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Sun and Zhao [5] obtained the boundedness of Marcinkiewicz integrals with rough kernel associated with Schrödinger operators $\mu_{j,\Omega}^L$ and its commutator $\mu_{j,\Omega,b}^L$ on weighted Morrey spaces. Wang and Ren [6] established the boundedness of the m -order commutators $\left[b^m,\mu_j^L\right]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $\frac{1}{q}=\frac{1}{p}-\frac{m\beta}{n}$ and $1<p<\frac{n}{m\beta}$, as well as, on the generalized Morrey spaces related to certain nonnegative potentials. Therefore, it will be interesting to study the boundedness of the high order commutator $\left[b^m,\mu_{j,\Omega}^L\right]$ on Lebesgue spaces. The main result of this paper is as follows.

Theorem Let $V\in RH_n$, $\Omega\in L^d(S^{n-1})$. Then for any $b\in\Lambda_\beta^\theta(\rho)$, $0<\beta<1$, the commutator $\left[b^m,\mu_{j,\Omega}^L\right]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $\frac{1}{q}=\frac{1}{p}-\frac{m\beta}{n}$, $1<p<\frac{n}{m\beta}$. Moreover,

$$\left\|\left[b^m,\mu_{j,\Omega}^L\right]f\right\|_{L^q(\mathbb{R}^n)}\le C\left([b]_\beta^\theta\right)^m\|f\|_{L^p(\mathbb{R}^n)}.$$

PRELIMINARIES

Lemma 2.1 [1] There exist constants C and $k_0\ge 1$ such that

$$C^{-1}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0}\le\frac{\rho(y)}{\rho(x)}\le C\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}$$

for all $x,y\in\mathbb{R}^n$.

In particular, assume that $Q=B(x_0,\rho(x_0))$, for $x\in Q$, Lemma 2.1 tells us that $\rho(x)\approx\rho(y)$, if $|x-y|<C\rho(x)$.

Lemma 2.2 [7] There exist a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, so that, for any ball $B_k=B(x_k,\rho(x_k)), k\ge 1$, satisfies:

(i) $\bigcup_k B_k=\mathbb{R}^n$;

(ii) There exist $N=N(\rho)$ such that for every $k\in\mathbb{N}$, card$\{j:4B_j\cap 4B_k\ne\phi\}\le N$.

Lemma 2.3 [8] Let $k\in\mathbb{N}$. Then

$$\frac{1}{\left(1+2^k r/\rho(x)\right)^N}\le C\frac{1}{\left(1+2^k r/\rho(x_0)\right)^{N/(k_0+1)}},\quad\text{for } x\in 2^{k+1}B(x_0,r)\setminus 2^k B(x_0,r).$$

Let $\alpha>0$, $f\in L_{loc}^1(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$. The maximal functions are defined as

$$M_{\rho,\alpha}f(x)=\sup_{x\in B\in \mathcal{B}_{\rho,\alpha}}\frac{1}{|B|}\int_B|f(y)|dy,\quad M_{\rho,\alpha}^{\#}f(x)=\sup_{x\in B\in\mathcal{B}_{\rho,\alpha}}\frac{1}{|B|}\int_B|f(y)-f_B|dy,$$

Where $\mathcal{B}_{\rho,\alpha}=\{B(z,r):z\in\mathbb{R}^n, r\le\alpha\rho(y)\}$.

Lemma 2.4 [2] For $1<p<\infty$, there exist δ and β such that if $\{B_k\}_{k=1}^\infty$ is a sequence of balls as in Lemma 2.2, then

$$\int_{\mathbb{R}^n}\left|M_{\rho,\delta}f(x)\right|^p dx\le C\int_{\mathbb{R}^n}\left|M_{\rho,\beta}^{\#}f(x)\right|^p dx+\sum_k|B_k|\left(\frac{1}{|B_k|}\int_{2B_k}|f|dx\right)^p$$

for all $f\in L_{loc}^1(\mathbb{R}^n)$.

Lemma 2.5 [9] Let $1\le s<\infty$, $b\in\Lambda_\beta^\theta(\rho)$, and $B=B(x,r)$. Then

$$\left(\frac{1}{|2^kB|}\int_{2^kB}|b(y)-b_B|^s dy\right)^{\frac{1}{s}}\le C[b]_\beta^\theta(2^kr)^\beta\left(1+\frac{2^kr}{\rho(x)}\right)^{\theta'}$$

for all $k\in\mathbb{N}$ where $\theta'=(k_0+1)\theta$ and k_0 is the same constant as in Lemma2.1.

Lemma 2.6 [1] Let $V\in RH_q(q>1)$.Then

(i) for every N, there exists a constant C such that

$$\left|K_j^L(x,z)\right|\le\frac{C\left(1+|x-z|/\rho(x)\right)^{-N}}{|x-z|^{n-1}}.$$

(ii) If $q_0 \ge n$, for every N and $0 < \theta < 1 - \frac{n}{q_0}$, there exists a constant C such that

$$\left|K_j^L(x,z) - K_j^L(y,z)\right| \le \frac{C|x-y|^\theta \left(1 + {|x-z|}/{\rho(x)}\right)^{-N}}{|x-z|^{n-1+\theta}},$$

where $|x-y| < \frac{2}{3}|x-z|$.

Lemma 2.7 [10] Suppose that Ω satisfies (1.1), (1.2) and $V \in RH_n$. If $\Omega \in L^d(S^{n-1})$, $1 < d < \infty$ then the operators $\mu_{j,\Omega}^L$, $j = 1,\cdots,n$ are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

PROOF OF THEOREM

We first prove the following Lemmas.

Lemma 3.1 Let $V \in RH_n$, $b \in \Lambda_\beta^\theta(\rho)$, $\Omega \in L^d(S^{n-1})$ and $Q = B(x_0, \rho(x_0))$. Then for any $1 < s < \infty$,

$$\frac{1}{|Q|}\int_Q \left|\left[b^m, \mu_{j,\Omega}^L\right] f\right| dy \le C\left([b]_\beta^\theta\right)^m \inf_{x\in Q} \mathrm{M}_{m\beta,s}(f)(x) + \sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{m-\gamma} \inf_{x\in Q} \mathrm{M}_{(m-\gamma)\beta,s}\left(\left[b^\gamma, \mu_{j,\Omega}^L\right] f\right)(x)$$

holds for all $f \in L^s(\mathbb{R}^n)$, where

$$\mathrm{M}_{m\beta,s}(f)(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-{m\beta s}/{n}}}\int_B |f(y)|^s dy\right)^{\frac{1}{s}}.$$

Proof: By Binomial Theorem, we get that

$$\begin{aligned}
(b(y)-b(z))^m &= \sum_{l=1}^m C_{l,m}(b(y)-\lambda)^l(\lambda-b(z))^{m-l} + (\lambda-b(z))^m \\
&= \sum_{l=1}^m C_{l,m}(b(y)-\lambda)^l(\lambda-b(y)+b(y)-b(z))^{m-l} + (\lambda-b(z))^m \\
&= \sum_{l=1}^m \sum_{h=0}^{m-l} C_{l,m,h}(b(y)-\lambda)^{l+h}(b(y)-b(z))^{m-l-h} + (\lambda-b(z))^m \\
&= \sum_{\gamma=0}^{m-1} C_{\gamma,m}(b(y)-\lambda)^{m-\gamma}(b(y)-b(z))^{\gamma} + (\lambda-b(z))^m .
\end{aligned}$$

where $C_{l,m} = C_m^l, C_{l,m,h} = C_{m-l}^h, C_{\gamma,m} = C_m^\gamma$ is positive constant, l,m,h,γ are nonnegative integers.

Then

$$\begin{aligned}
[b^m, \mu_{j,\Omega}^L] f(y) &= \left(\int_0^\infty \left|\int_{|y-z|\le t} \left|\Omega(y-z)\right| K_j^L(y,z)(b(y)-b(z))^m f(z) dz\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\le \sum_{\gamma=0}^{m-1} C_{\gamma,m}|b(y)-\lambda|^{m-\gamma}[b^\gamma, \mu_{j,\Omega}^L](f)(y) + \mu_{j,\Omega}^L((\lambda-b)^m f)(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \left|[b^m, \mu_{j,\Omega}^L] f\right| dy &\le \frac{1}{|Q|}\int_Q \left|\sum_{\gamma=0}^{m-1} C_{\gamma,m}(b(y)-\lambda)^{m-\gamma}[b^\gamma, \mu_{j,\Omega}^L](f)(y)\right| dy + \frac{1}{|Q|}\int_Q \left|\mu_{j,\Omega}^L((\lambda-b)^m f)(y)\right| dy \\
&= A_1 + A_2
\end{aligned}$$

Let $\lambda = b_{2Q}$. For A_1, by Hölder's inequality and Lemma2.5, we have

$$\begin{aligned}
A_1 &\le C\sum_{\gamma=0}^{m-1} \frac{1}{|Q|}\int_Q \left|(b(y)-b_{2Q})^{m-\gamma}[b^\gamma, \mu_{j,\Omega}^L](f)(y)\right| dy \\
&\le C\sum_{\gamma=0}^{m-1}\left(\frac{1}{|Q|}\int_Q |b(y)-b_{2Q}|^{(m-\gamma)s'} dy\right)^{\frac{1}{s'}}\left(\frac{1}{|Q|}\int_Q \left|[b^\gamma, \mu_{j,\Omega}^L] f(y)\right|^s dy\right)^{\frac{1}{s}} \\
&\le C\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{m-\gamma}(\rho(x_0))^{\beta(m-\gamma)}\left(\frac{1}{|Q|}\int_Q \left|[b^\gamma, \mu_{j,\Omega}^L] f(y)\right|^s dy\right)^{\frac{1}{s}} \\
&\le C\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{m-\gamma} \inf_{x\in Q} \mathrm{M}_{(m-\gamma)\beta,s}\left(\left[b^\gamma, \mu_{j,\Omega}^L\right] f\right)(x)
\end{aligned}$$

where $1 < s < \infty$, and $\frac{1}{s} + \frac{1}{s'} = 1$.

Let $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $1 < \tilde{s} < s < \infty$ and $u = \frac{s\tilde{s}}{s-\tilde{s}}$. For A_2, then

$$A_2 \le \frac{1}{|Q|}\int_Q \left|\mu_{j,\Omega}^L((b_{2Q}-b)^m f_1)(y)\right| dy + \frac{1}{|Q|}\int_Q \left|\mu_{j,\Omega}^L((b_{2Q}-b)^m f_2)(y)\right| dy$$

$$= A_{21} + A_{22}$$

By Lemma2.7, Hölder's inequality and Lemma2.5, we have that

$$A_{21} \le C\left(\frac{1}{|Q|}\int_Q \left|\mu_{j,\Omega}^L\left((b_{2Q}-b)^m f_1\right)(y)\right|^{\tilde{s}}\right)^{\frac{1}{\tilde{s}}} dy$$

$$\le C\left(\frac{1}{|Q|}\int_{2Q}\left|(b(y)-b_{2Q})^m f(y)\right|^{\tilde{s}} dy\right)^{\frac{1}{\tilde{s}}}$$

$$\le C\left(\frac{1}{|Q|}\int_{2Q}|f(y)|^s dy\right)^{\frac{1}{s}}\left(\frac{1}{|Q|}\int_{2Q}\left|b(y)-b_{2Q}\right|^{mu} dy\right)^{\frac{1}{u}}$$

$$\le C\left([b]_\beta^\theta\right)^m \inf_{x\in Q} \mathrm{M}_{m\beta,s} f(x)\cdot$$

For A_{22}, note that $\rho(y)\approx\rho(x_0)$ for any $y\in Q$, by Lemma2.6, the Minkowski inequality, and Lemma2.5, we obtain

$$\left|\mu_{j,\Omega}^L\left((b_{2Q}-b)^m f_2\right)(y)\right| = \left|\left(\int_0^\infty\left|\int_{|y-z|\le t}|\Omega(y-z)|K_j^L(y,z)(b(z)-b_{2Q})^m f_2(z)dz\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}\right|$$

$$\le\left(\int_0^\infty\left|\int_{|y-z|\le t}\frac{|\Omega(y-z)||f_2(z)||b(z)-b_{2Q}|^m}{|y-z|^{n-1}\left(1+{|y-z|}/{\rho(y)}\right)^N}dz\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}$$

$$\le\int_{\mathbb{R}^n}\frac{|\Omega(y-z)||f_2(z)||b(z)-b_{2Q}|^m}{|y-z|^{n-1}\left(1+{|y-z|}/{\rho(y)}\right)^N}\left(\int_{|y-z|\le t}\frac{dt}{t^3}\right)^{\frac{1}{2}}dz$$

$$\le C\rho(x_0)^N\int_{(2Q)^c}\frac{|\Omega(y-z)||f(z)||b(z)-b_{2Q}|^m}{|y-z|^{n+N}}dz$$

$$\le C\rho(x_0)^N\sum_{k=1}^\infty\frac{\left(2^k\rho(x_0)\right)^{-N}}{|2^{k+1}Q|}\int_{2^{k+1}Q\setminus 2^kQ}|\Omega(y-z)||f(z)||b(z)-b_{2Q}|^m dz$$

$$\le C\sum_{k=1}^\infty 2^{-kN}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|\Omega(y-z)||f(z)||b(z)-b_{2Q}|^m dz\ .$$

Note that

$$\|\Omega(x-\cdot)\|_{L^q(B(x_0,t))} = \left(\int_{B(0,t)}|\Omega(y)|^q dy\right)^{\frac{1}{q}} = \left(\int_0^t r^{n-1}dr\int_{S^{n-1}}|\Omega(y')|^q d\sigma(y')\right)^{\frac{1}{q}} \le C_0\|\Omega\|_{L^q(S^{n-1})}|B(x_0,t)|^{\frac{1}{q}}\ .$$

By choosing $N\ge m\theta'$, we obtain

$$\sum_{k=1}^\infty 2^{-kN}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|\Omega(y-z)||f(z)||b(z)-b_{2Q}|^m dz$$

$$\le C\sum_{k=1}^\infty 2^{-kN}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|\Omega(y-z)|^d dz\right)^{\frac{1}{d}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b(z)-b_{2Q}|^{mc} dz\right)^{\frac{1}{c}}\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(z)|^s dz\right)^{\frac{1}{s}}$$

$$\le C\sum_{k=1}^\infty 2^{-kN}\|\Omega\|_{L^d(S^{n-1})}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b(z)-b_{2Q}|^{mc} dz\right)^{\frac{1}{c}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(z)|^s dz\right)^{\frac{1}{s}}$$

$$\le C\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m\sum_{k=1}^\infty 2^{-kN}\left(2^k r\right)^{m\beta}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{m\theta'}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(z)|^s dz\right)^{\frac{1}{s}}$$

$$\le C\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m\sum_{k=1}^\infty 2^{-k(N-m\theta')}\inf_{x\in Q}\mathrm{M}_{m\beta,s}(f)(x)$$

$$\le C\left([b]_\beta^\theta\right)^m\inf_{x\in Q}\mathrm{M}_{m\beta,s} f(x)$$

Where $\frac{1}{d}+\frac{1}{c}+\frac{1}{s}=1$.

This finishes the proof of Lemma 3.1.

Lemma 3.2 Let $V\in RH_n$, $b\in\Lambda_\beta^\theta(\rho)$, and $\Omega\in L^d(S^{n-1})$, then for any $s>1$ and $\gamma\ge 1$, there exists a constant C such that

$$\left|\mu_{j,\Omega}^{L}\left((b-b_B)^m f_2\right)(u)-\mu_{j,\Omega}^{L}\left((b-b_B)^m f_2\right)(z)\right|\le C\left([b]_\beta^\theta\right)^m \inf_{x\in B} M_{m\beta,s} f(x)$$

holds for all $f\in L_{loc}^s(\mathbb{R}^n)$, $u,z\in B=B(x_0,r)$ with $r<\gamma\rho(x_0)$ and $f_2=f\chi_{(2B)^c}$.

Proof: We write

$$\begin{aligned}
&\left|\mu_{j,\Omega}^{L}\left((b-b_B)^m f_2\right)(u)-\mu_{j,\Omega}^{L}\left((b-b_B)^m f_2\right)(z)\right|\\
&\le\left(\int_0^\infty\left|\int_{|u-y|\le t<|z-y|}\left|\Omega(u-y)K_j^L(u,y)f_2(y)(b(y)-b_{2B})^m\right|dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\quad+\left(\int_0^\infty\left|\int_{|z-y|\le t<|u-y|}\left|\Omega(z-y)K_j^L(z,y)f_2(y)(b(y)-b_{2B})^m\right|dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\quad+\left(\int_0^\infty\left|\int_{|u-y|\le t,|z-y|\le t}\left|\Omega(u-y)K_j^L(u,y)-\Omega(z-y)K_j^L(z,y)\right|\left|f_2(y)(b(y)-b_{2B})^m\right|dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&=I_1+I_2+I_3
\end{aligned}$$

Let $Q=B(x_0,\gamma\rho(x_0))$. Since $u,z\in Q$, obviously that $\rho(u)\approx\rho(x_0)$ and $|u-y|\approx|z-y|$ .By Minkowski's inequality and Lemma2.6, we have that

$$\begin{aligned}
I_1&\le\int_{(2B)^c}\left|\Omega(u-y)K_j^L(u,y)f(y)(b(y)-b_{2B})^m\right|\left(\int_{|u-y|\le t<|z-y|}\frac{dt}{t^3}\right)^{\frac12}dy\\
&\le Cr^{\frac12}\int_{(2B)^c}\frac{|\Omega(u-y)|\left|K_j^L(u,y)f(y)(b(y)-b_{2B})^m\right|}{|u-y|^{\frac32}}dy\\
&\le Cr^{\frac12}\int_{Q\setminus 2B}\frac{|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\frac12}}dy+r^{\frac12}\rho(x_0)^N\int_{Q^c}\frac{|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\frac12+N}}dy\\
&=I_{11}+I_{12}.
\end{aligned}$$

Let j_0 be the least integer such that $2^{j_0}\ge\frac{\gamma\rho(x_0)}{r}$. Splitting into annuli, we obtain

$$I_{11}\le\sum_{j=2}^{j_0}2^{-\frac j2}\frac{1}{|2^jB|}\int_{2^jB}|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m dy\ .$$

By Hölder's inequality, Lemma2.5, and noting that $2^jr\le\gamma\rho(x_0)$ for $j<j_0$, then

$$\begin{aligned}
&\frac{1}{|2^jB|}\int_{2^jB}|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m dy\\
&\le C\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m(2^jr)^{m\beta}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(y)|^s dy\right)^{\frac1s}\\
&\le C\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m\inf_{x\in B}M_{m\beta,s}f(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_{11}&\le\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m\sum_{j=2}^{j_0}2^{-\frac j2}\inf_{x\in B}M_{m\beta,s}f(x)\\
&\le C\left([b]_\beta^\theta\right)^m\inf_{x\in B}M_{m\beta,s}f(x).
\end{aligned}$$

Note that for $j\ge j_0$, there is

$$\begin{aligned}
&\frac{1}{|2^jB|}\int_{2^jB}|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m dy\\
&\le\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m(2^jr)^{m\beta}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\left(\frac{1}{|2^jB|}\int_{2^jB}|f(y)|^s dy\right)^{\frac1s}\\
&\le C\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m\left(\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\inf_{x\in B}M_{m\beta,s}f(x).
\end{aligned}$$

Then, by choosing $N \geq m\theta'$, we have that

$$
\begin{aligned}
I_{12} &\leq r^{\frac{1}{2}}\rho(x_0)^N \int_{Q^c} \frac{|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\frac{1}{2}+N}}dy \\
&\leq r^{-N}\rho(x_0)^N \sum_{j=j_0}^{\infty} 2^{-j\left(\frac{1}{2}+N\right)} \frac{1}{|2^jB|}\int_{2^jB}|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m dy \\
&\leq C\|\Omega\|_{L^d(S^{n-1})}\left([b]_\beta^\theta\right)^m \sum_{j=j_0}^{\infty} 2^{-\frac{j}{2}}\left(\frac{\rho(x_0)}{2^jr}\right)^{N-m\theta'} \inf_{x\in B} M_{m\beta,s} f(x) \\
&\leq C\left([b]_\beta^\theta\right)^m \inf_{x\in B} M_{m\beta,s} f(x).
\end{aligned}
$$

Due to the estimates for I_1, similarly, we obtain the same estimates for I_2. For I_3, note that $\rho(u)\approx\rho(x_0)$, $|u-y|\approx|z-y|$, and $|u-z|<\frac{2}{3}|u-y|$, then by Minkowski's inequality and Lemma2.6, similar to the estimates for I_{11} and I_{12}, we have

$$
\begin{aligned}
I_3 &\leq \int_{(2B)^c}|f(y)||b(y)-b_{2B}|^m\left|\Omega(u-y)K_j^L(u,y)-\Omega(z-y)K_j^L(z,y)\right|\left(\int_{\{|u-y|\leq t,|z-y|\leq t\}}\frac{dt}{t^3}\right)^{\frac{1}{2}}dy \\
&\leq C\int_{(2B)^c}\frac{|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m\left|K_j^L(u,y)-K_j^L(z,y)\right|}{|u-y|}dy \\
&\leq Cr^\delta\int_{Q\setminus 2B}\frac{|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\delta}}dy + r^\delta\rho(x_0)^N\int_{Q^c}\frac{|\Omega(u-y)||f(y)||b(y)-b_{2B}|^m}{|u-y|^{n+\delta+N}}dy \\
&\leq C\left([b]_\beta^\theta\right)^m \inf_{x\in B} M_{m\beta,s} f(x).
\end{aligned}
$$

Thus, the proof of Lemma3.2 is completed.

Lemma 3.3 Let $s>1$, $b\in\Lambda_\beta^\theta(\rho)$, and $\Omega\in L^d(S^{n-1})$. Set $B=B(x_0,r)$ with $r\leq\gamma\eta(x_0)$, and $x\in B$. Then

$$
M_{\rho,\eta}^{\#}\left(\left[b^m,\mu_{j,\Omega}^L\right]f\right)(x)\leq C\left([b]_\beta^\theta\right)^m M_{m\beta,s}(f)(x)+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{m-\gamma} M_{(m-\gamma)\beta,s}\left(\left[b^\gamma,\mu_{j,\Omega}^L\right]f\right)(x).
$$

Proof: We write

$$
\left[b^m,\mu_{j,\Omega}^L\right]f(y)=\sum_{\gamma=0}^{m-1}C_{\gamma,m}(b(y)-b_{2B})^{m-\gamma}\left[b^\gamma,\mu_{j,\Omega}^L\right]f(y)+\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)(y).
$$

Then

$$
\begin{aligned}
&\frac{1}{|B|}\int_B\left|\left[b^m,\mu_{j,\Omega}^L\right]f(y)-\left(\left[b^m,\mu_{j,\Omega}^L\right]f\right)_B\right|dy \\
&\leq C\sum_{\gamma=0}^{m-1}\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})^{m-\gamma}\left[b^\gamma,\mu_{j,\Omega}^L\right](f)(y)-\left((b(y)-b_{2B})^{m-\gamma}\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right)_B\right|dy \\
&\quad+\frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)(y)-\left(\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)\right)_B\right|dy \\
&\leq C\sum_{\gamma=0}^{m-1}\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})^{m-\gamma}\left[b^\gamma,\mu_{j,\Omega}^L\right](f)(y)\right|dy+\frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)(y)-\left(\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)\right)_B\right|dy \\
&=J_1+J_2.
\end{aligned}
$$

Thus, Hölder's inequality and Lemma2.5 tell us

$$
\begin{aligned}
J_1 &\leq C\sum_{\gamma=0}^{m-1}\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})^{m-\gamma}\left[b^\gamma,\mu_{j,\Omega}^L\right](f)(y)\right|dy \\
&\leq C\sum_{\gamma=0}^{m-1}\left(\frac{1}{|B|}\int_B|b(y)-b_{2B}|^{(m-\gamma)s'}dy\right)^{\frac{1}{s'}}\left(\frac{1}{|B|}\int_B\left|\left[b^\gamma,\mu_{j,\Omega}^L\right](f)(y)\right|^s dy\right)^{\frac{1}{s}} \\
&\leq C\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)} M_{(m-\gamma)\beta,s}\left(\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right)(x).
\end{aligned}
$$

For J_2, we split $f=f_1+f_2$ with $f_1=f\chi_{2B}$. Then

$$
\begin{aligned}
J_2 &\leq \frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L\left((b-b_{2B})^m f_1\right)(y)\right|dy+\frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)(y)-\left(\mu_{j,\Omega}^L\left((b-b_{2B})^m f\right)\right)_B\right|dy \\
&=J_{21}+J_{22}.
\end{aligned}
$$

As the proof in Lemma3.1 we obtain

$$J_{21} \le C\left([b]_\beta^\theta\right)^m M_{m\beta,s} f(x).$$

For J_{22}, the Lemma3.2 tells us

$$J_{22} \le C\frac{1}{|B|^2}\int_B\int_B\left|\mu_{j,\Omega}^L\left((b-b_B)^m f_2\right)(u)-\mu_{j,\Omega}^L\left((b-b_B)^m f_2\right)(y)\right|dudy$$

$$\le C\left([b]_\beta^\theta\right)^m M_{m\beta,s} f(x).$$

The proof of Theorem

Let t_γ satisfy $\frac{1}{t_\gamma}=\frac{1}{p}-\frac{\gamma\beta}{n}$, $\gamma=0,1,\cdots,m-1$. Then $\frac{1}{q}=\frac{1}{t_\gamma}-\frac{(m-\gamma)\beta}{n}$. We only need to prove the inequality

$$\left\|\left[b^m,\mu_{j,\Omega}^L\right]f\right\|_{L^q(\mathbb{R}^n)}^q \le C\left([b]_\beta^\theta\right)^{mq}\|f\|_{L^p(\mathbb{R}^n)}^q+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)q}\left\|\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right\|_{L^{t_\gamma}(\mathbb{R}^n)}^q. \tag{3.1}$$

Once (3.1) holds, by the mathematical induction, then Theorem will be proved. In fact, when $m=1$,we have $\gamma=0$ and $p=t_\gamma$. Note that $\left[b^0,\mu_{j,\Omega}^L\right]=\mu_{j,\Omega}^L$, by Lemma2.7, thus $\left[b,\mu_{j,\Omega}^L\right]$ is bounded from L^p (P^n) to L^q (P^n). Suppose that the $L^p-L^{t_\gamma}$ boundedness of $\left[b^\gamma,\mu_{j,\Omega}^L\right]$ holds for $\frac{1}{t_\gamma}=\frac{1}{p}-\frac{\gamma\beta}{n}$, that is

$$\left\|\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right\|_{L^{t_\gamma}(\mathbb{R}^n)} \le C\left([b]_\beta^\theta\right)^\gamma\|f\|_{L^p(\mathbb{R}^n)},$$

where $\gamma=1,2,\cdots,m-1$. Then by (3.1) we obtain Theorem.

In the following, we show (3.1).

Let $1<s<p<\infty$, $f\in L^p$ (P^n). By Lemma2.4, we have

$$\left\|\left[b^m,\mu_{j,\Omega}^L\right]f\right\|_{L^q(\mathbb{R}^n)}^q \le \int_{\mathbb{R}^n}\left|M_{\rho,\delta}\left(\left[b^m,\mu_{j,\Omega}^L\right]f\right)(x)\right|^q dx$$

$$\le \int_{\mathbb{R}^n}\left|M_{\rho,\eta}^{\#}\left(\left[b^m,\mu_{j,\Omega}^L\right]f\right)(x)\right|^q dx+\sum_k|Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}\left|\left[b^m,\mu_{j,\Omega}^L\right]f(x)\right|dx\right)^q.$$

Using Lemma 3.3, there is

$$M_{\rho,\eta}^{\#}\left(\left[b^m,\mu_{j,\Omega}^L\right]f\right)(x)\le C\left([b]_\beta^\theta\right)^m M_{m\beta,s}(f)(x)+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{m-\gamma}M_{(m-\gamma)\beta,s}\left(\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right)(x).$$

Since $\frac{1}{q}=\frac{1}{t_\gamma}-\frac{(m-\gamma)\beta}{n}$,and $t_\gamma=p$, where $\gamma=0$, we obtain

$$\int_{\mathbb{R}^n}\left|M_{\rho,\eta}^{\#}\left(\left[b^m,\mu_{j,\Omega}^L\right]f\right)(x)\right|^q dx\le C\left([b]_\beta^\theta\right)^{mq}\int_{\mathbb{R}^n}\left|M_{m\beta,s}(f)(x)\right|^q dx+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)q}\int_{\mathbb{R}^n}\left|M_{(m-\gamma)\beta,s}\left(\left[b^\gamma,\mu_{j,\Omega}^L\right]f\right)(x)\right|^q dx$$

$$\le C\left([b]_\beta^\theta\right)^{mq}\|f\|_{L^p(\mathbb{R}^n)}^q+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)q}\left\|\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right\|_{L^{t_\gamma}(\mathbb{R}^n)}^q.$$

Therefore, Lemma2.2 and Lemma3.1 can tell us

$$\sum_k|Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}\left|\left[b^m,\mu_{j,\Omega}^L\right]f(x)\right|dx\right)^q\le C\left([b]_\beta^\theta\right)^{mq}\sum_k\int_{2Q_k}\left|M_{m\beta,s}(f)\right|^q dx+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)q}\sum_k\int_{2Q_k}\left|M_{(m-\gamma)\beta,s}\left(\left[b^\gamma,\mu_{j,\Omega}^L\right]f\right)\right|^q dx$$

$$\le C\left([b]_\beta^\theta\right)^{mq}\int_{\mathbb{R}^n}\left|M_{m\beta,s}(f)(x)\right|^q dx+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)q}\int_{\mathbb{R}^n}\left|M_{(m-\gamma)\beta,s}\left(\left[b^\gamma,\mu_{j,\Omega}^L\right]f\right)(x)\right|^q dx$$

$$\le C\left([b]_\beta^\theta\right)^{mq}\|f\|_{L^p(\mathbb{R}^n)}^q+\sum_{\gamma=0}^{m-1}\left([b]_\beta^\theta\right)^{(m-\gamma)q}\left\|\left[b^\gamma,\mu_{j,\Omega}^L\right](f)\right\|_{L^{t_\gamma}(\mathbb{R}^n)}^q.$$

The proof of (3.1) is finished.

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