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# Eigenvalue Problem with Fuzzy Coefficients of Boundary Conditions

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**Abstract:** In this article two point fuzzy boundary value problem is defined under the approach generalized Hukuhara differentiability (gH-differentiability). The solution method of the fuzzy boundary value problem is examined with the aid of initial value problems for boundary values. Then results of the propesed method are illustrated with a numerical example.

**Keywords:** gH-difference, gH-derivative, Fuzzy eigenvalue, Fuzzy eigenfunction, Fuzzy Coefficients.

## INTRODUCTION

The term "fuzzy differential equation" was first introduced in 1978 by Kandel and Byatt [1] and later an extended version of this equation was published many papers in [5, 7, 12, 13]. There are many suggestions to define a fuzzy derivative and to study fuzzy differential equation [2, 6, 8, 9, 10, 11,]. One of the most well-known definitions of difference and derivative for fuzzy set value functions was given bye Hukuhara in [10]. By using the H-derivative, Kaleva in [3] started to develop a theory for fuzzy differential equations. Many works have been done by several authers in theoretical and applied fields for fuzzy differential equations with the Hukuhara derivative [6, 10, 11]. But in some cases this approach suffers certain disadvantages since the diameter of the solutions is unbounded as time t increases [3, 12]. So here we use gH-difference and gH-derivative to solve FDE under much less restrictive conditions [10].

In this paper we consider the two point fuzzy boundary value problem

$$L = -\frac{d^2}{dx^2}$$

$$L\hat{u} = \lambda\hat{u}, x \in [a, b] \tag{1}$$

which satisfy the conditions

$$\hat{a}_1\hat{u}(a) = \hat{a}_2\hat{u}'(a) \tag{2}$$

$$\hat{b}_1\hat{u}(b) = \hat{b}_2\hat{u}'(b) \tag{3}$$

where $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2 \geq 0$, $\lambda > 0$, $\hat{a}_1 + \hat{a}_2 \neq 0$ and $\hat{b}_1 + \hat{b}_2 \neq 0$, $\hat{u}(x)$ fuzzy functions.

### Preliminaries

In this section, we give some concepts and results besides the essential notations which will be used throughout the paper.

Let $\hat{u}$ be a fuzzy subset on $\mathbb{R}$, i.e. a mapping $\hat{u}: \mathbb{R} \to [0,1]$ associating with each real number $t$ its grade of membership $\hat{u}(t)$.

In this paper, the concept of fuzzy real numbers (fuzzy intervals) is considered in the sense of Xiaoand Zhu which is defined below:

**Definition 1.** [14] A fuzzy subset $\hat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha -$ cut set is denoted by $[\hat{u}]^\alpha$, i.e., $[\hat{u}]^\alpha = \{t: \hat{u}(t) \geq 0\}$, if it satisfies two axioms:

i. There exists $r \in \mathbb{R}$ such that $\hat{u}(r) = 1$,
ii. For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_\alpha^- \leq u_\alpha^+ < +\infty$ such that $[\hat{u}]^\alpha$ is equal to the closed interval $[u_\alpha^-, u_\alpha^+]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{F}(\mathbb{R})$. $\mathbb{F}_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha -$ cuts are nonempty compact convex subsets of $\mathbb{R}$. If $\hat{u} \in \mathbb{F}(\mathbb{R})$ and $\hat{u}(t) = 0$ whenever $t < 0$, then $\hat{u}$ is called a non-negative fuzzy real number and $\mathbb{F}^+(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. For all $\hat{u} \in F^+(\mathbb{R})$ and each $\alpha \in (0,1]$, real number $u_\alpha^-$ is positive.

The fuzzy real number $\hat{r} \in \mathbb{F}(\mathbb{R})$ defined by

$$\hat{r}(t) = \begin{cases} 1, \mathrm{t = r} \\ 0, \mathrm{t} \neq r \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $\mathbb{F}(\mathbb{R})$, that is if $\hat{r} \in (-\infty, +\infty)$, then $\hat{r} \in \mathbb{F}(\mathbb{R})$ satisfies $\hat{r}(t) = \hat{0}(t - r)$ and $\alpha -$ cut of $\hat{r}$ is given by $[\hat{r}]^\alpha = [r, r]$, $\alpha \in (0,1]$.

**Definition 2.** [2] An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(u_\alpha^-, u_\alpha^+)$, $0 \leq \alpha \leq 1$, which satisfy the following requirements

i. $u_\alpha^-$ is bounded non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,
ii. $u_\alpha^+$ is bounded non- increasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,
iii. $u_\alpha^- \leq u_\alpha^+$, $0 \leq \alpha \leq 1$.

**Definition 3.** [2] For $\hat{u}, \hat{v} \in \mathbb{F}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the sum $\hat{u} = \hat{v} \oplus \hat{w}$ and the product $\lambda \odot \hat{u}$ are defined by for all $\alpha \in [0,1]$,

$$[\hat{u} \oplus \hat{v}]^\alpha = [\hat{u}]^\alpha + [\hat{v}]^\alpha = \{x + y : x \in [\hat{u}]^\alpha, y \in [\hat{v}]^\alpha\},$$
$$[\lambda \odot \hat{u}]^\alpha = \lambda \odot [\hat{u}]^\alpha = \{\lambda x : x \in [\hat{u}]^\alpha\},$$

Define $D: \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ by the equation

$$D(\hat{u}, \hat{v}) = \sup_{0<\alpha\leq 1} \{\max[|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|]\}$$

where $[\hat{u}]^\alpha = [u_\alpha^-, u_\alpha^-]$, $[\hat{v}]^\alpha = [v_\alpha^-, v_\alpha^-]$. Then it is easy to show that D is a metric in $\mathbb{F}(\mathbb{R})$.

**Definition 4.** [9] Let $\hat{u}, \hat{v} \in \mathbb{F}(\mathbb{R})$. If there exist $\hat{w} \in \mathbb{F}(\mathbb{R})$ such that $\hat{u} = \hat{v} \oplus \hat{w}$, then $\hat{w}$ is called the Hukuhara difference of $\hat{u}$ and $\hat{v}$ and it is denoted by $\hat{u}\ \Theta_H \hat{v}$. If $\hat{u}\ \Theta_H \hat{v}$ exists, its $\alpha -$ cuts are

$$[\hat{u}\ \Theta_H \hat{v}]^\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+]$$

for all $\alpha \in [0,1]$.

**Definition 5.** [10] The generalized Hukuhara difference of two fuzzy numbers $\hat{u}, \hat{v} \in \mathbb{F}(\mathbb{R})$ is defined as follows

$$[\hat{u}\ \Theta_{gH} \hat{v}] = \hat{w} \Leftrightarrow \begin{cases} (i) \hat{u} = \hat{v} \oplus \hat{w} \\ or\ (ii) \hat{v} = \hat{u} \oplus (-1)\hat{w}. \end{cases}$$

In terms of α-cuts we have

$$[\hat{u}\ \Theta_{gH} \hat{v}]^\alpha = [min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}, max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}]$$

and if the H -difference exists, then $\hat{u}\ \Theta_H \hat{v} = \hat{u}\ \Theta_{gH} \hat{v}$; the conditions for the existence of $\hat{w} = \hat{u}\Theta_{gH}\hat{v} \in \mathbb{F}(\mathbb{R})$ are

$$case(i) \begin{cases} w_\alpha^- = u_\alpha^- - v_\alpha^-\ and\ w_\alpha^+ = u_\alpha^+ - v_\alpha^+, \\ with\ w_\alpha^-\ increasing,\ w_\alpha^+\ decreasing,\ w_\alpha^- \leq w_\alpha^+ \end{cases}$$
$$case(ii) \begin{cases} w_\alpha^- = u_\alpha^+ - v_\alpha^+\ and\ w_\alpha^+ = u_\alpha^- - v_\alpha^-, \\ with\ w_\alpha^-\ increasing,\ w_\alpha^+ decreasing,\ w_\alpha^- \leq w_\alpha^+. \end{cases}$$

for all $\alpha \in [0,1]$

It is easy to show that (i) and (ii) are both valid if and only if $\hat{w}$ is a crisp number.

**Remark 1.** Throughout the rest of this paper, we assume that $\hat{u}\ \Theta_{gH} \hat{v} \in \mathbb{F}(\mathbb{R})$and $\alpha -$ cut representation of fuzzy-valued function $\hat{f}: (a, b) \to \mathbb{F}(\mathbb{R})$ is expressed by $[\hat{f}(t)]^\alpha = [(f_\alpha^-)(t), (f_\alpha^+)(t)]$, $t \in [a, b]$ for each $\alpha \in [0,1]$.

**Definition 6.** [10] Let $t_0 \in (a, b)$ and $h$ be such that $t_0 + h \in (a, b)$, then the gH-derivative of a function $\hat{f}: (a, b) \to \mathbb{F}(\mathbb{R})$ at $t_0$ is defined as

$$\hat{f}'_{gH}(t_0) = \lim_{h\to 0} \frac{\hat{f}(t_0 + h)\Theta_{gH}\hat{f}(t_0)}{h} \qquad (4)$$

If $\hat{f}'_{gH}(t_0) \in \mathbb{F}(\mathbb{R})$ satisfying (4) exist, we say that $\hat{f}$ is generalized Hukuhara differentiable (gH-differentiable for short) at $t_0$.

**Definition 7.** [10] Let $\hat{f}: (a, b) \to \mathbb{F}(\mathbb{R})$ and $t_0 \in (a, b)$ with $f_\alpha^-(t)$ and $f_\alpha^+(t)$ both differentiable at $t_0$. We say that

- $\hat{f}$ is $[(i) - gH]$-differentiable at $t_0$ if
$$[\hat{f}'_{gH}(t)]^\alpha = [\{(f_\alpha^-)'(t), (f_\alpha^+)'(t)\}] \qquad (5)$$

- $\hat{f}$ is $[(ii)-gH]$-differentiable at $t_0$ if
$$\left[\hat{f}'_{gH}(t)\right]^{\alpha} = [\{(f_{\alpha}^{+})'(t), (f_{\alpha}^{-})'(t)\}] \tag{6}$$

for all $\alpha \in [0,1]$

**Definition 8.** [4] The second generalized Hukuhara derivative of a fuzzy function $\hat{f}: (a,b) \to \mathbb{F}(\mathbb{R})$ at $t_0$ is defined as
$$\hat{f}''_{gH}(t_0) = \lim_{h\to 0} \frac{\hat{f}'(t_0+h)\Theta_{gH}\hat{f}'(t_0)}{h},$$
if $\hat{f}''_{gH}(t_0) \in \mathbb{F}(\mathbb{R})$, we say that $\hat{f}'_{gH}(t)$ is generalized Hukuhara derivative at $t_0$.

Also we say that $\hat{f}'_{gH}(t)$ is $[(i)-gH]$-differentiable at $t_0$ if
$$\hat{f}''_{i.gH}(t_0,\alpha) = \begin{cases} [(f_{\alpha}^{-})''(t_0), (f_{\alpha}^{+})''(t_0)], iff be[(i)-gH] - differentiable on (a,b) \\ [(f_{\alpha}^{+})''(t_0), (f_{\alpha}^{-})''(t_0)], iff be[(ii)-gH] - differentiable on (a,b) \end{cases}$$
for all $\alpha \in [0,1]$ and that $\hat{f}'_{gH}(t)$ is $[(ii)-gH]$-differentiable at $t_0$ if
$$\hat{f}''_{ii.gH}(t_0,\alpha) = \begin{cases} [(f_{\alpha}^{+})''(t_0), (f_{\alpha}^{-})''(t_0)], iff be[(i)-gH] - differentiable on (a,b) \\ [(f_{\alpha}^{-})''(t_0), (f_{\alpha}^{+})''(t_0)], iff be[(ii)-gH] - differentiable on (a,b) \end{cases}$$
for all $\alpha \in [0,1]$.

**Solution Method of the Fuzzy Problem**

In this section we concern with fuzzy eigenvalues and eigenfunctions of two-point fuzzy boundary value problems. To do this, at first we need to use fuzzy derivatives. So here we use gH-difference and gH-derivative to solve fuzzy problem [10].

**Definition 9.** Let $\hat{u}: [a,b] \subset \mathbb{R} \to \mathbb{F}(\mathbb{R})$ be a fuzzy function and $[\hat{u}(t,\lambda)]^{\alpha} = [u_{\alpha}^{-}(t,\lambda), u_{\alpha}^{+}(t,\lambda)]$ be the $\alpha-$cut representation of the fuzzy function $\hat{u}(t)$ for all $t \in [0,1]$ and $\alpha \in [0,1]$. If the fuzzy differential equation (1) has the nontrivial solutions such that $u_{\alpha}^{-}(t,\lambda) \neq 0$, $u_{\alpha}^{+}(t,\lambda) \neq 0$ then $\lambda$ is eigenvalue of (1).

Consider the eigenvalues of the fuzzy boundary value problem (1)-(3). Using the $\alpha-$cut sets and $[(i)-gH]$ $-$differentiable of $\hat{u}(t)$, $[(ii)-gH]$ $-$differentiable of $\hat{u}'(t)$, we get from (1)-(3) for $\lambda = k^2, k>0$:
$$[-(u_{\alpha}^{-})''(t), -(u_{\alpha}^{+})''(t)] = \lambda[u_{\alpha}^{-}(t), u_{\alpha}^{+}(t)] \tag{7}$$
$$[(a_1)_{\alpha}^{-}, (a_1)_{\alpha}^{+}][u_{\alpha}^{-}(a), u_{\alpha}^{+}(a)] = [(a_2)_{\alpha}^{-}, (a_2)_{\alpha}^{+}][(u_{\alpha}^{-})'(a), (u_{\alpha}^{+})'(a)] \tag{8}$$
$$[(b_1)_{\alpha}^{-}, (b_1)_{\alpha}^{+}][u_{\alpha}^{-}(b), u_{\alpha}^{+}(b)] = [(b_2)_{\alpha}^{-}, (b_2)_{\alpha}^{+}][(u_{\alpha}^{-})'(b), (u_{\alpha}^{+})'(b)] \tag{9}$$

Suppose that the two linearly independent solutions of $u'' + \lambda u = 0$ classic differential equation are $u_1(t,\lambda)$ and $u_2(t,\lambda)$. The general solution of the fuzzy differential equation (7) is
$$[\hat{u}(t,\lambda)]^{\alpha} = [u_{\alpha}^{-}(t,\lambda), u_{\alpha}^{+}(t,\lambda)]$$
where
$$u_{\alpha}^{-}(t,\lambda) = C_1(\alpha,\lambda)u_1(t,\lambda) + C_2(\alpha,\lambda)u_2(t,\lambda)$$
$$u_{\alpha}^{+}(t,\lambda) = C_3(\alpha,\lambda)u_1(t,\lambda) + C_4(\alpha,\lambda)u_2(t,\lambda).$$
Let $\left[\hat{\phi}(\mathrm{t},\lambda)\right]^{\alpha}$ be a solution which is satisfying
$$\begin{aligned} [u_{\alpha}^{-}(a), u_{\alpha}^{+}(a)] &= [(a_2)_{\alpha}^{-}, (a_2)_{\alpha}^{+}] \\ [(u_{\alpha}^{-})'(a), (u_{\alpha}^{+})'(a)] &= [(a_1)_{\alpha}^{-}, (a_1)_{\alpha}^{+}] \end{aligned} \tag{10}$$
initial conditions of fuzzy differential equations (7). This solution can be expressed as
$$\begin{aligned} \phi_{\alpha}^{-}(x,\lambda) &= C_{11}(\alpha,\lambda)cos(kx) + C_{12}(\alpha,\lambda)sin(kx) \\ \phi_{\alpha}^{+}(x,\lambda) &= C_{13}(\alpha,\lambda)cos(kx) + C_{14}(\alpha,\lambda)sin(kx) \end{aligned} \tag{11}$$
and we write (10) conditions in (11) such that
$$\phi_{\alpha}^{-}(a,\lambda) = C_{11}(\alpha,\lambda)cos(ka) + C_{12}(\alpha,\lambda)sin(ka) = (a_2)_{\alpha}^{-}$$
$$(\phi_{\alpha}^{-})'(a,\lambda) = -kC_{11}(\alpha,\lambda)sin(ka) + kC_{12}(\alpha,\lambda)cos(ka) = (a_1)_{\alpha}^{-}.$$
From the determinant of the coefficients matrix of the above linear system, we get $C_{11}(\alpha,\lambda)$ and $C_{12}(\alpha,\lambda)$ such that
$$C_{11}(\alpha,\lambda) = \frac{\begin{vmatrix} (a_2)_{\alpha}^{-} & sin(ka) \\ (a_1)_{\alpha}^{-} & kcos(ka) \end{vmatrix}}{\begin{vmatrix} cos(ka) & sin(ka) \\ -ksin(ka) & kcos(ka) \end{vmatrix}} = (a_2)_{\alpha}^{-}cos(ka) - (a_1)_{\alpha}^{-}\frac{sin(ka)}{k} \tag{12}$$

$$C_{12}(\alpha,\lambda)=\frac{\begin{vmatrix} cos(ka) & (a_2)_\alpha^- \\ -ksin(ka) & (a_1)_\alpha^- \end{vmatrix}}{\begin{vmatrix} cos(ka) & sin(ka) \\ -ksin(ka) & kcos(ka) \end{vmatrix}}=(a_2)_\alpha^- sin(ka)+(a_1)_\alpha^- \frac{cos(ka)}{k}. \tag{13}$$

Substituting this (12) and (13) coefficients the above equations in (11), the general solution is obtained as

$$\begin{aligned}\phi_\alpha^-(x,\lambda)&=\left(a_2^-(\alpha)cos(ka)-a_1^-(\alpha)\frac{sin(ka)}{k}\right)cos(kx)\\&+\left(a_2^-(\alpha)sin(ka)+a_1^-(\alpha)\frac{cos(ka)}{k}\right)sin(kx).\end{aligned} \tag{14}$$

Similarly we find $\phi_\alpha^+(x,\lambda)$ as

$$\begin{aligned}\phi_\alpha^+(x,\lambda)&=\left((a_2)_\alpha^+cos(ka)-(a_1)_\alpha^+\frac{sin(ka)}{k}\right)cos(kx)\\&+\left((a_2)_\alpha^+sin(ka)+(a_1)_\alpha^+\frac{cos(ka)}{k}\right)sin(kx).\end{aligned} \tag{15}$$

Let $[\hat{\chi}(x,\lambda)]^\alpha$ be a solution which is satisfying

$$\begin{aligned}&[u_\alpha^-(b),u_\alpha^+(b)]=[(b_2)_\alpha^-,(b_2)_\alpha^+]\\&[(u_\alpha^-)'(b),(u_\alpha^+)'(b)]=[(b_1)_\alpha^-,(b_1)_\alpha^+]\end{aligned} \tag{16}$$

Initial conditions of fuzzy differential equations (7). This solution can be expressed as

$$\begin{aligned}\chi_\alpha^-(x,\lambda)&=C_{21}(\alpha,\lambda)cos(kx)+C_{22}(\alpha,\lambda)sin(kx)\\\chi_\alpha^+(x,\lambda)&=C_{23}(\alpha,\lambda)cos(kx)+C_{24}(\alpha,\lambda)sin(kx)\end{aligned} \tag{17}$$

and we write (16) conditions in (17) such that

$$\begin{aligned}\chi_\alpha^-(b,\lambda)&=C_{21}cos(kb)+C_{22}sin(kb)=(b_2)_\alpha^-\\(\chi_\alpha^-)'(b,\lambda)&=-kC_{21}sin(kb)+kC_{22}cos(kb)=(b_1)_\alpha^-\end{aligned}$$

From the determinant of the coefficients matrix of the above linear system we get $C_{21}(\alpha,\lambda)$ and $C_{22}(\alpha,\lambda)$ such that

$$C_{21}(\alpha,\lambda)=\frac{\begin{vmatrix} k^2(b_2)_\alpha^- & sin(kb) \\ (b_1)_\alpha^- & kcos(kb) \end{vmatrix}}{\begin{vmatrix} cos(kb) & sin(kb) \\ -ksin(kb) & kcos(kb) \end{vmatrix}}=k^2(b_2)_\alpha^- cos(kb)-(b_1)_\alpha^-\frac{sin(kb)}{k} \tag{18}$$

$$C_{22}(\alpha,\lambda)=\frac{\begin{vmatrix} cos(kb)k^2 & (b_2)_\alpha^- \\ -ksin(kb) & (b_1)_\alpha^- \end{vmatrix}}{\begin{vmatrix} cos(kb) & sin(kb) \\ -ksin(kb) & kcos(kb) \end{vmatrix}}=k^2(b_2)_\alpha^- sin(kb)+(b_1)_\alpha^-\frac{cos(kb)}{k} \tag{19}$$

Substituting this (18) and (19) coefficients the above equations in (17), the general solution is obtained as

$$\begin{aligned}\chi_\alpha^-(x,\lambda)&=\left(k^2(b_2)_\alpha^-cos(kb)-(b_1)_\alpha^-\frac{sin(kb)}{k}\right)cos(kx)\\&+\left(k^2(b_2)_\alpha^-sin(kb)+(b_1)_\alpha^-\frac{cos(kb)}{k}\right)sin(kx)\end{aligned} \tag{20}$$

Similarly we find $\chi_\alpha^+(x,\lambda)$ as

$$\begin{aligned}\chi_\alpha^+(x,\lambda)&=\left(k^2b_2^+(\alpha)cos(kb)-b_1^+(\alpha)\frac{sin(kb)}{k}\right)cos(kx)\\&+\left(k^2b_2^+(\alpha)sin(kb)+b_1^+(\alpha)\frac{cos(kb)}{k}\right)sin(kx)\end{aligned} \tag{21}$$

Then from (14)-(15) and (20)-(21) we find Wronskian function as

$$\begin{aligned}W(\phi_\alpha^-,\chi_\alpha^-)(t,\lambda)&=\phi_\alpha^-(x,\lambda)(\chi_\alpha^-)'(x,\lambda)-\chi_\alpha^-(x,\lambda)(\phi_\alpha^-)'(x,\lambda)\\&=\left(a_2^-(\alpha)b_1^-(\alpha)-a_1^-(\alpha)b_2^-(\alpha)\right)cosk(a-b)\\&-\left(ka_2^-(\alpha)b_2^-(\alpha)+\frac{a_1^-(\alpha)b_1^-(\alpha)}{k}\right)sink(a-b)\end{aligned}$$

and

$$\begin{aligned}W(\phi_\alpha^+,\chi_\alpha^+)(x,\lambda)&=\phi_\alpha^+(x,\lambda)(\chi_\alpha^+)'(x,\lambda)-\chi_\alpha^+(x,\lambda)(\phi_\alpha^+)'(x,\lambda)\\&=\left(a_2^+(\alpha)b_1^+(\alpha)-a_1^+(\alpha)b_2^+(\alpha)\right)cosk(a-b)\\&-\left(ka_2^+(\alpha)b_2^+(\alpha)+\frac{a_1^+(\alpha)b_1^+(\alpha)}{k}\right)sink(a-b).\end{aligned}$$

**Teorem 1.** [11] The Wronskian functions $W(\phi_\alpha^-,\chi_\alpha^-)(t,\lambda)$ and $W(\phi_\alpha^+,\chi_\alpha^+)(t,\lambda)$ are independent of variable x for $x \in [a,b]$, where functions $\phi_\alpha^-,\chi_\alpha^-,\phi_\alpha^+,\chi_\alpha^+$ are the solution of the fuzzy boundary value problem (1)-(3) and it is show that

$$[\widehat{W}]^\alpha = [W_\alpha^-(\lambda), W_\alpha^+(\lambda)]$$

for each $\alpha \in [0,1]$.

**Teorem 2.** [11] The eigenvalues of the fuzzy boundary value problem (1)-(3) if and only if are consist of the zeros of functions $W_\alpha^-(\lambda)$ and $W_\alpha^+(\lambda)$.

**Example 1.** Consider the two point fuzzy boundary problem

$$-\hat{u}'' = \lambda\hat{u} \tag{22}$$
$$\hat{u}(0) = 0 \tag{23}$$
$$\hat{1}\hat{u}(1) = \hat{u}'(1) \tag{24}$$

where $[\hat{1}]^\alpha = [\alpha, 2-\alpha]$, $\lambda = k^2, k > 0$ and $\hat{u}(t)$ is $[(i)-gH]$ −differentiable and $\hat{u}'(t)$ is $[(ii)-gH]-$ differentiable fuzzy functions.

Let $[\hat{\phi}(x,\lambda)]^\alpha$ be the solution which is satisfying

$$[u_\alpha^-(0), u_\alpha^+(0)] = 0$$

initial conditions of fuzzy differential equations (22). Then we find $[\hat{\phi}(x,\lambda)]^\alpha$ as

$$[\hat{\phi}(x,\lambda)]^\alpha = sin(kx). \tag{25}$$

Similarly $[\hat{\chi}(x,\lambda)]^\alpha$ be the solution which is satisfying

$$[u_\alpha^-(1), u_\alpha^+(1)] = 1$$
$$[(u_\alpha^-)'(1), (u_\alpha^+)'(1)] = [\alpha, 2-\alpha]$$

initial conditions of fuzzy differential equations (22). Then we find $[\hat{\chi}(x,\lambda)]^\alpha$ as

$$\begin{aligned} \chi_\alpha^-(x,k) &= cos(kx-k) + \frac{\alpha}{k} sin(kx-k) \\ \chi_\alpha^+(x,k) &= cos(kx-k) + \frac{(2-\alpha)}{k} sin(kx-k) \end{aligned} \tag{26}$$

From Theorem 2, eigenvalues of the fuzzy problem(22)-(24) are zeros of the functions $W_\alpha^-(\lambda)$ and $W_\alpha^+(\lambda)$. So we get

$$W_\alpha^-(\lambda) = \alpha sink - kcosk = 0 \tag{27}$$
$$W_\alpha^+(\lambda) = (2-\alpha)sink - kcosk = 0. \tag{28}$$

For each $\alpha \in [0,1]$, if the k values satisfying (27) and (28) equations compute with Matlab Program, then an infinite number of eigenvalues are obtained. So we show k values of (27) with $k_{1n}$ in Table 1 such that and k values of (28) with $k_{2n}$ in Table 2 such that

**Table-1: For $W_\alpha^-(\lambda)$ $k_{1,n}$ eigenvalues corresponding to α**

| $\alpha \in [0,1]$ | $k_{1,0}$ | $k_{1,1}$ | $k_{1,2}$ | $k_{1,3}$ | $k_{1,4}\ldots$ |
|---|---|---|---|---|---|
| $\alpha = 0$ | 4.2748 | 7.5965 | 10.8127 | 13.9952 | 17.1628 |
| $\alpha = 0.2$ | 4.3174 | 7.6221 | 10.8309 | 14.0094 | 17.1743 |
| $\alpha = 0.5$ | 4.3826 | 7.6606 | 10.8583 | 14.0307 | 17.1917 |
| $\alpha = 0.8$ | 4.4489 | 7.6994 | 10.8858 | 14.0520 | 17.2091 |
| $\alpha = 1$ | 4.4934 | 7.7253 | 10.9041 | 14.0662 | 17.2208 |

**Table-2: For $W_\alpha^+(\lambda)$ $k_{2,n}$ eigenvalues corresponding to α**

| $\alpha \in [0,1]$ | $k_{2,0}$ | $k_{2,1}$ | $k_{2,2}$ | $k_{2,3}$ | $k_{2,4}\ldots$ |
|---|---|---|---|---|---|
| $\alpha = 0$ | 4.7124 | 7.8540 | 10.9956 | 14.1372 | 17.2788 |
| $\alpha = 0.2$ | 4.6696 | 7.8284 | 10.9774 | 14.1230 | 17.2672 |
| $\alpha = 0.5$ | 4.6042 | 7.7899 | 10.9499 | 14.1017 | 17.2498 |
| $\alpha = 0.8$ | 4.5379 | 7.7511 | 10.9225 | 14.0804 | 17.2324 |
| $\alpha = 1$ | 4.4934 | 7.7253 | 10.9041 | 14.0662 | 17.2208 |

So if we write $k_{1n}$ and $k_{2n}$ eigenvalues in (25) and (26) equations, then $[\hat{\phi}(x,\lambda)]^{\alpha}$ and $[\hat{\chi}(x,\lambda)]^{\alpha}$ are

$$[\hat{\phi}_n(x,\lambda)]^{\alpha} = [(\phi_n)_{\alpha}^{-}(x), (\phi_n)_{\alpha}^{+}(x)] = [sin(k_{1,n}x), sin(k_{2,n}x)] \tag{29}$$

$$\begin{aligned}[\hat{\chi}_n(x,\lambda)]^{\alpha} &= [(\chi_n)_{\alpha}^{-}(x,k), (\chi_n)_{\alpha}^{+}(x,k)] \\ &= \Big[cos(k_{1,n}x - k_{1,n}) + \frac{\alpha}{k_{1,n}} sin(k_{1,n}x - k_{1,n}), \\ &\quad cos(k_{2,n}x - k_{2,n}) + \frac{(2-\alpha)}{k_{2,n}} sin(k_{2,n}x - k_{2,n})\Big].\end{aligned} \tag{30}$$

Then for all $\alpha \in [0,1]$, (29) are eigenfunctions corresponding to $\lambda_{1n} = (k_{1n})^2$ eigenvalues satisfying (27) equation and (30) are eigenfunctions corresponding to $\lambda_{2n} = (k_{2n})^2$ eigenvalues satisfying (28) equation.

Consider that eigenvalues of $[\hat{\phi}_n(x,\lambda)]^{\alpha}$ and $[\hat{\chi}_n(x,\lambda)]^{\alpha}$ eigenfunctions depend on $\alpha -$ cut. So if we change α, then this eigenvalues change and eigenfunctions corresponding to λ change.

In particular, we select $k_{1,0} = 4.2748$ in Table 1 and $k_{2,0} = 4.7124$ in Table 2 for $\alpha = 0$. If we substitute this values respectively in (29) and (30), we have the following figures.

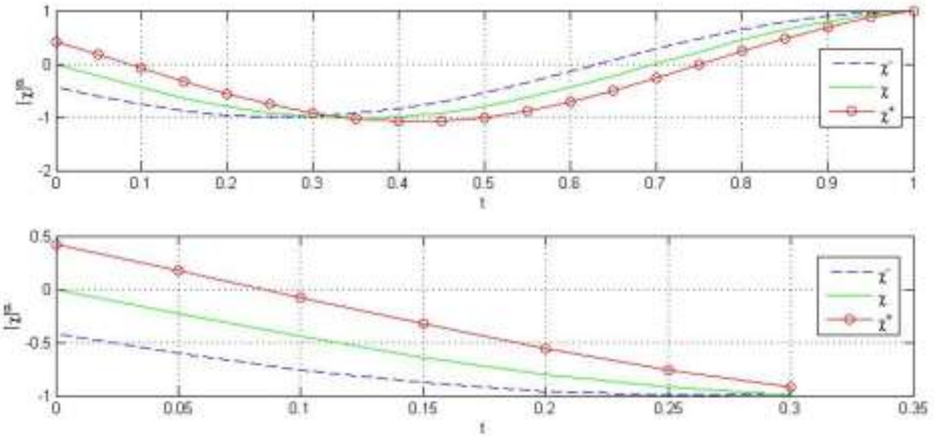


**Fig-1: Fuzzy Solution of $\hat{\chi}(x,\lambda)$eigenfunctions for $\alpha = 0$**

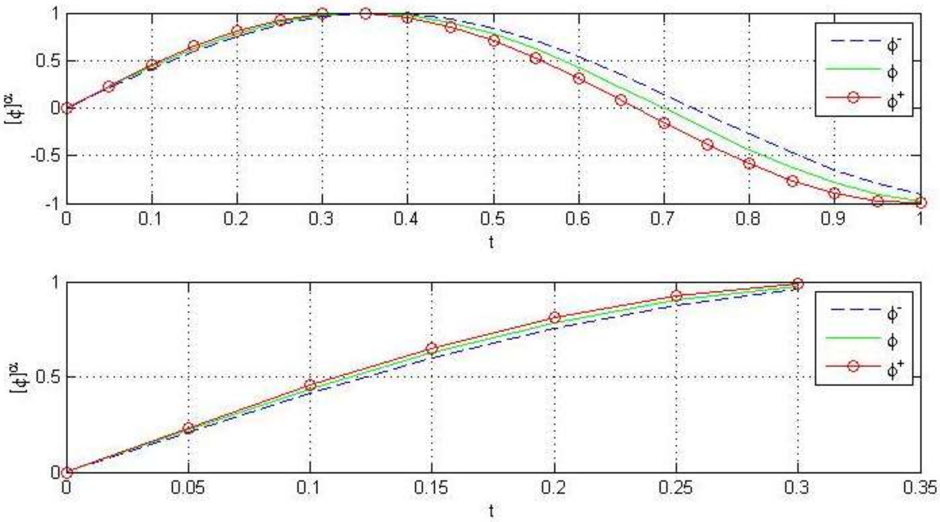


**Fig-2: Fuzzy Solution of $\hat{\phi}(x,\lambda)$eigenfunctions for $\alpha = 0$**

From Definition 2, we see $\chi_{\alpha}^{-}(x)$ and $\chi_{\alpha}^{+}(x)$ represent a valid fuzzy number for $t \in (0,0.3)$in figure 1 and we see $\phi_{\alpha}^{-}(x)$ and $\phi_{\alpha}^{+}(x)$ represent a valid fuzzy number for $t \in (0,3)$in figure 2.

## CONCLUSIONS

Eigenvalues and eigenfunctions of the fuzzy boundary problem are introduced and examined by using generalized Hukuhara differentiability concept. Then we give numerical example. Here we solve initial value problems for boundary values. Then we get eigenfunctions corresponding to eigenvalues. This is different method to find eigenvalues for fuzzy problem. In future study the analogous problem for two point boundary value problems with eigenvalue parameter in the boundary conditions is an important subject in this field.

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