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Boundedness of Fractional Integral Operators with Variable Kernels Associate to Variable Exponents

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Abstract: Let $\alpha(\cdot)$ satisfy the log-Hölder continuity condition and $1<\alpha(\cdot)<n$. Suppose $T_{\Omega,\alpha(\cdot)}$ is the fractional integral operator with variable kernel associate to variable exponent. In this paper, using the properties of weighted Morrey spaces, we prove that $T_{\Omega,\alpha(\cdot)}$ is bounded from $L^{p,\kappa}(\omega^p,\omega^q)$ to $L^{p,\kappa q/p}(\omega^q)$.

Keywords: fractional integral operator; variable exponent; variable kernel; weighted Morrey space.

INTRODUCTION

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$, $n>2$ and $d\sigma$ is the normalized Lebesgue measure on S^{n-1}. A function $\Omega(x,z)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n)\times L^r(S^{n-1}), r\ge 1$, if it satisfies the following conditions.

(i) $\Omega(x,\lambda z)=\Omega(x,z)$, for any $\lambda>0$ and $x,z\in\mathbb{R}^n$;

(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} := \sup\limits_{x\in\mathbb{R}^n}\left(\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z')\right)^{1/r}<\infty$;

(iii) for any $x\in\mathbb{R}^n$, $\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z')=0$, where $z'=\dfrac{z}{|z|}$ and $z\neq 0$.

Assume $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1}), r\ge 1$. We say that Ω satisfies the L^r-Dini condition if the conditions (i),(ii),(iii) above hold and $\int_0^1\frac{\omega_r(\delta)}{\delta}d\delta<\infty$, where $\omega_r(\delta)$ is defined by

$$\omega_r(\delta):=\sup_{x\in\mathbb{R}^n,\|\rho\|\le\delta}\left(\int_{S^{n-1}}|\Omega(x,\rho x')-\Omega(x,x')|^r d\sigma(x')\right)^{1/r},$$

which ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\|=\sup\limits_{x'\in S^{n-1}}|\rho x'-x'|$.

A measurable function $\alpha(\cdot)$ is called a variable exponent if $\alpha(\cdot):\mathbb{R}^n\to(0,\infty)$. For a measurable subset $G\subset\mathbb{R}^n$, we write $\alpha_-=\inf\limits_{x\in G}\alpha(x)$, $\alpha_+=\sup\limits_{x\in G}\alpha(x)$. Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of functions $\alpha(\cdot)$ satisfying $1<\alpha_-\le\alpha_+<n$.

We say that $\alpha(\cdot)$ satisfies the log-Hölder continuity condition, if

$$|\alpha(x)-\alpha(y)|\le\frac{C}{\log(1/|x-y|)},\quad |x-y|\le\frac{1}{2};$$

$$|\alpha(x)-\alpha(y)|\le\frac{C}{\log(e+|x|)},\quad |y|\ge|x|.$$

Set $\Omega(x,z)\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1}), r\ge 1$, satisfying the L^r-Dini condition. The fractional integral operator with variable kernels associate to variable exponents is defined by

$$T_{\Omega,\alpha(\cdot)}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\alpha(\cdot)}}f(y)dy,$$

where $\alpha(\cdot)$ is a variable exponent , satisfying the log-Hölder continuity condition and $\alpha(\cdot)\in\mathcal{P}(\mathbb{R}^n)$.

Obviously, if $\alpha(\cdot)=\alpha$ being a constant, and $\Omega\equiv 1$, the fractional integral operator with variable kernels associate to variable exponents is become to the classical fractional integral $T_{\Omega,\alpha}$. In 1974, Muckenhoupt and Wheeden[1] studied the boundedness of $T_{\Omega,\alpha}$ on Lebesgue spaces for $0<\alpha<n$. In 2009, Komori and Shirai[2] defined the weighted Morrey space, which is a generalized weighted Lebesgue space. Then, the boundedness of fractional integral operators and fractional maximal operators, and their commutators on weighted Morrey spaces were discussed by Wang[3].

In this paper, we discussed the fractional integral operators with variable kernels associate to variable exponents on weighted Morrey spaces.

PRELIMINARIES

We recall several useful lemmas and definitions.

Lemma 2.1(see [4] and [5]). Let $\alpha(\cdot)$ satisfy log-Hölder condition and $\alpha(\cdot)\in\mathcal{P}$. Suppose that $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1}),\, r\geq\dfrac{n}{n-\alpha(\cdot)}$, satisfying the L^r-Dini condition. Set $1/q=1/p-\alpha(\cdot)/n$. Then there exists a positive constant C , such that for all $f\in L_p(\Omega)$,

$$\|T_{\Omega,\alpha(\cdot)}f\|_{L^q}\leq C\|f\|_{L^p} .$$

Lemma 2.2(see [6]). Let $\omega\in A_p$. Then for any ball B , there exists a constant $C>0$, such that $\omega(2B)\leq C\omega(B)$. In fact, $\omega(\lambda B)\leq C\lambda^{np}\omega(B)$ for $\lambda>1$, where the constant C is independent of B and λ .

Let $1<p<q<\infty$. We say that a weighted function ω is belong to weighted set $A(p,q)$, if for any ball $B\subset\mathbb{R}^n$, there exists a constant $C>0$ independent of B , such that

$$\left(\frac{1}{|B|}\int_B\omega(x)^q\,dx\right)^{1/q}\left(\frac{1}{|B|}\int_B\omega(x)^{-p'}\,dx\right)^{1/p'}\leq C .$$

We say a weighted function belong to the inverse Hölder inequality RH_r , if there exist constants $s>1$ and $C>0$, such that

$$\left(\frac{1}{|B|}\int_B\omega(x)^s\,dx\right)^{1/s}\leq C\left(\frac{1}{|B|}\int_B\omega(x)\,dx\right).$$

As we all know that if $\omega\in A_p$, then for all $s>p$, $\omega\in A_s$. If $\omega\in A_p$, then there exists $s>1$, such that $\omega\in RH_s$.

Lemma 2.3(see [7]). Let $\omega\in RH_s$, $s>1$. Then there exists a positive constant C , such that for any measurable subset $E\subset B$,

$$\frac{\omega(E)}{\omega(B)}\leq C\left(\frac{|E|}{|B|}\right)^{(s-1)/s} .$$

Definition 2.4(see [2]). Let $1\leq p<\infty$, $0<\kappa<1$ and ω be a weighted function. Then a weighted Morrey space $L^{p,\kappa}(\omega)$ is defined by

$$L^{p,\kappa}(\omega):=\left\{\, f\in L^p_{\mathrm{loc}}(\omega):\|f\|_{L^{p,\kappa}(\omega)}<\infty\right\},$$

where $\|f\|_{L^{p,\kappa}(\omega)}:=\sup\limits_{B\in\mathbb{R}^n}\left(\dfrac{1}{\omega(B)^\kappa}\displaystyle\int_B|f(x)|^p\,\omega(x)\,dx\right)^{1/p}$.

Definition 2.5(see [2]). Let $1\leq p<\infty$, $0<\kappa<1$, u and v be weighted functions. Then a weighted Morrey space $L^{p,\kappa}(u,v)$ is defined by

$$L^{p,\kappa}(u,v):=\left\{\, f\in L^p_{loc}(u):\|f\|_{L^{p,\kappa}(u,v)}<\infty\right\},$$

where $\|f\|_{L^{p,\kappa}(u,v)}:=\sup\limits_{B\in\mathbb{R}^n}\left(\dfrac{1}{v(B)^\kappa}\displaystyle\int_B|f(x)|^p\,u(x)\,dx\right)^{1/p}$.

Lemma 2.6 (see [8] and [9]). Let $\alpha(\cdot)\in\mathcal{P}$. If $\alpha(\cdot)$ is log-Hölder continuous at origin, then $C^{-1}|x|^{\alpha(0)}\le|x|^{\alpha(x)}\le C|x|^{\alpha(0)}$, $|x|<1$. If $\alpha(\cdot)$ is log-Hölder continuous at the infinity, then $C^{-1}|x|^{\alpha(\infty)}\le|x|^{\alpha(x)}\le C|x|^{\alpha(\infty)}$, $|x|\ge 1$, where $\alpha(\infty)=\lim_{x\to\infty}\alpha(x)$.

THE MAIN RESULT

The main result of this paper is as follows.

Theorem 3.1 Set $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$, $r\in(1,\infty)$. Let $\alpha(\cdot)$ be a variable exponent satisfying the log-Hölder continuity condition. Suppose $r'<p<n/\alpha(\cdot)$, $1/q=1/p-\alpha(\cdot)/n$, $0<\kappa<p/q$ and $\omega^{r'}\in A(p/r',q/r')$. Then

$$\left\|T_{\Omega,\alpha(\cdot)}f\right\|_{L^{q,\kappa q/p}(\omega^q)}\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}. \tag{1}$$

Proof: Fix a ball $B=B(x_0,r_B)\subset\mathbb{R}^n$. Let $f=f_1+f_2$, where $f_1=f_{\chi_{2B}}$, χ_{2B} denoting the characteristic function of $2B$.

Since $T_{\Omega,\alpha(\cdot)}$ is a linear operator, we write

$$\begin{aligned}
&\frac{1}{\omega^q(B)^{\kappa/p}}\left(\int_B\left|T_{\Omega,\alpha(\cdot)}f(x)\right|^q\omega(x)^q dx\right)^{1/q}\\
&\le\frac{1}{\omega^q(B)^{\kappa/p}}\left(\int_B\left|T_{\Omega,\alpha(\cdot)}f_1(x)\right|^q\omega(x)^q dx\right)^{1/q}+\frac{1}{\omega^q(B)^{\kappa/p}}\left(\int_B\left|T_{\Omega,\alpha(\cdot)}f_2(x)\right|^q\omega(x)^q dx\right)^{1/q}\\
&=I_1+I_2.
\end{aligned}$$

Set $p_1=p/r'$, $q_1=q/r'$ and $\upsilon=\omega^{r'}$. Since $\upsilon\in A(p_1,q_1)$, we can get(see [1])

$$\upsilon^{q_1}=\omega^q\in A_{1+q_1/p_1'}.$$

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
I_1&\le C\frac{1}{\omega^q(B)^{\kappa/p}}\left(\int_{2B}\left|f(x)\right|^p\omega(x)^p dx\right)^{1/p}\\
&\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}\frac{\omega^q(2B)^{\kappa/p}}{\omega^q(B)^{\kappa/p}}\\
&\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}.
\end{aligned}$$

To estimate I_2, using the Hölder inequality, we obtain

$$\begin{aligned}
\left|T_{\Omega,\alpha(\cdot)}f_2(x)\right|&\le\int_{(2B)^c}\frac{\Omega(x,x-y)}{|x-y|^{n-\alpha(\cdot)}}\left|f(y)\right|dy\\
&\le\sum_{k=1}^{\infty}\left(\int_{2^{k+1}B\setminus 2^kB}\left|\Omega(x,x-y)\right|^r dy\right)^{1/r}\times\left(\int_{2^{k+1}B\setminus 2^kB}\frac{\left|f(y)\right|^{r'}}{|x-y|^{(n-\alpha(\cdot))r'}}dy\right)^{1/r'}.
\end{aligned}\tag{2}$$

Since $x\in B$, $y\in 2^{k+1}B\setminus 2^kB$, we see that $|x-y|\sim|x_0-y|\sim 2^{k+1}r_B$, where x_0 is the center of B. Hence,

$$\left(\int_{2^{k+1}B\setminus 2^kB}\left|\Omega(x,x-y)\right|^r dy\right)^{1/r}\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\left|2^{k+1}B\right|^{1/r}. \tag{3}$$

By Lemma 2.6, if $2^{k+1}r_B\le 1$, then

$$\left(\int_{2^{k+1}B\setminus 2^kB}\frac{\left|f(y)\right|^{r'}}{|x-y|^{(n-\alpha(\cdot))r'}}dy\right)^{1/r'}\le C\frac{1}{\left|2^{k+1}B\right|^{1-\alpha(0)/n}}\left(\int_{2^{k+1}B}\left|f(y)\right|^{r'}dy\right)^{1/r'}. \tag{4}$$

If $2^{k+1}r_B>1$, then

$$\left(\int_{2^{k+1}B\setminus 2^kB}\frac{\left|f(y)\right|^{r'}}{|x-y|^{(n-\alpha(\cdot))r'}}dy\right)^{1/r'}\le C\frac{1}{\left|2^{k+1}B\right|^{1-\alpha(\infty)/n}}\left(\int_{2^{k+1}B}\left|f(y)\right|^{r'}dy\right)^{1/r'}.$$

Here and below we only prove the case that $2^{k+1}r_B \le 1$. The other one is similar and simple. (2), (3) and (4) tell us that

$$\left|T_{\Omega,\alpha(\cdot)}f_2(x)\right| \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{1-\alpha(0)/n-1/r}}\left(\int_{2^{k+1}B}|f(y)|^{r'}dy\right)^{1/r'}.$$

By the Hölder inequality and the definition of $\upsilon \in A(p_1,q_1)$, we have

$$\begin{aligned}\left(\int_{2^{k+1}B}|f(y)|^{r'}dy\right)^{1/r'} &\le \left(\int_{2^{k+1}B}|f(y)|^{p_1r'}\upsilon(y)^{p_1}dy\right)^{1/(p_1r')}\left(\int_{2^{k+1}B}\upsilon(y)^{-p_1'}dy\right)^{1/(p_1'r')}\\ &\le C\left(\int_{2^{k+1}B}|f(y)|^{p}\omega(y)^{p}dy\right)^{1/p}\left(\frac{\left|2^{k+1}B\right|^{1-1/p_1+1/q_1}}{\upsilon^{q_1}\left(2^{k+1}B\right)^{1/q_1}}\right)^{1/r'}\\ &\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}\,\omega^q\left(2^{k+1}B\right)^{\kappa/p}\cdot\frac{\left|2^{k+1}B\right|^{1/r'-1/p+1/q}}{\omega^q\left(2^{k+1}B\right)^{1/q}}\\ &\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}\,\omega^q\left|2^{k+1}B\right|^{1-1/r-\alpha(0)/n}\omega^q\left(2^{k+1}B\right)^{\kappa/p-1/q}.\end{aligned}$$

Thus,

$$\left|T_{\Omega,\alpha(\cdot)}f_2(x)\right| \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}\sum_{k=1}^{\infty}\omega^q\left(2^{k+1}B\right)^{\kappa/p-1/q}.$$

Therefore, it can be obtained that

$$I_2 \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}\sum_{k=1}^{\infty}\frac{\omega^q(B)^{1/q-\kappa/p}}{\omega^q\left(2^{k+1}B\right)^{1/q-\kappa/p}}.$$

Noting that

$$\omega^q = \upsilon^{q_1} \in A_{1+q_1/p_1'},$$

there exists a constant $s>1$, such that $\omega^q \in RH_s$. From Lemma 2.3, it can be obtained that

$$\frac{\omega^q(B)}{\omega^q\left(2^{k+1}B\right)} \le C\left(\frac{|B|}{\left|2^{k+1}B\right|}\right)^{(s-1)/s}.$$

Finally, since $s>1$ and $0<\kappa<p/q$, we see that

$$I_2 \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}\sum_{k=1}^{\infty}\left(\frac{1}{2^{kn}}\right)^{(1-1/s)(1/q-\kappa/p)} \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)}.$$

Hence, (1) is proved. The proof of Theorem 1.1 is now completed.

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REFERENCES

1. Muckenhoupt B, Wheeden R. Weighted norm inequalities for fractional integrals. Transactions of the American Mathematical Society. 1974;192:261-74.
2. Komori Y, Shirai S. Weighted Morrey spaces and a singular integral operator. Math. Nachr, 2009, 282(2):219-231.
3. Wang H. Boundedness of fractional integral operators with rough kernels on weighted Morrey spaces. Acta Mathematica Sinica(Chinese Series), 2013, 56(2):175-186.
4. Almeida A, Hasanov J, Samko S. Maximal and potential operators in variable exponent Morrey spaces.Georgian Mathematical Journal, 2008, 15(2):1-14.
5. Tang C, Wu Q, Xu J. Estimates of Fractional Integral Operators on Variable Exponent Lebesgue Spaces. Journal of Function Spaces. 2016;2016.
6. García-Cuerva J, De Francia JR. Weighted norm inequalities and related topics. Elsevier; 2011 Aug 18.
7. Gundy R, Wheeden R. Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh-Paley series. Studia Mathematica. 1974;49(2):107-24.
8. Cheng XX, Shu LS. Boundedness for some Hardy type operators on Herz-Morrey spaces with variable exponent. Journal of Anhui Normal University, 2015, 38(1):19-24.

9. Izuki M. Fractional integrals on Herz-Morrey spaces with variable exponent. Hiroshima Mathematical Journal. 2010;40(3):343-55.