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On Pseudo Hopfian R-modules

Nada Khalid[1*], Adwar H. Salih[2]
[1]Department of Mathematics, University of Tikrit –Iraq
[2]Department of Mathematics, University of Tikrit –Iraq

***Corresponding author**
Nada Khalid

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Abstract: In the present paper we introduced and investigate a new definition of pseudo Hopfian R-Module. An R-Module M is said to be pseudo Hopfian when any surjection f ∈ End(M) with closed kernel, (i.e) any surjective f ∈ End(M) is split. Several characterizations and fundamental properties concerning of pseudo Hopfian R-Module are obtained, furthermore some interesting examples are discussed.
Keywords: Hopfian R- module, semi Hopfian R- module, pseudo Hopfian R-Module, closed kernel, extension R-module.

INTRODUCTION

Hopfian groups rings and modules studied by a lot of authors. From [1] That an R-mod. M is assumed Hopfian if any surjective $f \in End(M)$is a homology. Notice, every Noetherian R- Module is Hopfian. In this paper, we are concerned with pseudo Hopfian R-mod. R-Module M is said to be pseudo Hopfian when any surjective $f \in End(M)$ has a closed kernel, so any surjective $f \in End(M)$ is splits. In this paper, we will study some properties of pseudo Hopfian R-Module and their relationship with other R-Module, Recall from [2] that An R- Module M is named semi Hopfian when any surjective $f \in End(M)$ hold direct summand kernel, i.e. any surjective $f \in End(M)$ is splits, and clearly every Hopfian R- Module is semi Hopfian R-Module.

Recall from [3] that A sub- Module N of M is named closed sub- Module if N with no proper essential extension in M .denoted by $N \subseteq_c M$. i.e If $N \subseteq_c K \subseteq M$ then N = K . It is well-Known every direct summand of an R- mod. M was closed sub- Module but converse is not real in general [4]. This lead to introduce the following concept, namely pseudo Hopfian R-Module.

In the following study we give the main results which we obtained

MAIN RESULTS

Now, we state the new definition of pseudo Hopfian module and examples of this definition.

Definition.2.1. R- Module M is pseudo Hopfian when any surjective $f \in End$ (M) has a closed kernel, so any surjective $f \in End(M)$ is a split.

Example.2.2. Let Z_6 be a Z- Module, $Z_6 = 2Z \oplus 3Z$ .Then Z_6 is pseudo Hopfian Z- Module.

Example.2.3. If M is a direct sum of copies of Z_{p^2} , where P is aprime number. Then we suppose that M is a pseudo Hopfian Z – Module. Consider an epimorphism g: M → M Since p^2 M $= 0$, and f is an Z_{p^2} epimorphism. When M is a free Z_{p^2} – Module $g$is split.

This make that M is a pseudo Hopfian Z – Module. Whereas not a Hopfian Z – Module.

Remark.2.4. A sub- Module of pseudo Hopfian R-modules need not be pseudo Hopfian R- Module. See the following example:

Example.2.5. Let $M_1 = Z_p \; where \; p \; be \; a \; prime \; and$ M_2 an infinite sum of the copies of Z_{p^2}. So M_1 is simple M_2 is pseudo Hopfian R- Module by example (2.3).

When $M = M_1 \oplus M_2$ is not pseudo Hopfian Z – Module. Define: $M \to M$ by $f\ (a_1 + pZ\ , a_2 + p^2Z\ , a_3 + p^2Z\ , \ldots) = (\ a_2 + pZ\ , a_3 + p^2Z\ , \ldots)$ .Then g is a Z –epimorphism and Ker$g = Z_p \oplus p\ Z_{p^2} \oplus 0 \oplus 0 \ldots$ is not a direct summand and hance is not closed of M since pZ_{p^2} is not a direct summand of Z_{p^2}.

Proposition.2.6. every straight summand of a pseudo Hopfian R- Module is pseudo Hopfian R- Module.

Proof: Suppose M be a pseudo Hopfian R- Module and K be summand sub- Module of M and g: K ⟶ K be a surjection. So, $M = K \oplus N$ for N of M, and $g \oplus I_N : M \to M$ is again a surjection. So Ker $(g \oplus I_N)$ =Ker $g \subseteq_c M$ and hence Ker $\subseteq_c K$. So , K is pseudo Hopfian R- Module.

Recall from [5] that M is a fully stable if for every sub- Module N of M, $g: N \to M$ is then $g(N) \subseteq N$.

Proposition.2.7. Suppose M an R- Module such $M = M_1 \oplus M_2$, M is a stable. So, M is pseudo Hopfian R-Module iff M_1, M_2 are pseudo Hopfian R- Module.

Proof: The necessity is by prop.(2.6) . Let $f: M \longrightarrow M$ be a surjective. And the restriction of $f \in End(M_i)$ ($i \in I$,i=1,2), is again surjection. By theory Ker $(f | M_1)$ and Ker $(f | M_2)$ are closed in M_1, M_2 respectively. Then $Ker(f| M_1) \oplus Ker\ (f| M_2)$ is closed in M [3] .It easy to see that $Kerf = Ker(f| M_1) \oplus Ker\ (f| M_2)$ .Thus $Ker\ f$ is closed in M. So, M is pseudo HopfianR- Module

Corollary.2.8. Suppose $M = \oplus_{i \in I} M_i$, M is a stable. Then M is pseudo Hopfian R- Module iff $M_i is$ pseudo Hopfian R-module for all $\in I$.

Proof: Necessity is by prop.(2.6) . Let $f: M \longrightarrow M$ be a surjection . The restriction of $f \in End(M_i)$ ($i \in I$), is again surjection. By theory, $Ker\ (f| M_i) \subseteq_c M_i (i \in I)$ thus implies that $Ker\ f = \oplus_{i \in I} Ker(f| M_i) \subseteq_c M$ [3]. So, M is pseudo f. R- Module.

We study the relationship between pseudo Hopfian R- Module and (Hopfian , semi Hopfian , semi simple , simple , Ascending chain condition) R- Module.

Proposition.2.9. Every Hopfian R- Module is pseudo Hopfian R- Module.

Proof: the proof is obvious thus deleted.

Remark.2.10. The Converse of proposition (2.9) not true in general, as it appears in the following example.

Example.2.11. See example (2.3)

Recall from [6] that M is Dedekind finite if $M = M \oplus K$, for some sub- Module K, K=0.
Now, we can put some conditions to get the opposite of propos ion.(2.4) is true.

Proposition.2.12. Suppose M be a Dedekind finite R- Module. If M is pseudo Hopfian R- Module, so M is Hopfian R-Module.

Proof: Put $g: M \to M$ be asurjective ,so M is pseudo Hopfian R- Module , g is splits(from def.2.1) then ∃ $f: M \to M$ such $gf = I$,and since M is Dedekind finiteness of $End_R(M)$ implies $fg = I$, Hance g is an injection. Therefore, M is Hopfian R- Module.

We can get the following result from proposion. (2.9) and proposion (2.12).

Corollary.2.13. Suppose M be a Dedekind finite R- Module. So M is Hopfian R- Module iff M is pseudo Hopfian R-Module.

Proposition:-2.14. Each semi Hopfian R- Module is pseudo Hopfian R- Module.

Proof: the proof is obvious thus deleted.

Remark.2.15. The Converse of proposition (2.14) not true in general, as it appears in the following example.

Example.2.16. Let the module Ɱ = $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module, Let A=((2, 1)) be the sub-module generated by (2,1). A={(2,1) , (4,0),(6,1),(0,0)}.
So, A is closed in Ɱ but not a direct summand.

Recall from [7] that Ɱ is an extension R-module Iff every closed sub- Module of Ɱ is a straight summand of Ɱ.

Now, we can put some conditions to get the opposite of proposition (2.14) is true.

Proposition.2.17. Let Ɱ be an extending R- Module .Then any pseudo Hopfian R- Module is semi Hopfian R-module.

Proof: The proof is obvious thus deleted.
We can get the following result from proposion (2.14) and proposion (2.17).

Corollary: 2.18. Suppose Ɱ be an extending R- Module. So, Ɱ is pseudo Hopfian R- Module iff Ɱ is semi Hopfian R-Module.

Recall from [6] an R- Hopfian Ɱ is called semi simple module if sub- Module Ɲ of Ɱ is a direct summand of Ɱ.

Proposition.2.19. Any semi simple R- Module Ɱ is pseudo Hopfian R- Module.
Proof: the proof is obvious thus deleted.

Remark.2.20. The Converse of proposition (2.19) not true in general, as it appears in the following example.

Example.2.21. See example (2.16)
Recall from [6] that an R-mod. Ɱ is named simple Module if Ɱ has no proper sub- Module.

Proposition.2.22. Any simple R- Module Ɱ is pseudo Hopfian R- Module.

Proof: The proof is obvious thus deleted.

Remark.2.23. The Converse of proposition (2.22) not true in general, as it appears in the following example.

Example.2.24. Let Z_6 be a Z-module, $Z_6 = 2Z \oplus 3Z$ .Then Z_6 is pseudo Hopfian Z- Module, but not simple Z- Module.

Next, we introduce the relationship between Ascending chain condition and pseudo Hop. R- Module.

Proposition:-2.25. If Ɱ has Ascending chain condition on non–closed sub- Module Then Ɱ is pseudo Hopfian R-Module.

Proof: - Let g: Ɱ ⟶ Ɱ is a surjective and Ker$g$ is a non–closed of Ɱ . So $Ker\ g \subseteq Ker\ g^2 \subseteq Ker\ g^3 \subseteq \cdots$ be an ascend chain of non–closed sub-mod. of Ɱ . By theory there will be a number n such as $(g^n) = Ker\ (g^{n+1})$. Now we claim that $Kerg$= 0 .Let $0 \neq x \in$ Ɱ such that $g(x) = 0$. Since g is surjective, $g(a_1) = x$ for some $a_1 \in$ Ɱ . Also, $g(a_2) = a_1$ for some $a_2 \in$ Ɱ . By repeating this argument, we have $g(a_n) = a_{n-1}$ for some $a_n \in$ Ɱ . Then $g(a_1) = g^2\ (a_2) = \cdots = g^n\ (a_n) = x\ .hence\ g(x) = g^{n+1}(a_n)$ implies that $a_n \in Ker\ (g^{n+1}) = \ker(g^n)$. The result, $x = 0$. So we have opposites. Therefore, Ɱ is pseudo Hopfian R-module.

Remark:-2.26. Every Noetherian R- Module is pseudo Hopfian R- Module.

Proposition:-2.27. Let Ɱ/Ɲ is pseudo Hopfian R- Module for any non–closed sub-module Ɲ of a R- Module Ɱ. So Ɱ is pseudo Hopfian R- Module.

Proof: - Assume Ɱ is not pseudo Hopfian R- Module Then it is a surjection endomorphism g : Ɱ → Ɱ such that Ker$g$ is not a closed sub- Module of Ɱ. But by assumption Ɱ /Ker$g$ ≅ Ɱ is pseudo Hopfian, this opposite. So, Ɱ is pseudo Hopfian R- Module.

Lemma:-2.28. For each an R- Module Ɱ, the terms are equivalent.

- Ɱ is pseudo Hopfian R- Module.
- Each sub- Module K of Ɱ that satisfies Ɱ/K ≅ Ɱ , K is a closed sub- Module of Ɱ

Proposition:-2.29. Assume P be a value of R- Module under isomorphism. If a R- Module M has the value P and satisfies Ascending chain condition on closed sub-mod. K as M/K has the property P. Then M is pseudo Hop. R-Module.

Proof: - Assume M is not pseudo Hopfian R-module. So there is a non–closed sub-Module $K_1$of M as M $/K_1 \cong$ M .Since M/K_1has the value P but is not pseudo Hopfian, there is a non–closed sub- Module K_2/K_1 of M $/K_1$ such that M $/K_2 \cong$ M $/K_1$, K_2 is agian a non–closed of M. Doing this we get an ascending chain $0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of non–closed sub- Module of M. But this is a opposite. So, M is pseudo Hopfian R- Module.

Theorem:-2.30. Let M be an R – Module and M [X] be an extending R[X]- Module. When M [X] is pseudo Hopfian R [X] – Module, so M is pseudo Hopfian R – Module

Proof: - Assume f: M → M be a serjective endomorphism of M. so f [X] : M [X] → M [X] with $f[X](\sum M_i X^i) = \sum f(M_i) X^i$ is a serjective endomorphism of M [X]. when M [X] is pseudo Hopfian, Ker (f [X]) = (Ker f)[X] $\subseteq_c$ M [X] , since M [X] an extending R[X]-module .Then Ker (f [X]) is a total of M [X] . so we can claim Ker $f \subseteq_c$ M . let M [X] = (Ker$f$)[X] ⊕ K for sub- Module K of M [X] and N become the sub- Module of M that is calculated by the constant polynomials of K . Note that N ≠ 0 if M ≠ Ker$f$. We shall show M = Ker$f$ ⊕ N . Let m ∈ M. Then m ∈ M [X] so m = g (X) + k (X) since g (X) ∈(Kerf)[X], k (X) ∈K. when m is a constant in M [X], we get m =g (0) + k (0) where g (0) ∈ Ker f and k (0) ∈ N . Then, take ∈ $Ker\ f \cap$ N . But $n \in (Ker\ f)[X] \cap K = 0$ .So, $Ker\ f$ is direct summand, and hence is closed sub- Module of M. Therefore, M is pseudo Hopfian R- Module.

CONCLUSION

In this study, we present a new concept of pseudo Hopfian R-Module some properties of pseudo Hopfian R-Module are obtained, moreover the relationships between pseudo Hopfian R-Module and other R-Module are discussed.

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